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Mathematica Bohemica, Vol. 145 (2020), No. 3, 325–336

Persistent URL: http://dml.cz/dmlcz/148352

ON THE GROWTH OF ENTIRE SOLUTION OF A NONLINEAR DIFFERENTIAL EQUATION

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Received August 25, 2018. Published online November 21, 2019.
Communicated by Grigore Sălăgean

Abstract. In the paper we consider the growth of entire solution of a nonlinear differential equation and improve some existing results.

Keywords: entire function; nonlinear differential equation; growth of entire solution

MSC 2010: 30D15, 34A34, 30D35

1. Introduction, definition and results

Let f be an entire function and $M(r,f)$ the maximum modulus function of f. Also we denote by $T(r,f)$ the Nevanlinna characteristic function of f. Then

$$\sigma(f)=\limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log r}=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}$$

and

$$\mu(f)=\liminf_{r\to\infty}\frac{\log\log M(r,f)}{\log r}=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}$$

are respectively called the *order* and *lower order* of f.

Also

$$\sigma_2(f)=\limsup_{r\to\infty}\frac{\log\log\log M(r,f)}{\log r}=\limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r}$$

and

$$\mu_2(f)=\liminf_{r\to\infty}\frac{\log\log\log M(r,f)}{\log r}=\liminf_{r\to\infty}\frac{\log\log T(r,f)}{\log r}$$

are respectively called the *hyper-order* and *lower hyper-order* of f.

The work of Shubhashish Das is supported by CSIR fellowship, India (File No. 09/106(0144)/2013-EMR-I)

DOI: 10.21136/MB.2019.0105-18

A conjecture of Brück (see [2]) on the value sharing of an entire function with its derivative gives rise to a stream of research on the growth of entire solutions of some differential equations.

Let f be an entire function. We consider a differential polynomial of the form

$$L(f) = f^{(p)} + a_{p-1}f^{(p-1)} + \ldots + a_1 f^{(1)} + a_0 f, \tag{1.1}$$

where p is a positive integer and $a_0, a_1, \ldots, a_{p-1}$ are complex numbers.

In 2008, Li and Yi (see [6]) proved the following result on the growth of an entire solution of a linear differential equation.

Theorem A ([6]). *Let $A = A(z)$ be a nonconstant polynomial and let a $(\neq 0, \infty)$ be a complex number. If f is a nonconstant solution of the differential equation*

$$L(f) - a = (f - a)\mathrm{e}^{A},$$

where $L(f)$ is defined by (1.1), *then one of the following two cases will occur:*

(i) *If $\mu(f) > 1$, then $\mu(f) = \infty$ and $\mu_2(f) = \sigma_2(f) = \deg A$.*

(ii) *If $\mu(f) \leqslant 1$, then $\mu(f) = 1$ and $A = az + b$, where a $(\neq 0)$ and b are complex numbers and $a_0, a_1, \ldots, a_{p-1}$ are not all zero.*

In 2009, Li and Yi (see [7]) extended Theorem A and proved the following result.

Theorem B ([7]). *If f is a transcendental entire solution of the differential equation*

$$L(f) - \alpha_1 = (f - \alpha_2)\mathrm{e}^{A},$$

where $L(f)$ is defined by (1.1), *$A = A(z)$ is a nonconstant polynomial, α_1 and α_2 are entire functions such that $\sigma(\alpha_j) < 1$ for $j = 1, 2$, then the conclusion of Theorem A holds.*

In 2013, Bouabdelli and Belaïdi (see [1]) also extended Theorem A and Theorem B and proved the following result.

Theorem C ([1]). *Let $A = A(z)$ be a nonconstant polynomial and let α_1, α_2 be entire functions with $\sigma(\alpha_j) < 1$ for $j = 1, 2$. If f is a nonconstant solution of the differential equation*

$$(L(f))^l - \alpha_1 = (f^l - \alpha_2)\mathrm{e}^{A},$$

where $L(f)$ is defined by (1.1) *and l $(\geqslant 1)$ is an integer, then the conclusion of Theorem A holds.*

We note that Theorem C uses a special type of nonlinear homogeneous differential polynomial $(L(f))^l$. So one may naturally ask: *what will happen if* $(L(f))^l$ *is replaced by a general homogeneous differential polynomial?*

In the paper we consider this problem and improve Theorem A, Theorem B and Theorem C. We now require the following well known definition.

