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ON A SINGULAR MULTI-POINT THIRD-ORDER BOUNDARY VALUE PROBLEM ON THE HALF-LINE

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Abstract. We establish not only sufficient but also necessary conditions for existence of solutions to a singular multi-point third-order boundary value problem posed on the half-line. Our existence results are based on the Krasnosel'skii fixed point theorem on cone compression and expansion. Nonexistence results are proved under suitable a priori estimates. The nonlinearity $f = f(t, x, y)$ which satisfies upper and lower-homogeneity conditions in the space variables x, y may be also singular at time $t = 0$. Two examples of applications are included to illustrate the existence theorems.

Keywords: singular nonlinear boundary value problem; positive solution; Krasnosel'skii fixed point theorem; multi-point; half-line

MSC 2010: 34B10, 34B16, 34B18, 34B40

1. Introduction

This work is concerned with the following multi-point third-order boundary value problem posed on $(0, \infty)$:

$$\begin{aligned} x(0) &= \sum_{i=1}^{n_1} \alpha_i x(\xi_i), \\ x'(0) &= \sum_{i=1}^{n_2} \beta_i x'(\eta_i), \\ \lim_{t\to\infty} x''(t) &= 0, \\ -x'''(t) &= f(t, x(t), x'(t)), \quad t > 0, \end{aligned} \tag{1.1}$$

where $0 \leqslant \alpha_j \leqslant \sum\limits_{i=1}^{n_1} \alpha_i < 1$ $(j = 1, 2, \ldots, n_1)$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{n_1} < \infty$,

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$0 \leqslant \beta_j \leqslant \sum\limits_{i=1}^{n_2} \beta_i < 1$ $(j = 1, 2, \ldots, n_2)$, $0 < \eta_1 < \eta_2 < \ldots < \eta_{n_2} < \infty$. The nonlinearity $f\colon (0,\infty) \times [0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous and there exist $0 < \alpha < \beta < \infty$ such that $I_{\alpha,\beta} = \int_\alpha^\beta f(t, 1+t^2, 1+t)\,\mathrm{d}t > 0$. Moreover,

$$(\mathcal{H}) \qquad \begin{gathered} \text{there exist constants } \lambda_1, \lambda_2, \mu_1, \mu_2, \\ 0 < \lambda_1 \leqslant \mu_1,\ 0 \leqslant \lambda_2 \leqslant \mu_2 < 1,\ \lambda_1 + \lambda_2 > 1 \\ \text{such that for all } t > 0,\ x, y \geqslant 0 \text{ and for all } 0 < c,\ d \leqslant 1, \\ c^{\mu_1} d^{\mu_2} f(t,x,y) \leqslant f(t,cx,dy) \leqslant c^{\lambda_1} d^{\lambda_2} f(t,x,y). \end{gathered}$$

Taking $c = x(t)/y(t)$ and $d = x'(t)/y'(t)$ in Hypothesis $(\mathcal{H})$, we obtain a monotonicity property for the nonlinearity f

$$f(t, x(t), x'(t)) \leqslant f(t, y(t), y'(t))$$

whenever

$$0 \leqslant x(t) \leqslant y(t) \quad \text{and} \quad 0 \leqslant x'(t) \leqslant y'(t).$$

By time-singularity, we mean that the function f in (1.1) is allowed to be unbounded at the point $t = 0$.

Boundary value problems (BVPs for short) on the half-line arise naturally in many applications in physics and engineering. Since the solution may represent a density, temperature or a concentration, its positivity is required for physical considerations. This motivates the study of such BVPs on positive cones of some functional Banach spaces. Also, the nonlinearity f, which represents a physical law, is generally positive, depends on t, x, and may depend on the first derivative. Some general existence results for different classes of BVPs may be found in [1]. The particular case of BVPs associated with second-order operators has recently received great attention (see, e.g., [4], [11]). However, only few papers have considered problems for higher order differential equations on infinite intervals of the real line (we refer to [5], [7] for some specific results). In [9], a fourth-order m-point BVP is studied on the bounded interval $[0,1]$ and existence of solutions is obtained by application of a fixed point theorem on a cone while in [10], a third-order multi-point BVP is treated via comparison arguments. Necessary and sufficient conditions for existence of solutions are then provided. Some singular BVPs are discussed in [2], [8]. Following [9], [10], our aim in this paper is to provide sufficient and necessary conditions for solutions of the singular third-order BVP (1.1) to exist on the half-line. The adequate functional space is

$$X = \Bigl\{x \in C^1([0,\infty), \mathbb{R})\colon \lim_{t\to\infty} \frac{x'(t)}{1+t} = 0\Bigr\}.$$

It is a Banach space with the norm

$$\|x\| = \max\{\|x\|_0, \|x\|_1\},$$

where $\|x\|_0 = \sup_{t\in\mathbb{R}^+} |x(t)|/(1+t^2)$ and $\|x\|_1 = \sup_{t\in\mathbb{R}^+} |x'(t)|/(1+t)$. Notice that $\lim_{t\to\infty} x'(t)/(1+t) = 0$ implies that $\lim_{t\to\infty} x(t)/(1+t^2) = 0$, which justifies the norm $\|\cdot\|$.

Definition 1.1. By a solution we mean a function $x \in C^3(0,\infty)$ satisfying problem (1.1). If further $x''(0^+) := \lim_{t\to 0^+} x''(t)$ exists, then x is said to be a $C^2[0,\infty)\cap C^3(0,\infty)$ solution.

The basic tool to be used in this work is the classical Krasnosel'skii fixed point theorem on cone compression and expansion.

Lemma 1.1 ([6]). *Let E be a Banach space and $\mathcal{P}\subset E$ a cone. Assume that Ω_1 and Ω_2 are bounded open subsets of E with $\theta\in\Omega_1$, $\overline{\Omega}_1\subset\Omega_2$, where θ is the zero element in E. Let $A\colon \mathcal{P}\cap(\overline{\Omega}_2\setminus\Omega_1)\to P$ be a completely continuous operator such that either*

$$\|Ax\|\leqslant\|x\|\ \forall\, x\in\mathcal{P}\cap\partial\Omega_1,\quad \|Ax\|\geqslant\|x\|\ \forall\, x\in\mathcal{P}\cap\partial\Omega_2,$$

or

$$\|Ax\|\geqslant\|x\|\ \forall\, x\in\mathcal{P}\cap\partial\Omega_1,\quad \|Ax\|\leqslant\|x\|\ \forall\, x\in\mathcal{P}\cap\partial\Omega_2.$$

Then A has a fixed point in $\mathcal{P}\cap(\overline{\Omega}_2\setminus\Omega_1)$.