Definition 1.1. Let f be an entire function and let $a_1, a_2, \ldots, a_p$ be polynomials. An expression form

$$P(f) = \sum_{j=1}^{p} P_j(f), \tag{1.2}$$

is called a *homogeneous differential polynomial of degree* $\gamma_P = \sum_{k=0}^{m_j} n_{jk}$ for $j = 1, 2, \ldots, p$, where

$$P_j(f) = a_j(f)^{n_{j0}}(f^{(1)})^{n_{j1}} \ldots (f^{(m_j)})^{n_{jm_j}}$$

is called a *differential monomial*.

The number $\Gamma_P = \max\{\Gamma_j\colon 1 \leqslant j \leqslant p\}$ is called the *weight of* $P(f)$, where $\Gamma_j = \sum_{k=0}^{m_j} (k+1)n_{jk}$ is called the *weight of* $P_j(f)$ for $j = 1, 2, \ldots, p$.

Let $P(f)$ be given by (1.2). We divide the set of coefficients $C = \{a_1, a_2, \ldots, a_p\}$ of $P(f)$ into two subsets as follows: Let $A = \{a_j\colon a_j \in C$ such that $\Gamma_j = \Gamma_P\}$ and $B = C \setminus A$.

We denote by $a = a(z)$ a polynomial of the subset A that has the maximum degree among the members of A. If there are more than one a_j's in A with maximum degree we denote by $a = a(z)$ any one of those. Further, let $\chi_j = (\deg a_j - \deg a)/(\Gamma_P - \Gamma_j)$ if $a_j \in B$ and $\chi_j = 0$ if $a_j \in A$.

We now state the main result of the paper.

Theorem 1.1. *Let f, α_1, α_2 be three entire functions such that $f^n \not\equiv \alpha_2$ and $\sigma(\alpha_j) < 1$ for $j = 1, 2$. Suppose that $P(f)$ is given by (1.2) and $A = A(z)$ is a nonconstant polynomial such that f satisfies the differential equation*

$$P(f) - \alpha_1 = (f^n - \alpha_2)\mathrm{e}^A, \tag{1.3}$$

where $n = \gamma_P$.

(i) *If $\mu(f) > 1 + \max\limits_{1 \leqslant j \leqslant p}\{\chi_j, 0\}$, then $\mu(f) = \infty$ and $\mu_2(f) = \sigma_2(f) = \deg A$.*
(ii) *If $\mu(f) \leqslant 1$, then $\mu(f) = 1$ and $A = az + b$, where a $(\neq 0)$ and b are two finite complex numbers and at least two of $a_1, a_2, \ldots, a_p$ are not identically zero.*

The following example shows that Theorem 1.1 does not admit the case $\mu(f) = 1 + \max\limits_{1\leqslant j\leqslant p}\{\chi_j, 0\}$, but the case $1 < \mu(f) < 1 + \max\limits_{1\leqslant j\leqslant p}\{\chi_j, 0\}$ is unanswered and so remains as an open problem. However, if all the coefficients a_j's are constants, then $\max\limits_{1\leqslant j\leqslant p}\{\chi_j, 0\} = 0$ and so the case $1 < \mu(f) < 1 + \max\limits_{1\leqslant j\leqslant p}\{\chi_j, 0\}$ does not arise.

Example 1.1 ([8]). Let $f = \mathrm{e}^{-z^2/2} + z^2$, $\alpha_1 = \alpha_2 = z^2$ and $P(f) = \frac{1}{3}f^{(2)} + \frac{1}{3}zf^{(1)} + \frac{1}{3}f$. Then $\mu(f) = 2 = 1 + \max\limits_{1\leqslant j\leqslant 3}\{\chi_j, 0\}$ and $P(f) - \alpha_1 = \frac{2}{3}\mathrm{e}^{z^2/2}(f - \alpha_2)$.

For standard definitions and notation we refer the reader to [4] and [5].

2. Lemmas

In this section we present some necessary lemmas. Let $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ be an entire function. Then $\mu(r, f) = \max\{|a_n|r^n\colon n = 0, 1, 2, \ldots\}$ is called the *maximum term of* f and $\nu(r, f) = \max\{n\colon \mu(r, f) = |a_n|r^n\}$ is called the *central index of* f.

Lemma 2.1 ([5], page 51). *If f is an entire function of order $\sigma(f)$, then*

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log \nu(r, f)}{\log r}.$$

Lemma 2.2 ([5], page 9). *Let $A(z) = b_n z^n + b_{n-1}z^{n-1} + \ldots + b_0$, $b_n \neq 0$ be a polynomial of degree n with constant coefficients. Then for a given $\varepsilon > 0$ there exists $R > 0$ such that for all $|z| = r > R$ we have*

$$(1 - \varepsilon)|b_n|r^n \leqslant |A(z)| \leqslant (1 + \varepsilon)|b_n|r^n.$$

Lemma 2.3 ([5], page 51). *Let f be a transcendental entire function. Then there exists a set $E \subset (1, \infty)$ with finite logarithmic measure such that for $|z| = r \notin [0, 1] \cup E$ and $|f(z)| = M(r, f)$ we have*

$$\frac{f^{(j)}(z)}{f(z)} = (1 + o(1))\Big(\frac{\nu(r, f)}{z}\Big)^j$$

for $j = 1, 2, \ldots, k$, where k is a positive integer.

Lemma 2.4 ([5], page 36). *Let f be a transcendental entire function and let $p \geqslant 1$ be an integer. Then*

$$m\Big(r, \frac{f^{(p)}}{f}\Big) = O\Big(\log T(r, f) + \log r\Big)$$

possibly outside a set of finite linear measure.

Lemma 2.5 ([5], page 5). *Let $g\colon (0,\infty)\to\mathbb{R}$ and $h\colon (0,\infty)\to\mathbb{R}$ be monotone increasing functions such that $g(r)\leqslant h(r)$ outside a set of finite logarithmic measure. Then for a given $\alpha>1$ there exists $R>0$ such that $g(r)\leqslant h(r^{\alpha})$ for all $r>R$.*

Lemma 2.6 ([5], page 5). *Let $g\colon (0,\infty)\to\mathbb{R}$ and $h\colon (0,\infty)\to\mathbb{R}$ be monotone increasing functions such that $g(r)\leqslant h(r)$ outside a set of finite linear measure. Then for a given $\alpha>1$ there exists $R>0$ such that $g(r)\leqslant h(\alpha r)$ for all $r>R$.*

Lemma 2.7 ([6]). *For an entire function f*

$$\mu(f)=\liminf_{r\to\infty}\frac{\log\nu(r,f)}{\log r}\quad\text{and}\quad \mu_2(f)=\liminf_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r}.$$

Lemma 2.8 ([3]). *For an entire function f*

$$\sigma_2(f)=\limsup_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r}.$$

3. Proof of Theorem 1.1

P r o o f. First we verify that an entire function f that satisfies (1.3) with $f^n\not\equiv\alpha_2$ must be transcendental. On the contrary we suppose that f is a polynomial and satisfies (1.3). Then $P(f)$ and f^n are also polynomials. So we have $1\leqslant\deg A=\sigma(\mathrm{e}^A)=\sigma((P(f)-\alpha_1)/(f^n-\alpha_2))\leqslant\max\{\sigma(\alpha_1),\sigma(\alpha_2)\}<1$, a contradiction.

Now by Lemma 2.3 there exists $E\subset[1,\infty)$ with finite logarithmic measure such that for $|z|=r\notin E\cup[0,1]$ and $|f(z)|=M(r,f)$ we have

$$\frac{f^{(j)}(z)}{f(z)}=\Bigl(\frac{\nu(r,f)}{z}\Bigr)^{j}(1+o(1)),\tag{3.1}$$

for $j=1,2,\dots u$, where $u=\max\{m_j\colon 1\leqslant j\leqslant p\}$.

Now for all z with $|z|=r\notin E\cup[0,1]$ and $|f(z)|=M(r,f)$ we get by (3.1)

$$\frac{P_j(f)}{f^n}=a_j\Bigl(\frac{\nu(r,f)}{z}\Bigr)^{\Gamma_j-n}(1+o(1)),\tag{3.2}$$

where $\Gamma_j=\Gamma_{P_j}$ for $j=1,2,\dots p$.