To show the compactness of a fixed point operator, we need the following compactness criterion on unbounded intervals of the real line which can be easily derived from Corduneanu's Compactness Criterion (see [3]):

Lemma 1.2. *Let W be a bounded subset of X. Then W is relatively compact if the following two conditions hold:*

(a) *the sets $\{v(t)/(1+t^2), v\in W\}$ and $\{v'(t)/(1+t), v\in W\}$ are equicontinuous on any finite subinterval of $(0,\infty)$, i.e.*

$$\forall\, I=[a,b]\subset\mathbb{R}^+,\ \forall\,\varepsilon>0,\ \exists\,\delta>0,\ \forall\, t_1,t_2\in I,\ \forall\, u\in W,$$
$$|t_1-t_2|<\delta\Rightarrow\left|\frac{u(t_1)}{1+t_1^2}-\frac{u(t_2)}{1+t_2^2}\right|<\varepsilon\quad\text{and}\quad\left|\frac{u'(t_1)}{1+t_1}-\frac{u'(t_2)}{1+t_2}\right|<\varepsilon.$$

(b) *for all $\varepsilon>0$, there exists $T=T(\varepsilon)>0$ such that for all $t\geqslant T$ and $u\in W$*

$$\left|\frac{u(t)}{1+t^2}\right|<\varepsilon\quad\text{and}\quad\left|\frac{u'(t)}{1+t}\right|<\varepsilon.$$

2. Preliminaries

Before we state our main existence result, we need some auxiliary lemmas.

Lemma 2.1. *Suppose that Hypothesis* $(\mathcal{H})$ *holds and let* x *be a solution of problem* (1.1). *Then*

$$x(t) = B_f(x) + tA_f(x) + \int_0^\infty G(t,s) f(s, x(s), x'(s))\,\mathrm{d}s$$

and

$$x'(t) = A_f(x) + \int_0^\infty K(t,s) f(s, x(s), x'(s))\,\mathrm{d}s,$$

where the constants

$$A_f(x) = \sum_{i=1}^{n_2} \beta_i x'(\eta_i) = \frac{1}{\breve{B}} \sum_{i=1}^{n_2} \beta_i \int_0^\infty K(\eta_i, s) f(s, x(s), x'(s))\,\mathrm{d}s, \quad \breve{B} = 1 - \sum_{i=1}^{n_2} \beta_i$$

and

$$B_f(x) = \sum_{i=1}^{n_1} \alpha_i x(\xi_i) = \frac{1}{\breve{A}} \sum_{i=1}^{n_1} \alpha_i \Big(\xi_i A_f(x) + \int_0^\infty G(\xi_i, s) f(s, x(s), x'(s)) \Big),$$

$$\breve{A} = 1 - \sum_{i=1}^{n_1} \alpha_i$$

are positive and where the kernels are the Green functions defined by

$$G(t,s) = \begin{cases} \frac{1}{2} t^2 & \text{if } s \geqslant t, \\ \frac{1}{2} s(2t - s) & \text{if } s \leqslant t \end{cases} \quad \text{and} \quad K(t,s) = \min(s,t). \tag{2.1}$$

P r o o f. Integrating the equation in (1.1) three times yields

$$x(t) = x(0) + tx'(0) + \frac{t^2}{2} x''(\infty) + \frac{1}{2} \int_0^t s(2t - s) f(s, x(s), x'(s))\,\mathrm{d}s + \frac{1}{2} t^2 \int_t^\infty f(s, x(s), x'(s))\,\mathrm{d}s. \tag{2.2}$$

Applying the boundary conditions, we get

$$x(t) = \sum_{i=1}^{n_1} \alpha_i x(\xi_i) + t \sum_{i=1}^{n_2} \beta_i x'(\eta_i) + \frac{1}{2} \int_0^t s(2t - s) f(s, x(s), x'(s))\,\mathrm{d}s + \frac{1}{2} t^2 \int_t^\infty f(s, x(s), x'(s))\,\mathrm{d}s \tag{2.3}$$

and

$$x'(t) = \sum_{i=1}^{n_2} \beta_i x'(\eta_i) + \int_0^t s f(s, x(s), x'(s))\, \mathrm{d}s + t \int_t^\infty f(s, x(s), x'(s)). \tag{2.4}$$

Substituting into (2.4) gives

$$\sum_{i=1}^{n_2} \beta_i x'(\eta_i) = \frac{1}{\breve{B}} \sum_{i=1}^{n_2} \beta_i \int_0^\infty K(\eta_i, s) f(s, x(s), x'(s))\, \mathrm{d}s = A_f(x) > 0.$$

Back to (2.3), we find

$$\begin{aligned} x(t) = {} & \sum_{i=1}^{n_1} \alpha_i x(\xi_i) + t A_f(x) + \frac{1}{2} \int_0^t s(2t - s) f(s, x(s), x'(s))\, \mathrm{d}s \\ & + \frac{1}{2} t^2 \int_t^\infty f(s, x(s), x'(s))\, \mathrm{d}s. \end{aligned}$$

By substitution, we get

$$\begin{aligned} \sum_{i=1}^{n_1} \alpha_i x(\xi_i) = \frac{1}{\breve{A}} \Bigg(& \sum_{i=1}^{n_1} \alpha_i \xi_i A_f(x) + \sum_{i=1}^{n_1} \alpha_i \frac{\xi_i^2}{2} \int_{\xi_i}^\infty f(s, x(s), x'(s))\, \mathrm{d}s \\ & + \sum_{i=1}^{n_1} \frac{\alpha_i}{2} \int_0^{\xi_i} s(2\xi_i - s) f(s, x(s), x'(s))\, \mathrm{d}s \Bigg) \end{aligned}$$

and

$$\begin{aligned} x(t) = {} & \frac{1}{\breve{A}} \Bigg(\sum_{i=1}^{n_1} \alpha_i \xi_i A_f(x) + \sum_{i=1}^{n_1} \alpha_i \frac{\xi_i^2}{2} \int_{\xi_i}^\infty f(s, x(s), x'(s))\, \mathrm{d}s \\ & + \sum_{i=1}^{n_1} \frac{\alpha_i}{2} \int_0^{\xi_i} s(2\xi_i - s) f(s, x(s), x'(s))\, \mathrm{d}s \Bigg) + t A_f(x) \\ & + \frac{t^2}{2} \int_t^\infty f(s, x(s), x'(s))\, \mathrm{d}s + \frac{1}{2} \int_0^t s(2t - s) f(s, x(s), x'(s))\, \mathrm{d}s \\ = {} & \frac{1}{\breve{A}} \Bigg(\sum_{i=1}^{n_1} \alpha_i \bigg(\xi_i A_f(x) + \frac{\xi_i^2}{2} \int_{\xi_i}^\infty f(s, x(s), x'(s))\, \mathrm{d}s \\ & + \frac{1}{2} \int_0^{\xi_i} s(2\xi_i - s) f(s, x(s), x'(s))\, \mathrm{d}s \bigg) \Bigg) + t A_f(x) \\ & + \frac{t^2}{2} \int_t^\infty f(s, x(s), x'(s))\, \mathrm{d}s + \frac{1}{2} \int_0^t s(2t - s) f(s, x(s), x'(s))\, \mathrm{d}s \\ = {} & \frac{1}{\breve{A}} \sum_{i=1}^{n_1} \alpha_i \bigg(\xi_i A_f(x) + \int_0^\infty G(\xi_i, s) f(s, x(s), x'(s))\, \mathrm{d}s \bigg) + t A_f(x) \\ & + \int_0^\infty G(t, s) f(s, x(s), x'(s))\, \mathrm{d}s. \end{aligned}$$

Hence,

$$x(t) = B_f(x) + tA_f(x) + \int_0^\infty G(t,s) f(s, x(s), x'(s))\,\mathrm{d}s.$$

□

The Green functions G and K satisfy $G(t,s) \leqslant \frac{1}{2}t^2$ for all $s, t \geqslant 0$ and the following estimates.

Lemma 2.2. *For all positive s, t, we have*

$$\frac{G(t,s)}{K(t,s)} \leqslant t, \qquad \frac{\gamma(t)}{1+s} K(s,s) \leqslant K(t,s) \leqslant \frac{1+t}{1+s} K(s,s),$$

where $\gamma(t) = \min\{t, 1\}$.

Proof. Since the function $g(t) = t/(1+t)$ is nondecreasing, we have $t/(1+t) \leqslant s/(1+s)$ for $t \leqslant s$ while $s/(1+t) \leqslant s/(1+s)$ for $s \leqslant t$. Then

$$\frac{K(t,s)}{1+t} \leqslant \frac{K(s,s)}{1+s} \quad \forall\, t, s \in [0, \infty).$$

On the other hand, if $t \in [0,1]$, then

$$\frac{K(t,s)}{K(s,s)} = \begin{cases} \dfrac{t}{s} \geqslant \dfrac{t}{1+s} & \text{if } t \leqslant s, \\[2ex] \dfrac{s}{s} = 1 \geqslant \dfrac{t}{1+s} & \text{if } t \geqslant s \end{cases}$$

while if $t \in [1, \infty)$, then

$$\frac{K(t,s)}{K(s,s)} = \begin{cases} \dfrac{t}{s} \geqslant \dfrac{1}{1+s} & \text{if } t \leqslant s, \\[2ex] \dfrac{s}{s} = 1 \geqslant \dfrac{1}{1+s} & \text{if } t \geqslant s. \end{cases}$$

□

Lemma 2.3. *Suppose that Hypothesis* $(\mathcal{H})$ *holds and let x be a solution of problem* (1.1). *Then*

$$x'(t) \geqslant \gamma(t)\|x\|_1 \quad \forall\, t \geqslant 0.$$

P r o o f. We have

$$\begin{aligned} x'(t) &= A_f(x) + \int_0^\infty K(t,s) f(s,x(s),x'(s))\,\mathrm{d}s \\ &\geqslant A_f(x) + \gamma(t) \int_0^\infty \frac{K(s,s)}{1+s} f(s,x(s),x'(s))\,\mathrm{d}s \\ &\geqslant \gamma(t)\Bigl(\frac{A_f(x)}{\gamma(t)} + \int_0^\infty \frac{K(s,s)}{1+s} f(s,x(s),x'(s))\,\mathrm{d}s\Bigr) \\ &\geqslant \gamma(t)\Bigl(A_f(x) + \int_0^\infty \frac{K(s,s)}{1+s} f(s,x(s),x'(s))\,\mathrm{d}s\Bigr). \end{aligned}$$

Since

$$\begin{aligned} \frac{x'(t)}{1+t} &\leqslant \frac{A_f(x)}{1+t} + \int_0^\infty \frac{K(s,s)}{1+s} f(s,x(s),x'(s))\,\mathrm{d}s \\ &\leqslant A_f(x) + \int_0^\infty \frac{K(s,s)}{1+s} f(s,x(s),x'(s))\,\mathrm{d}s, \end{aligned}$$

hence,

$$\|x\|_1 \leqslant A_f(x) + \int_0^\infty \frac{K(s,s)}{1+s} f(s,x(s),x'(s))\,\mathrm{d}s.$$

Finally,

$$x'(t) \geqslant \gamma(t)\|x\|_1 \quad \forall\, t \geqslant 0.$$

□

3. Existence and nonexistence results

3.1. $C^3(0,\infty)$ solutions. We first prove a nonexistence result.

Theorem 3.1. *Suppose that Hypothesis ($\mathcal{H}$) holds. Then a necessary condition for problem* (1.1) *to have a nontrivial solution is:*

$$\int_0^\infty \frac{t f(t,1+t^2,1+t)}{(1+t)^{\mu_2+1}(1+t^2)^{\mu_1}}\,\mathrm{d}t < \infty. \tag{3.1}$$

P r o o f. Let x be a nontrivial solution of problem (1.1) and let $c_0 = c_0(x)$ be a constant such that $0 < c_0 \leqslant \min\{1, 1/\|x\|\}$. By Hypothesis $(\mathcal{H})$, we have

$$\begin{aligned} f(t,x(t),x'(t)) &= f\Big(t, \frac{c_0(1+t^2)x(t)}{c_0(1+t^2)}, \frac{c_0(1+t)x'(t)}{c_0(1+t)}\Big) \\ &\geqslant \Big(\frac{1}{c_0}\Big)^{\lambda_1}\Big(\frac{1}{c_0}\Big)^{\lambda_2} f\Big(t, \frac{c_0(1+t^2)x(t)}{1+t^2}, \frac{c_0(1+t)x'(t)}{1+t}\Big) \\ &\geqslant {c_0}^{-\lambda_1-\lambda_2}\Big(\frac{c_0 x(t)}{1+t^2}\Big)^{\mu_1}\Big(\frac{c_0 x'(t)}{1+t}\Big)^{\mu_2} f(t,1+t^2,1+t) \\ &\geqslant c_0^{\mu_1-\lambda_1}\Big(\frac{x(t)}{1+t^2}\Big)^{\mu_1} c_0^{\mu_2-\lambda_2}\Big(\frac{x'(t)}{1+t}\Big)^{\mu_2} f(t,1+t^2,1+t). \\ &\geqslant c_0^{\mu_1+\mu_2-\lambda_1-\lambda_2}\frac{B_f^{\mu_1}(x)}{(1+t^2)^{\mu_1}}\frac{A_f^{\mu_2}(x)}{(1+t)^{\mu_2}} f(t,1+t^2,1+t). \end{aligned}$$

So,

$$\begin{aligned} \frac{f(t,1+t^2,1+t)}{(1+t^2)^{\mu_1}(1+t)^{\mu_2}} &\leqslant c_0^{\lambda_1+\lambda_2-\mu_1-\mu_2} B_f(x)^{-\mu_1} A_f(x)^{-\mu_2} f(t,x(t),x'(t)) \\ &\leqslant C B_f(x)^{-\mu_1} A_f(x)^{-\mu_2}(-x'''(t)). \end{aligned}$$

Integrating both sides yields

$$\int_t^\infty \frac{f(s,1+s^2,1+s)}{(1+s^2)^{\mu_1}(1+s)^{\mu_2}}\,\mathrm{d}s \leqslant C B_f(x)^{-\mu_1} A_f(x)^{-\mu_2}(x''(t))$$

and

$$\int_0^\infty \frac{1}{(1+t)^2}\,\mathrm{d}t \int_t^\infty \frac{f(s,1+s^2,1+s)}{(1+s^2)^{\mu_1}(1+s)^{\mu_2}}\,\mathrm{d}s \leqslant \int_0^\infty C B_f^{-\mu_1} A_f^{-\mu_2}\frac{x''(t)}{(1+t)^2}\,\mathrm{d}t.$$

Then,

$$\int_0^\infty \frac{t f(t,1+t^2,1+t)}{(1+t)^{\mu_2+1}(1+t^2)^{\mu_1}}\,\mathrm{d}t \leqslant C B_f^{-\mu_1} A_f^{-\mu_2}\int_0^\infty \frac{x''(t)}{(1+t)^2}\,\mathrm{d}t.$$