Therefore from (3.2) we get for all z with $|z|=r\notin E\cup[0,1]$ and $|f(z)|=M(r,f)$

$$\frac{P(f)}{f^n}=\sum_{j=1}^{p}a_j\Bigl(\frac{\nu(r,f)}{z}\Bigr)^{\Gamma_j-n}(1+o(1)).\tag{3.3}$$

We now consider the following cases.

Case I. Let $\mu(f) > 1 + \max_{1 \leqslant j \leqslant p} \{\chi_j, 0\}$. In this case we see that $\sigma(\alpha_j) < \mu(f)$ for $j = 1, 2$. Hence there exists r_0 (> 0) such that $M(r, \alpha_j) < \frac{1}{2} M(r, f)$ for all $r \geqslant r_0$ and $j = 1, 2$.

Since $M(r, f) > 1$ for all sufficiently large values of r, we get

$$\frac{M(r, \alpha_j)}{M(r, f^n)} = \frac{M(r, \alpha_j)}{(M(r, f))^n} < \frac{1}{2} \tag{3.4}$$

for all sufficiently large values of r and $j = 1, 2$. Also we note that (3.4) is obvious if α_j is constant for some $j \in \{1, 2\}$.

Let $\Gamma_1 = \Gamma_2 = \ldots = \Gamma_t = \Gamma_{t+1} = \Gamma_P = \Gamma$ and $\Gamma_j < \Gamma$ for $j = t+2, t+3, \ldots, p$. If any two or more of $a_1, a_2, \ldots, a_t, a_{t+1}$ have the same degree, then in view of (3.3) we can add them to obtain a term like $b(\nu(r, f)/z)^{\Gamma - n}(1 + o(1))$, where b is a polynomial with degree not exceeding that of a_j's having the same degree. So without loss of generality we suppose that the degrees of no two polynomials of $a_1, a_2, \ldots, a_t, a_{t+1}$ are the same. Also, by rearranging the terms if necessary, we suppose that $\deg a_{t+1} > \deg a_t > \deg a_j$ for $j = 1, 2, \ldots, t-1$. Then from (3.3) we get for all sufficiently large $|z| = r \notin E \cup [0, 1]$ and $|f(z)| = M(r, f)$

$$\begin{aligned} \frac{P(f)}{f^n} &= a_t \left(1 + \sum_{j=1}^{t-1} \frac{a_j}{a_t}\right) \left(\frac{\nu(r, f)}{z}\right)^{\Gamma - n} (1 + o(1)) \\ &\quad + \sum_{j=t+1}^{p} a_j \left(\frac{\nu(r, f)}{z}\right)^{\Gamma_j - n} (1 + o(1)) = F_1(z) + F_2(z), \quad \text{say.} \end{aligned} \tag{3.5}$$

Since $\deg a_j < \deg a_t$ for $j = 1, 2, \ldots, t-1$, by Lemma 2.2 we have $a_j(z)/a_t(z) \to 0$ as $z \to \infty$ for $j = 1, 2, \ldots, t-1$. So for sufficiently large $|z| = r \notin E \cup [0, 1]$ and $|f(z)| = M(r, f)$

$$F_1(z) = a_t(z) \left(\frac{\nu(r, f)}{z}\right)^{\Gamma - n} (1 + o(1)). \tag{3.6}$$

We now show that for sufficiently large $|z| = r \notin E \cup [0, 1]$ and $|f(z)| = M(r, f)$

$$F_2(z) = a_{t+1}(z) \left(\frac{\nu(r, f)}{z}\right)^{\Gamma - n} (1 + o(1)). \tag{3.7}$$

Let $d_j = \deg a_j$ for $j = 1, 2, \ldots, p$. Since $\mu = \mu(f) > 1 + (d_j - d_{t+1})/(\Gamma - \Gamma_j)$ for $j = t+2, t+3, \ldots, p$, we can choose an ε such that

$$0 < \varepsilon < \min_{t+2 \leqslant j \leqslant p} \frac{(\Gamma - \Gamma_j)(\mu - 1) + d_{t+1} - d_j}{2(\Gamma - \Gamma_j)}.$$

Since $\mu(f) > 1 + (d_j - d_{t+1})/(\Gamma - \Gamma_j) + \varepsilon$ for $t+2 \leqslant j \leqslant p$, we get by Lemma 2.7 for all sufficiently large values of r

$$\nu(r,f) > r^{1+(d_j - d_{t+1})/(\Gamma-\Gamma_j)+\varepsilon}, \tag{3.8}$$

for $j = t+2, t+3, \ldots, p$.