On the other hand,

$$\begin{aligned} \int_0^\infty \frac{x''(t)}{(1+t)^2}\,\mathrm{d}t &= \int_0^\infty \frac{1}{1+t}\Big(\frac{x'(t)}{1+t}\Big)'\,\mathrm{d}t + \int_0^\infty \frac{x'(t)}{(1+t)^3}\,\mathrm{d}t \\ &\leqslant \int_0^\infty \Big(\frac{x'(t)}{1+t}\Big)'\,\mathrm{d}t + \|x\|_1\int_0^\infty \frac{1}{(1+t)^2}\,\mathrm{d}t \\ &\leqslant -x'(0) + \|x\|_1\frac{\pi}{2} < \infty \end{aligned}$$

proving our claim. □

Define a cone in X by

$$\mathcal{P}=\Big\{x\in X\colon\ x(0)=\sum_{i=1}^{n_1}\alpha_i x(\xi_i)\ \text{and}\ x'(t)\geqslant\gamma(t)\|x\|_1\ \text{ for all }\ t\geqslant 0\Big\}.$$

The following estimates hold:

Lemma 3.1. *For $x\in\mathcal{P}$ we have*

$$M_1\|x\|_1\leqslant\|x\|_0\leqslant M_2\|x\|_1,$$

where

$$M_2=1+\frac{1}{\breve{A}}\sum_{i=1}^{n_1}\alpha_i\Big(\xi_i+\frac{\xi_i^2}{2}\Big)\quad\textit{and}\quad M_1=\frac{1}{\breve{A}}\sum_{i=1}^{n_1}\alpha_i\delta(\xi_i).$$

Proof. For each $x\in X$ we have

$$x(t)=\int_0^t x'(s)\,\mathrm{d}s+\frac{1}{\breve{A}}\sum_{i=1}^{n_1}\alpha_i\int_0^{\xi_i}x'(s)\,\mathrm{d}s,$$

and for $x\in\mathcal{P}$

$$\Big(\delta(t)+\frac{1}{\breve{A}}\sum_{i=1}^{n_1}\alpha_i\delta(\xi_i)\Big)\|x\|_1\leqslant x(t)\leqslant\Big(t+\frac{t^2}{2}+\frac{1}{\breve{A}}\sum_{i=1}^{n_1}\alpha_i\Big(\xi_i+\frac{\xi_i^2}{2}\Big)\Big)\|x\|_1,$$

where $\delta(t)=\int_0^t\gamma(s)\,\mathrm{d}s$. So, for all $t\geqslant 0$,

$$\frac{1}{1+t^2}\Big(\delta(t)+\frac{1}{\breve{A}}\sum_{i=1}^{n_1}\alpha_i\delta(\xi_i)\Big)\|x\|_1$$
$$\leqslant\frac{x(t)}{1+t^2}\leqslant\frac{1}{1+t^2}\Big(t+\frac{t^2}{2}+\frac{1}{\breve{A}}\sum_{i=1}^{n_1}\alpha_i\Big(\xi_i+\frac{\xi_i^2}{2}\Big)\Big)\|x\|_1.$$

Then,

$$\frac{1}{1+t^2}\Big(\delta(t)+\frac{1}{\breve{A}}\sum_{i=1}^{n_1}\alpha_i\delta(\xi_i)\Big)\|x\|_1\leqslant\frac{x(t)}{1+t^2}\leqslant\Big(1+\frac{1}{\breve{A}}\sum_{i=1}^{n_1}\alpha_i\Big(\xi_i+\frac{\xi_i^2}{2}\Big)\Big)\|x\|_1.$$

Finally,

$$\frac{1}{\breve{A}}\sum_{i=1}^{n_1}\alpha_i\delta(\xi_i)\|x\|_1\leqslant\|x\|_0\leqslant\Big(1+\frac{1}{\breve{A}}\sum_{i=1}^{n_1}\alpha_i\Big(\xi_i+\frac{\xi_i^2}{2}\Big)\Big)\|x\|_1.$$

□

Now we are ready to state and prove our first existence result:

Theorem 3.2. *Suppose Hypothesis* $(\mathcal{H})$ *holds and*

$$\int_0^\infty \frac{sf(s,1+s^2,1+s)}{1+s}\,\mathrm{d}s < \infty, \tag{3.2}$$

$$\lim_{t\to\infty} \frac{1}{1+t}\int_0^t sf(s,1+s^2,1+s)\,\mathrm{d}s = 0. \tag{3.3}$$

Then problem (1.1) *has at least one positive solution.*

P r o o f. *Step 1. A fixed point formulation.* For each $x \in X$, let $0 < c_1 \leqslant 1$ be a positive constant such that $c_1\|x\| \leqslant 1$. For all $t \geqslant 0$ we have

$$\begin{aligned}
f(t,x(t),x'(t)) &\leqslant \left(\frac{1}{c_1}\right)^{\mu_1} f\left(t, c_1(1+t^2)\frac{x(t)}{1+t^2}, x'(t)\right)\\
&\leqslant c_1^{-\mu_1} f(t, c_1(1+t^2)\|x\|_0, x'(t))\\
&\leqslant c_1^{-\mu_1}(c_1\|x\|)^{\lambda_1} f(t, 1+t^2, x'(t))\\
&\leqslant c_1^{-\mu_1-\mu_2}(c_1\|x\|)^{\lambda_1} f(t, 1+t^2, c_1(1+t)\|x\|_1)\\
&\leqslant c_1^{-\mu_1-\mu_2}(c_1\|x\|)^{\lambda_1}(c_1\|x\|)^{\lambda_2} f(t, 1+t^2, 1+t)\\
&\leqslant c_1^{-\mu_1-\mu_2} f(t, 1+t^2, 1+t).
\end{aligned}$$

Then,

$$\begin{aligned}
\int_0^\infty G(t,s)f(s,x(s),x'(s))\,\mathrm{d}s &\leqslant t\int_0^\infty K(t,s)f(s,x(s),x'(s))\,\mathrm{d}s\\
&\leqslant t(1+t)\int_0^\infty \frac{K(s,s)f(s,x(s),x'(s))}{1+s}\,\mathrm{d}s\\
&\leqslant t(1+t)c_1^{-\mu_1-\mu_2}\int_0^\infty \frac{sf(s,1+s^2,1+s)}{1+s}\,\mathrm{d}s < \infty.
\end{aligned}$$

In particular, this implies that for all $x \in X$

$$\int_0^\infty G(t,s)f(s,x(s),x'(s))\,\mathrm{d}s < \infty \quad \text{and} \quad \int_0^\infty K(t,s)f(s,x(s),x'(s))\,\mathrm{d}s < \infty,$$

which allows us to define an operator A on X by

$$Ax(t) = B_f(x) + tA_f(x) + \int_0^\infty G(t,s)f(s,x(s),x'(s))\,\mathrm{d}s.$$

Step 2. $A\colon X \to X$ is well defined. Indeed, for all $t \geqslant 0$

$$\begin{aligned}(Ax)'(t) &= A_f(x) + \int_0^t sf(s,x(s),x'(s))\,\mathrm{d}s + t\int_t^\infty f(s,x(s),x'(s))\,\mathrm{d}s \\ &= A_f(x) + \int_0^\infty K(t,s)f(s,x(s),x'(s))\,\mathrm{d}s.\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{(Ax)'(t)}{1+t} &= \frac{A_f}{1+t} + \frac{1}{1+t}\int_0^\infty K(t,s)f(s,x(s),x'(s))\,\mathrm{d}s \\ &= \frac{A_f}{1+t} + \frac{1}{1+t}\int_0^t sf(s,x(s),x'(s))\,\mathrm{d}s + \frac{t}{1+t}\int_t^\infty f(s,x(s),x'(s))\,\mathrm{d}s \\ &\leqslant \frac{A_f}{1+t} + c_1^{-\mu_1-\mu_2}\biggl(\frac{1}{1+t}\int_0^t sf(s,1+s,1+s^2)\,\mathrm{d}s \\ &\qquad\qquad + \int_t^\infty \frac{sf(s,1+s,1+s^2)\,\mathrm{d}s}{1+s}\biggr).\end{aligned}$$