So by Lemma 2.2 and (3.8) we get for all sufficiently large values of r and $j = t+2, t+3, \ldots, p$

$$\begin{aligned}\frac{|a_j(z)/a_{t+1}(z) z^{\Gamma-\Gamma_j}(\nu(r,f))^{\Gamma_j-n}|}{(\nu(r,f))^{\Gamma-n}} &\leqslant M_1 r^{d_j-d_{t+1}+\Gamma-\Gamma_j}(\nu(r,f))^{-(\Gamma-\Gamma_j)} \\ &< M_1 r^{d_j-d_{t+1}+\Gamma-\Gamma_j-\Gamma+\Gamma_j-d_j+d_{t+1}-\varepsilon(\Gamma-\Gamma_j)} \\ &= M_1 r^{-\varepsilon(\Gamma-\Gamma_j)} \to 0 \quad \text{as } |z| = r \to \infty,\end{aligned}$$

where $M_1 > 0$ is a suitable constant.

Hence

$$\frac{a_j(z)}{a_{t+1}(z)} z^{\Gamma-\Gamma_j}(\nu(r,f))^{\Gamma_j-n} = o(\nu(r,f)^{\Gamma-n}) \tag{3.9}$$

as $r \to \infty$.

So for sufficiently large $|z| = r \notin E \cup [0,1]$ with $|f(z)| = M(r,f)$ we get by (3.9)

$$\begin{aligned}F_2(z) &= \frac{a_{t+1}(z)}{z^{\Gamma-n}}\left((\nu(r,f))^{\Gamma-n} + \sum_{j=t+2}^{p}\frac{a_j(z)}{a_{t+1}(z)} z^{\Gamma-\Gamma_j}(\nu(r,f))^{\Gamma_j-n}\right)(1+o(1)) \\ &= a_{t+1}(z)\Big(\frac{\nu(r,f)}{z}\Big)^{\Gamma-n}(1+o(1)).\end{aligned}$$

Now by (3.5) and (3.6) and Lemma 2.2 we get for sufficiently large $|z| = r \notin E \cup [0,1]$ and $|f(z)| = M(r,f)$

$$\begin{aligned}\frac{P(f)}{f^n} &= (a_t(z) + a_{t+1}(z))\Big(\frac{\nu(r,f)}{z}\Big)^{\Gamma-n}(1+o(1)) \\ &= a_{t+1}(z)\Big(\frac{\nu(r,f)}{z}\Big)^{\Gamma-n}(1+o(1)).\end{aligned} \tag{3.10}$$

Now from (3.10) and Lemma 2.2 we get for sufficiently large $|z| = r \notin E \cup [0,1]$ and $|f(z)| = M(r,f)$

$$\Big|\frac{P(f)}{f^n}\Big| = \Big|a_{t+1}(z)\Big(\frac{\nu(r,f)}{z}\Big)^{\Gamma-n}(1+o(1))\Big| \leqslant 4|\beta_{t+1}|\Big(\frac{\nu(r,f)}{r}\Big)^{\Gamma-n} r^{\deg a_{t+1}}, \tag{3.11}$$

$$\Big|\frac{P(f)}{f^n}\Big| = \Big|a_{t+1}(z)\Big(\frac{\nu(r,f)}{z}\Big)^{\Gamma-n}(1+o(1))\Big| \geqslant \frac{1}{4}|\beta_{t+1}|\Big(\frac{\nu(r,f)}{r}\Big)^{\Gamma-n} r^{\deg a_{t+1}}, \tag{3.12}$$

where β_{t+1} is the leading coefficient of $a_{t+1}(z)$.