Equations (3.3) and (3.2) imply that $\lim\limits_{t\to\infty}(Ax)'(t)/(1+t) = 0$. In addition, $A(\mathcal{P}) \subset \mathcal{P}$. Indeed, by simple calculation, we get $Ax(0) = \sum\limits_{i=1}^{n_1}\alpha_i Ax(\xi_i)$ for $x \in \mathcal{P}$. Following the same steps as in the proof of Lemma 2.3, we can check that

$$(Ax)'(t) \geqslant \gamma(t)\|Ax\|_1 \quad \forall\, t \geqslant 0.$$

Step 3. A fixed point of A is a solution of problem (1.1). Let

$$x(t) = Ax(t) = B_f(x) + tA_f(x) + \int_0^\infty G(t,s)f(s,x(s),x'(s))\,\mathrm{d}s.$$

Differentiating x three times, we get successively

$$\begin{aligned}x'(t) &= A_f(x) + \int_0^t sf(s,x(s),x'(s))\,\mathrm{d}s + t\int_t^\infty f(s,x(s),x'(s))\,\mathrm{d}s, \\ x''(t) &= \int_t^\infty f(s,x(s),x'(s))\,\mathrm{d}s,\end{aligned}$$

and

$$x'''(t) = -f(t,x(t),x'(t)).$$

So $x'(0) = A_f(x) = \sum\limits_{i=1}^{n_2}\beta_i x'(\eta_i)$ and from the fact that

$$x(0) = B_f(x) = \frac{1}{\breve{A}}\sum_{i=1}^{n_1}\alpha_i\biggl(\xi_i A_f(x) + \int_0^\infty G(\xi_i,s)f(s,x(s),x'(s))\,\mathrm{d}s\biggr),$$

we deduce that $B_f(x) = \sum\limits_{i=1}^{n_1} \alpha_i x(\xi_i)$. Finally, using (3.2) we find

$$0 \leqslant \frac{t}{1+t} x''(t) \leqslant c_1^{-\mu_1-\mu_2} \int_t^\infty \frac{s f(s, 1+s^2, 1+s)}{1+s} \, \mathrm{d}s \quad \forall\, t \geqslant 0.$$

Hence

$$\lim_{t\to\infty} x''(t) = \lim_{t\to\infty} \frac{t}{1+t} x''(t) = 0.$$

Step 4. Operator $A\colon \mathcal{P} \to \mathcal{P}$ is completely continuous.

(i) *A is bounded.* Indeed, let $B \subset \mathcal{P}$ be a bounded set. Then there exists a constant M such that for all $x \in B, \|x\| \leqslant M$. Let c_2 be a constant such that $0 < c_2 \leqslant \min(1, 1/M)$. We have

$$\begin{aligned} f(t, x(t), x'(t)) &\leqslant \Big(\frac{1}{c_2}\Big)^{\mu_1} f\Big(t, c_2(1+t^2)\frac{x(t)}{1+t^2}, x'(t)\Big) \\ &\leqslant c_2^{-\mu_1} f(t, c_2(1+t^2)\|x\|_0, x'(t)) \\ &\leqslant c_2^{-\mu_1} (c_2\|x\|)^{\lambda_1} f(t, 1+t^2, x'(t)) \\ &\leqslant c_2^{-\mu_1-\mu_2} (c_2 M)^{\lambda_1} f(t, 1+t^2, c_2(1+t)\|x\|_1) \\ &\leqslant c_2^{-\mu_1-\mu_2} (c_2 M)^{\lambda_1} (c_2 M)^{\lambda_2} f(t, 1+t^2, 1+t) \\ &\leqslant c_2^{-\mu_1-\mu_2} f(t, 1+t^2, 1+t). \end{aligned}$$

As a consequence

$$\begin{aligned} \frac{|(Ax)'(t)|}{1+t} &= \frac{A_f}{1+t} + \frac{1}{1+t}\int_0^\infty K(t,s) f(s, x(s), x'(s))\,\mathrm{d}s \\ &\leqslant \frac{1}{\breve{B}} \sum_{i=1}^{n_2} \beta_i \int_0^\infty (1+\eta_i)\frac{K(\eta_i, s)}{1+\eta_i} f(s, x(s), x'(s))\,\mathrm{d}s \\ &\quad + \int_0^\infty \frac{K(t,s)}{1+t} f(s, x(s), x'(s))\,\mathrm{d}s \\ &\leqslant M_3 c_2^{-\mu_1-\mu_2} \int_0^\infty \frac{s}{1+s} f(s, 1+s^2, 1+s)\,\mathrm{d}s < \infty, \end{aligned}$$

where $M_3 = 1 + \Big(\sum\limits_{i=1}^{n_2} \beta_i(1+\eta_i)\Big)/\breve{B}$. Since $\|Ax\|_0 \leqslant M_2\|Ax\|_1$, $A(B)$ is bounded, as claimed.

(ii) *A is continuous.* Let $x_n,\ x_0 \in \mathcal{P}$ be such that $\|x_n - x_0\| \to 0$ as $n \to \infty$; then $(x_n)_n$ is bounded. Let $L = \sup\{\|x_n\|, n = 1, 2, \ldots\}$ and let c_3 be a constant such that $0 < c_3 \leqslant \min(1, 1/L)$. We have

$$
\begin{aligned}
&\Big|\frac{(Ax_n)'(t)}{1+t} - \frac{(Ax_0)'(t)}{1+t}\Big| \\
&\qquad \leqslant \frac{1}{\breve{B}(1+t)} \sum_{i=1}^{n_2} \beta_i \int_0^\infty K(\eta_i, s)|(f(s, x_n(s), x_n'(s)) - f(s, x_0(s), x_0'(s)))|\,\mathrm{d}s \\
&\qquad\quad + \int_0^\infty \frac{K(t,s)}{1+t}|(f(s, x_n(s), x_n'(s)) - f(s, x_0(s), x_0'(s)))|\,\mathrm{d}s \\
&\qquad \leqslant M_3 \int_0^\infty \frac{s}{1+s}|f(s, x_n(s), x_n'(s)) - f(s, x_0(s), x_0'(s))|\,\mathrm{d}s
\end{aligned}
$$

and

$$
\begin{aligned}
f(t, x_n(t), x_n'(t)) &\leqslant \Big(\frac{1}{c_3}\Big)^{\mu_1} f\Big(t, c_3(1+t^2)\frac{x_n(t)}{1+t^2}, x_n'(t)\Big) \\
&\leqslant c_3^{-\mu_1} f(t, c_3(1+t^2)\|x_n\|_0, x_n'(t)) \\
&\leqslant c_3^{-\mu_1}(c_3\|x_n\|)^{\lambda_1} f(t, 1+t^2, x_n'(t)) \\
&\leqslant c_3^{-\mu_1-\mu_2}(c_3 L)^{\lambda_1} f(t, 1+t^2, c_3(1+t)\|x_n\|_1) \\
&\leqslant c_3^{-\mu_1-\mu_2}(c_3 L)^{\lambda_1}(c_3 L)^{\lambda_2} f(t, 1+t^2, 1+t) \\
&\leqslant c_3^{-\mu_1-\mu_2} f(t, 1+t^2, 1+t).
\end{aligned}
$$