Since $\mu = \mu(f) > 1$, we have for all large values of r, $\nu(r,f) > r^{1+\varepsilon_0}$, where $0 < 2\varepsilon_0 < \mu - 1$. Therefore for all large values of r we get

$$\left(\frac{\nu(r,f)}{r}\right)^{\Gamma-n} r^{\deg a_{t+1}} > r^{\varepsilon_0(\Gamma-n)+d_{t+1}}. \tag{3.13}$$

Now from (3.4) and (3.11) we get for sufficiently large $|z| = r \notin E \cup [0,1]$ and $|f(z)| = M(r,f)$

$$\begin{aligned}\left|\frac{P(f)-\alpha_1}{f^n-\alpha_2}\right| &\leqslant \frac{|P(f)f^{-n}| + |\alpha_1 f^{-n}|}{1-|\alpha_2 f^{-n}|} \\ &\leqslant \frac{4|\beta_{t+1}|(\nu(r,f)r^{-1})^{\Gamma-n} r^{d_{t+1}} + \frac{1}{2}}{1-\frac{1}{2}} = M_2\left(\frac{\nu(r,f)}{r}\right)^{\Gamma-n} r^{d_{t+1}},\end{aligned} \tag{3.14}$$

where $M_2 > 0$ is a constant.

Similarly, from (3.4), (3.12) and (3.13) we get for sufficiently large $|z| = r \notin E \cup [0,1]$ and $|f(z)| = M(r,f)$

$$\begin{aligned}\left|\frac{P(f)-\alpha_1}{f^n-\alpha_2}\right| &\geqslant \frac{|P(f)f^{-n}| - |\alpha_1 f^{-n}|}{1+|\alpha_2 f^{-n}|} \\ &\geqslant \frac{\frac{1}{4}|\beta_{t+1}|(\nu(r,f)r^{-1})^{\Gamma-n} r^{d_{t+1}} - \frac{1}{2}}{1+\frac{1}{2}} > M_3 r^{\varepsilon_0(\Gamma-n)+d_{t+1}},\end{aligned} \tag{3.15}$$

where $M_3 > 0$ is a constant.

By Lemma 2.2 we get for all sufficiently large $|z| = r$

$$\frac{1}{2}|\beta| r^{\deg A} \leqslant |A(z)|, \tag{3.16}$$

where β is the leading coefficient of $A = A(z)$.

Since $A(z) = \log{(P(f)-\alpha_1)}/(f^n-\alpha_2)$, we get from (3.14) in view of (3.15) for sufficiently large $|z| = r \notin E \cup [0,1]$ and $|f(z)| = M(r,f)$

$$\begin{aligned}\Big|A(z)\Big| &= \left|\log\frac{P(f)-\alpha_1}{f^n-\alpha_2}\right| \\ &\leqslant \left|\log\left|\frac{P(f)-\alpha_1}{f^n-\alpha_2}\right|\right| + 2\pi = \log\left|\frac{P(f)-\alpha_1}{f^n-\alpha_2}\right| + 2\pi \\ &\leqslant (\Gamma-n)\log\nu(r,f) + (\Gamma-n+d_{t+1})\log r + |\log M_2| + 2\pi \\ &\leqslant M_4 \log\nu(r,f),\end{aligned} \tag{3.17}$$

where $M_4 > 0$ is a constant.

Now from (3.16) and (3.17) we get for sufficiently large $r \notin E \cup [0,1]$

$$\frac{1}{2}|\beta| r^{\deg A} \leqslant M_4 \log \nu(r,f)$$

and so

$$\deg A \log r \leqslant \log\log \nu(r,f) + \log \frac{2M_4}{\beta}.$$

Therefore, by Lemma 2.5 for a given $\xi > 1$, there exists $r_0 > 0$ such that for all $r > r_0$

$$\deg A \log r \leqslant \log\log \nu(r^{\xi},f) + \log \frac{2M_4}{\beta}.$$

By Lemma 2.7 this implies $\deg A \leqslant \xi \mu_2(f)$. Since $\xi > 1$ is arbitrary, we get

$$\deg A \leqslant \mu_2(f). \tag{3.18}$$

Now for sufficiently large $|z| = r \notin E \cup [0,1]$ and $|f(z)| = M(r,f)$ we get from (1.3) and (3.15)

$$M_3\Big(\frac{\nu(r,f)}{r}\Big)^{\Gamma-n} r^{d_{t+1}} \leqslant \Big|\frac{P(f)-\alpha_1}{f^n-\alpha_2}\Big| = |\mathrm{e}^{A(z)}| \leqslant M(r,\mathrm{e}^A). \tag{3.19}$$