By the Lebesgue dominated convergence theorem and Lemma 3.1, we deduce that

$$\|Ax_n - Ax_0\|_1 \to 0 \quad \text{as } n \to \infty,$$

and by Lemma 3.1, we infer that

$$\|Ax_n - Ax_0\|_0 \to 0 \quad \text{as } n \to \infty,$$

proving that $A\colon \mathcal{P} \to \mathcal{P}$ is continuous.

(iii) *A is equicontinuous.* Let $B \subset \mathcal{P}$ be a bounded set, and let $t_1, t_2 \in [a, b]$ be such that $0 < t_1 < t_2$. Then for all $x \in B$, we have the estimates:

$$
\begin{aligned}
&\Big|\frac{(Ax)'(t_2)}{1+t_2} - \frac{(Ax)'(t_1)}{1+t_1}\Big| \\
&\qquad = \Big|\Big(\frac{A_f(x)}{1+t_2} - \frac{A_f(x)}{1+t_1}\Big) + \int_0^\infty \Big(\frac{K(t_2,s)}{1+t_2} - \frac{K(t_1,s)}{1+t_1}\Big) f(s, x(s), x'(s))\,\mathrm{d}s\Big| \\
&\qquad \leqslant |t_2 - t_1| \frac{1}{\breve{B}} \sum_{i=1}^{n_2} \beta_i \int_0^\infty K(\eta_i, s) f(s, x(s), x'(s))\,\mathrm{d}s \\
&\qquad\quad + \Big|\int_0^{t_2} \frac{s}{1+t_2} f(s, x(s), x'(s))\,\mathrm{d}s + \int_{t_2}^\infty \frac{t_2}{1+t_2} f(s, x(s), x'(s))\,\mathrm{d}s \\
&\qquad\quad - \int_0^{t_1} \frac{s}{1+t_1} f(s, x(s), x'(s))\,\mathrm{d}s - \int_{t_1}^\infty \frac{t_1}{1+t_1} f(s, x(s), x'(s))\,\mathrm{d}s\Big|,
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\Big|\frac{(Ax)'(t_2)}{1+t_2}-\frac{(Ax)'(t_1)}{1+t_1}\Big| \\
&\qquad\leqslant |t_2-t_1|\frac{1}{\breve{B}}\sum_{i=1}^{n_2}\beta_i\int_0^{\infty}(1+\eta_i)\frac{K(\eta_i,s)}{1+\eta_i}f(s,x(s),x'(s))\,\mathrm{d}s \\
&\qquad\quad+\frac{|t_1-t_2|}{(1+t_2)(1+t_1)}\int_0^{t_1}sf(s,x(s),x'(s))\,\mathrm{d}s \\
&\qquad\quad+\frac{1}{1+t_2}\int_{t_1}^{t_2}sf(s,x(s),x'(s))\,\mathrm{d}s \\
&\qquad\quad+\frac{|t_1-t_2|}{(1+t_2)(1+t_1)}\int_{t_2}^{\infty}f(s,x(s),x'(s))\,\mathrm{d}s \\
&\qquad\quad+\frac{t_1}{1+t_1}\int_{t_1}^{t_2}f(s,x(s),x'(s))\,\mathrm{d}s \\
&\qquad\leqslant |t_2-t_1|\frac{1}{\breve{B}}\sum_{i=1}^{n_2}\beta_i(1+\eta_i)\int_0^{\infty}\frac{s}{1+s}f(s,x(s),x'(s))\,\mathrm{d}s \\
&\qquad\quad+\frac{|t_1-t_2|}{1+t_2}\int_0^{t_1}\frac{s}{1+s}f(s,x(s),x'(s))\,\mathrm{d}s \\
&\qquad\quad+\frac{|t_1-t_2|}{(1+t_2)(1+t_1)}\int_{t_2}^{\infty}f(s,x(s),x'(s))\,\mathrm{d}s \\
&\qquad\quad+2\int_{t_1}^{t_2}\frac{s}{1+s}f(s,x(s),x'(s))\,\mathrm{d}s \\
&\qquad\leqslant C|t_2-t_1|\frac{1}{\breve{B}}\sum_{i=1}^{n_2}\beta_i(1+\eta_i)\int_0^{\infty}\frac{s}{1+s}f(s,1+s^2,1+s)\,\mathrm{d}s \\
&\qquad\quad+C\frac{|t_1-t_2|}{1+t_2}\int_0^{t_1}\frac{s}{1+s}f(s,1+s^2,1+s)\,\mathrm{d}s \\
&\qquad\quad+C\frac{|t_1-t_2|}{(1+t_2)(1+t_1)}\int_{t_2}^{\infty}f(s,1+s^2,1+s)\,\mathrm{d}s.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\Big|\frac{(Ax)'(t_2)}{1+t_2}-\frac{(Ax)'(t_1)}{1+t_1}\Big| &\leqslant C|t_2-t_1|\frac{1}{\breve{B}}\sum_{i=1}^{n_2}\beta_i(1+\eta_i)\int_0^{\infty}\frac{sf(s,1+s^2,1+s)}{1+s}\,\mathrm{d}s \\
&\quad+C\frac{|t_1-t_2|}{1+t_2}\int_0^{t_1}\frac{sf(s,1+s^2,1+s)}{1+s}\,\mathrm{d}s \\
&\quad+C\frac{|t_1-t_2|}{t_2(1+t_1)}\int_{t_2}^{\infty}\frac{sf(s,1+s^2,1+s)}{1+s}\,\mathrm{d}s\leqslant\varepsilon,
\end{aligned}
$$

where $C = c_2^{-\mu_1-\mu_2}$. Similarly, we can prove that

$$\Big|\frac{(Ax)(t_2)}{1+t_2^2} - \frac{(Ax)(t_1)}{1+t_1^2}\Big| \leqslant \varepsilon.$$

Therefore the operator A is equicontinuous.

(iv) A *is equiconvergent.* We have

$$\begin{aligned}
\Big|\frac{(Ax)'(t)}{1+t}\Big| &= \Big|\frac{A_f(x)}{1+t} + \frac{1}{1+t}\int_0^\infty K(t,s)f(s,x(s),x'(s))\,\mathrm{d}s\Big| \\
&\leqslant \frac{1}{1+t}\frac{1}{\breve{B}}\sum_{i=1}^{n_2}\beta_i(1+\eta_i)c_2^{-\mu_1-\mu_2}\int_0^\infty \frac{sf(s,1+s,1+s^2)}{1+s}\,\mathrm{d}s \\
&\quad + c_2^{-\mu_1-\mu_2}\bigg(\frac{1}{1+t}\int_0^t sf(s,1+s,1+s^2)\,\mathrm{d}s \\
&\qquad + \int_t^\infty \frac{sf(s,1+s,1+s^2)}{1+s}\,\mathrm{d}s\bigg),
\end{aligned}$$

which tends to 0 as $t\to\infty$. In the same way we can prove that $|(Ax)(t)/(1+t^2)| \to 0$ as $t\to\infty$. We conclude that $A\colon \mathcal{P}\to\mathcal{P}$ is completely continuous.

(v) Let $J_0 = [\alpha,\beta]$. Then $K(t,s)/(1+t) \geqslant \varepsilon_1 = \alpha/(1+\beta)$ for $(t,s)\in J_0\times J_0$. Hence, for all $x\in\mathcal{P}$ and for all $t\in J_0$ we have

$$\frac{(Ax)'(t)}{1+t} \geqslant \frac{1}{1+t}\int_\alpha^\beta K(t,s)f(s,x(s),x'(s))\,\mathrm{d}s \geqslant \varepsilon_1\int_\alpha^\beta f(s,x(s),x'(s))\,\mathrm{d}s$$

and

$$f(t,x(t),x'(t)) \geqslant f(t,(1+t^2)M_1\|x\|_1,(1+t)\varepsilon_2\|x\|_1),$$

where $\varepsilon_2 = \min(\alpha,1)/(1+\beta)$. For each $x\in\mathcal{P}$, let $c_4 = c_4(x)$ be a positive constant such that $c_4\min(M_1,\varepsilon_2)\|x\|_1 > 1$. For all $x\in\mathcal{P}$ and $t\in J_0$, we have

$$f(t,x(t),x'(t)) \geqslant c_4^{-\mu_1-\mu_2}(c_4M_1\|x\|_1)^{\lambda_1}(c_4\varepsilon_2\|x\|_1)^{\lambda_2}(f(t,(1+t^2),(1+t)).$$

Since

$$\begin{aligned}
\frac{(Ax)'(t)}{1+t} &\geqslant \varepsilon_1 c_4^{-\mu_1-\mu_2}(c_4M_1\|x\|_1)^{\lambda_1}(c_4\varepsilon_2\|x\|_1)^{\lambda_2}I_{\alpha\beta} \\
&\geqslant \varepsilon_1 c_4^{\lambda_1+\lambda_2-\mu_1-\mu_2}M_1^{\lambda_1}\|x\|_1^{\lambda_1+\lambda_2}\varepsilon_2^{\lambda_2}I_{\alpha\beta} \\
&\geqslant c_4^{\lambda_1+\lambda_2-\mu_1-\mu_2}\varepsilon_1\varepsilon_2^{\lambda_2}M_1^{\lambda_1}I_{\alpha\beta}\|x\|_1^{\lambda_1+\lambda_2}, \\
&\geqslant c_4^{\lambda_1+\lambda_2-\mu_1-\mu_2}\varepsilon_1\varepsilon_2^{\lambda_2}M_1^{\lambda_1}I_{\alpha\beta}\Big(\frac{1}{M_2}\|x\|_0\Big)^{\lambda_1+\lambda_2},
\end{aligned}$$

we deduce that

$$
\begin{aligned}
\|Ax\|_1 &\geqslant c_4^{\lambda_1+\lambda_2-\mu_1-\mu_2}\varepsilon_1\varepsilon_2^{\lambda_2}M_1^{\lambda_1}I_{\alpha\beta}\max\Big(\|x\|_1^{\lambda_1+\lambda_2},\Big(\frac{1}{M_2}\|x\|_0\Big)^{\lambda_1+\lambda_2}\Big)\\
&\geqslant c_4^{\lambda_1+\lambda_2-\mu_1-\mu_2}\varepsilon_1\varepsilon_2^{\lambda_2}M_1^{\lambda_1}\frac{I_{\alpha\beta}}{M_2^{\lambda_1+\lambda_2}}\max(\|x\|_1^{\lambda_1+\lambda_2},\|x\|_0^{\lambda_1+\lambda_2})\\
&\geqslant c_4^{\lambda_1+\lambda_2-\mu_1-\mu_2}\varepsilon_1\varepsilon_2^{\lambda_2}M_1^{\lambda_1}\frac{I_{\alpha\beta}}{M_2^{\lambda_1+\lambda_2}}\|x\|^{\lambda_1+\lambda_2}.
\end{aligned}
$$

By Lemma 3.1, we have

$$
\|Ax\|_0 \geqslant M_1\|Ax\|_1 \geqslant M_1c_4^{\lambda_1+\lambda_2-\mu_1-\mu_2}\varepsilon_1\varepsilon_2^{\lambda_2}M_1^{\lambda_1}\frac{I_{\alpha\beta}}{M_2^{\lambda_1+\lambda_2}}\|x\|^{\lambda_1+\lambda_2}.
$$

Then,

$$
\|Ax\| \geqslant \max(1,M_1)c_4^{\lambda_1+\lambda_2-\mu_1-\mu_2}\varepsilon_1\varepsilon_2^{\lambda_2}M_1^{\lambda_1}\frac{I_{\alpha\beta}}{M_2^{\lambda_1+\lambda_2}}\|x\|^{\lambda_1+\lambda_2}.
$$

Since $\lambda_1+\lambda_2>1$, we may choose

$$
R=\left(\max(1,M_1)c_4^{\lambda_1+\lambda_2-\mu_1-\mu_2}\varepsilon_1\varepsilon_2^{\lambda_2}M_1^{\lambda_1}\frac{I_{\alpha\beta}}{M_2^{\lambda_1+\lambda_2}}\right)^{-1/(\lambda_1+\lambda_2-1)}.
$$

As a consequence, for R large enough, we obtain

$$
\|Ax\| \geqslant \|x\| \quad \forall\, x\in\mathcal{P},\ \|x\|=R.
$$

Furthermore, let $0<r<1$ be selected sufficiently small and $B=B(0,r)$. Then for all $x\in\mathcal{P}\cap\partial B$, let $0<c_5=c_5(x)\leqslant 1$ be a positive constant such that $c_5\max(M_2,1)\|x\|_1\leqslant 1$. Hence for all positive t, we have

$$
f(t,x(t),x'(t)) \leqslant c_5^{\lambda_1+\lambda_2-\mu_1-\mu_2}(M_2\|x\|_1)^{\lambda_1}(\|x\|_1)^{\lambda_2}f(t,1+t^2,1+t).
$$

As a consequence

$$
\begin{aligned}
\frac{|(Ax)'(t)|}{1+t} &= \frac{A_f}{1+t}+\frac{1}{1+t}\int_0^\infty K(t,s)f(s,x(s),x'(s))\,\mathrm{d}s\\
&\leqslant M_3c_5^{\lambda_1+\lambda_2-\mu_1-\mu_2}(M_2\|x\|_1)^{\lambda_1}(\|x\|_1)^{\lambda_2}\int_0^\infty \frac{sf(s,1+s^2,1+s)}{1+s}\,\mathrm{d}s\\
&\leqslant M_3c_5^{\lambda_1+\lambda_2-\mu_1-\mu_2}M_2^{\lambda_1}\|x\|_1^{\lambda_1+\lambda_2}\int_0^\infty \frac{sf(s,1+s^2,1+s)}{1+s}\,\mathrm{d}s\\
&\leqslant M_3c_5^{\lambda_1+\lambda_2-\mu_1-\mu_2}M_2^{\lambda_1}\frac{1}{M_1^{\lambda_1+\lambda_2}}\|x\|_0^{\lambda_1+\lambda_2}\int_0^\infty \frac{sf(s,1+s^2,1+s)}{1+s}\,\mathrm{d}s.
\end{aligned}
$$