First we suppose that $d_{t+1} < \Gamma - n$. Then from (3.19) we get for sufficiently large $r \notin E \cup [0,1]$ that

$$M_3(\nu(r,f))^{\Gamma-n} \leqslant M(r,\mathrm{e}^A) r^{\Gamma-n-d_{t+1}}.$$

So by Lemma 2.5 for a given $\xi > 1$ there exists $r_0 > 0$ such that for all $r > r_0$

$$M_3(\nu(r,f))^{\Gamma-n} \leqslant M(r^{\xi},\mathrm{e}^A) r^{\xi(\Gamma-n-d_{t+1})}.$$

Hence by Lemma 2.8 we get

$$\sigma_2(f) \leqslant \xi \sigma(\mathrm{e}^A) = \xi \deg A.$$

Since $\xi > 1$ is arbitrary, we have

$$\sigma_2(f) \leqslant \deg A. \tag{3.20}$$

Next we suppose that $\Gamma - n \leqslant d_{t+1}$. Then from (3.19) we get for sufficiently large $r \notin E \cup [0,1]$ that

$$M_3(\nu(r,f))^{(\Gamma-n)} r^{d_{t+1}-(\Gamma-n)} \leqslant M(r,\mathrm{e}^A).$$

So by Lemma 2.5 for a given $\xi > 1$ there exists $r_0 > 0$ such that for all $r > r_0$ we get

$$M_3(\nu(r,f))^{(\Gamma-n)} r^{d_{t+1}-(\Gamma-n)} \leqslant M(r^{\xi},\mathrm{e}^A).$$

Now proceeding as above we obtain (3.20). Combining (3.18) and (3.20) we get

$$\mu_2(f) = \sigma_2(f) = \deg A.$$

Since $\deg A \geqslant 1$, it follows that $\mu(f) = \infty$.

Case II. Let $\mu(f) \leqslant 1$. Then by (1.3) and Lemma 2.4 we get

$$\begin{aligned} T(r, \mathrm{e}^A) &= m(r, \mathrm{e}^A) \leqslant m\Big(r, \frac{P(f)}{f^n}\Big) + T\Big(r, \frac{\alpha_1}{f^n}\Big) + T\Big(r, \frac{\alpha_2}{f^n}\Big) + O(1) \\ &= O(\log T(r,f)) + O(T(r,f)) + O(\log r) + O(T(r,\alpha_1)) \\ &\quad + O(T(r,\alpha_2)) + O(1) \\ &= O(T(r,f)) + O(T(r,\alpha_1)) + O(T(r,\alpha_2)), \end{aligned} \tag{3.21}$$

possibly outside a set of r of finite linear measure.

By Lemma 2.6 we get from (3.21) that for all sufficiently large values of r

$$T(r, \mathrm{e}^A) \leqslant M_5(T(2r,f) + T(2r,\alpha_1) + T(2r,\alpha_2)), \tag{3.22}$$

where $M_5 > 0$ is a constant.

Since $\sigma(\alpha_j) < 1$ for $j = 1, 2$, from (3.22) we get for all sufficiently large values of r

$$T(r, \mathrm{e}^A) \leqslant M_6(T(2r,f) + (2r)^\alpha), \tag{3.23}$$

where $M_6 > 0$ is a constant and $0 < \alpha < 1$.

Since $\deg A \geqslant 1$, we see that

$$\frac{(2r)^\alpha}{T(r,\mathrm{e}^A)} = \frac{(2r)^\alpha}{|\beta|\pi^{-1} r^{\deg A} + O(1)} \to 0 \quad \text{as } r \to \infty,$$

where β is the leading coefficient of A.

Hence from (3.23) we get for all sufficiently large values of r

$$T(r, \mathrm{e}^A)\Big(1 - \frac{M_6(2r)^\alpha}{T(r,\mathrm{e}^A)}\Big) \leqslant M_6 T(2r,f),$$

which implies

$$1 \leqslant \deg A = \mu(\mathrm{e}^A) \leqslant \mu(f) \leqslant 1.$$

Therefore $\mu(f) = 1$ and $A = A(z)$ is a linear polynomial of the form $A(z) = az + b$, where $a \neq 0$.

We shall now show that at least two of the coefficients $a_1, a_2, \ldots, a_p$ are not identically zero. Let $P(f) = a_1(f)^{n_{10}}(f^{(1)})^{n_{11}} \ldots (f^{(m_1)})^{n_{1m_1}}$. Then from (3.3) we get for all z with $|z| = r \notin E \cup [0,1]$ and $|f(z)| = M(r,f)$

$$\frac{P(f)}{f^n} = a_1(z)\Big(\frac{\nu(r,f)}{z}\Big)^{\Gamma - n}(1 + o(1)), \tag{3.24}$$

where $\Gamma_P = \Gamma$.

Since $\sigma(\alpha_j) < 1 = \mu(f)$ for $j = 1, 2$, we see that $M(r, \alpha_j)/M(r, f) \to 0$ as $r \to \infty$. Hence by (3.24) we get for large $|z| = r \notin E \cup [0, 1]$ and $|f(z)| = M(r, f)$

$$
\begin{aligned}
(3.25) \qquad \log\left|\frac{P(f) - \alpha_1}{f^n - \alpha_2}\right| &= \log \frac{|P(f)/f^n| + o(1)}{1 + o(1)} \\
&= \log\Big(|a_1(z)|\Big(\frac{\nu(r, f)}{r}\Big)^{\Gamma - n}(1 + o(1))\Big) \\
&= \log|a_1(z)| + (\Gamma - n)\log\frac{\nu(r, f)}{r} + o(1) \\
&= O(\log r) + (\Gamma - n)\log\nu(r, f).
\end{aligned}
$$

Now by (1.3) we have

$$
(3.26) \qquad A = \log\frac{P(f) - \alpha_1}{f^n - \alpha_2} = \log\left|\frac{P(f) - \alpha_1}{f^n - \alpha_2}\right| + \mathrm{i}\,\mathrm{Arg}\Big(\frac{P(f) - \alpha_1}{f^n - \alpha_2}\Big),
$$

where $\mathrm{Arg}((P(f) - \alpha_1)/(f^n - \alpha_2))$ denotes the principal argument of $(P(f) - \alpha_1)/(f^n - \alpha_2)$.

Since $|\mathrm{Arg}((P(f) - \alpha_1)/(f^n - \alpha_2))| \leqslant 2\pi$, we get from (3.25) and (3.26) and for large $|z| = r \notin E \cup [0, 1]$ and $|f(z)| = M(r, f)$

$$
(3.27) \qquad |A(z)| \leqslant M_7 \log r + (\Gamma - n)\log\nu(r, f),
$$

where $M_7 > 0$ is a constant.

Again by Lemma 2.2 we get for all large values of r

$$
(3.28) \qquad \frac{|a|}{2} r \leqslant |A(z)|.
$$

From (3.27) and (3.28) we get for large values of $r \notin E \cup [0, 1]$

$$
(3.29) \qquad \frac{|a|}{2} r \leqslant M_7 \log r + (\Gamma - n)\log\nu(r, f).
$$

By Lemma 2.5 for a given $\xi > 1$ there exists $r_0 > 0$ such that for all $r > r_0$ we get from (3.29)

$$
\frac{|a|}{2} r \leqslant \xi M_7 \log r + (\Gamma - n)\log\nu(r^{\xi}, f),
$$

which implies

$$
\lim_{r \to \infty} \frac{r}{\log r} \leqslant \frac{2\xi}{|a|}(M_7 + (\Gamma - n)\mu(f)) < \infty,
$$

a contradiction. Therefore at least two of $a_1, a_2, \ldots, a_p$ are not identically zero. This proves the theorem. $\square$

A c k n o w l e d g e m e n t The authors are thankful to the referee for his/her valuable observations and suggestions towards improvement of the paper.

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