Hence,

$$\begin{aligned}\|Ax\|_1 &\leqslant M_3 c_5^{\lambda_1+\lambda_2-\mu_1-\mu_2} M_2^{\lambda_1} \min\Big(\|x\|_1^{\lambda_1+\lambda_2}, \frac{1}{M_1^{\lambda_1+\lambda_2}}\|x\|_0^{\lambda_1+\lambda_2}\Big)\\ &\quad\times \int_0^\infty \frac{sf(s,1+s^2,1+s)}{1+s}\,\mathrm{d}s\\ &\leqslant C\|x\|^{\lambda_1+\lambda_2}\int_0^\infty \frac{sf(s,1+s^2,1+s)}{1+s}\,\mathrm{d}s.\end{aligned}$$

Now if we choose

$$0<r\leqslant\Big(\min\Big(1,\frac{1}{M_1^{\lambda_1+\lambda_2}}\Big)M_3M_2\int_0^\infty\frac{sf(s,(1+s^2),(1+s))}{1+s}\,\mathrm{d}s\Big)^{-1/(\lambda_1+\lambda_2-1)}$$

for r small enough, then we arrive at the estimate

$$\|Ax\|\leqslant\|x\|\quad\forall\, x\in\mathcal{P}\cap\partial B.$$

By Lemma 1.1, we conclude that A has a fixed point $x^*\in\mathcal{P}$ which satisfies $r\leqslant\|x^*\|\leqslant R$. □

Example 3.1. Consider the boundary value problem:

$$\begin{aligned}x(0)-\frac{1}{2}x\Big(\frac{1}{2}\Big)&=0,\\ x'(0)-\frac{1}{3}x'\Big(\frac{1}{3}\Big)&=0\\ \lim_{t\to\infty}x''(t)&=0,\\ x'''(t)+\frac{x(t)^\lambda x'(t)^\mu}{t(1+t^2)^{\lambda+1}}&=0,\quad t\geqslant0,\end{aligned}\tag{3.4}$$

where $\lambda>0$, $0\leqslant\mu<1$, $\lambda+\mu>1$.

We have

$$f(t,cx,dx')=\frac{1}{t(1+t^2)^{\lambda+1}}(cx(t))^\lambda(dx'(t))^\mu=c^\lambda d^\mu\frac{1}{t(1+t^2)^{\lambda+1}}x(t)^\lambda x'(t)^\mu$$

and

$$\int_0^\infty\frac{tf(t,1+t^2,1+t)}{1+t}\,\mathrm{d}t=\int_0^\infty\frac{1}{(1+t)^{1-\mu}(1+t^2)}\,\mathrm{d}t.$$

Since $1-\mu>0$, we get

$$\int_0^\infty\frac{tf(t,1+t^2,1+t)}{1+t}\,\mathrm{d}t\leqslant\int_0^\infty\frac{1}{1+t^2}\,\mathrm{d}t=\frac{\pi}{2}.$$

Also,

$$0 \leqslant \frac{1}{1+t}\int_0^t sf(s,1+s^2,1+s)\,\mathrm{d}s = \frac{1}{1+t}\int_0^t \frac{(1+s)^{\mu}}{1+s^2}\,\mathrm{d}s \leqslant \frac{1}{(1+t)^{1-\mu}}\arctan t.$$

Hence,

$$\lim_{t\to\infty}\frac{1}{1+t}\int_0^t sf(s,1+s^2,1+s)\,\mathrm{d}s = 0.$$

From Theorem 3.2, problem (3.4) has at least a $C^3(0,\infty)$ positive solution.

3.2. $C^2[0,\infty)\cap C^3(0,\infty)$ **solutions.** First, we prove a nonexistence result.

Theorem 3.3. *Suppose that Hypothesis* ($\mathcal{H}$) *holds. Then a necessary condition for problem* (1.1) *to have a* $C^2[0,\infty)\cap C^3(0,\infty)$ *positive solution is*

$$\int_0^{\infty}\frac{f(t,1+t^2,1+t)}{(1+t)^{\mu_2}(1+t^2)^{\mu_1}}\,\mathrm{d}t < \infty. \tag{3.5}$$

Proof. Suppose that x is a $C^2[0,\infty)\cap C^3(0,\infty)$ positive solution of problem (1.1). Then $x''(0^+)$ exists. By integration of (1.1) we obtain

$$\int_0^{\infty} f(s,x(s),x'(s))\,\mathrm{d}s = x''(0) < \infty.$$

Let c_0 be a constant such that $0 < c_0 \leqslant \min\{1, 1/\|x\|\}$. From Hypothesis ($\mathcal{H}$), we have the estimates

$$\begin{aligned}
f(t,x(t),x'(t)) &= f\Big(t,\frac{c_0(1+t^2)x(t)}{c_0(1+t^2)},\frac{c_0(1+t)x'(t)}{c_0(1+t)}\Big)\\
&\geqslant \Big(\frac{1}{c_0}\Big)^{\lambda_1}\Big(\frac{1}{c_0}\Big)^{\lambda_2} f\Big(t,\frac{c_0(1+t^2)x(t)}{1+t^2},\frac{c_0(1+t)x'(t)}{1+t}\Big)\\
&\geqslant c_0^{-\lambda_1-\lambda_2}\Big(\frac{c_0x(t)}{1+t^2}\Big)^{\mu_1}\Big(\frac{c_0x'(t)}{1+t}\Big)^{\mu_2} f(t,1+t^2,1+t)\\
&\geqslant (c_0)^{\mu_1-\lambda_1}\Big(\frac{x(t)}{1+t^2}\Big)^{\mu_1}(c_0)^{\mu_2-\lambda_2}\Big(\frac{x'(t)}{1+t}\Big)^{\mu_2} f(t,1+t^2,1+t).
\end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned}
&\Big(\frac{1}{1+t^2}\Big)^{\mu_1}\Big(\frac{1}{1+t}\Big)^{\mu_2} f(t,1+t^2,1+t)\\
&\qquad\leqslant \frac{1}{c_0^{\mu_1+\mu_2-\lambda_1-\lambda_2}}B_f(x)^{-\mu_1}A_f(x)^{-\mu_2} f(t,x(t),x'(t))\\
&\qquad\leqslant a_0B_f(x)^{-\mu_1}A_f(x)^{-\mu_2}(-x'''(t)).
\end{aligned}$$

Integrating both sides, we obtain

$$\int_0^\infty \frac{f(s,1+s^2,1+s)}{(1+s)^{\mu_2}(1+s^2)^{\mu_1}}\,\mathrm{d}s \leqslant a_0 B_f(x)^{-\mu_1} A_f(x)^{-\mu_2}(x''(0)) < \infty.$$

□

We end the paper by an existence result for a regular solution.

Theorem 3.4. *Suppose that Hypothesis* $(\mathcal{H})$ *holds and*

$$\int_0^\infty f(t,1+t^2,1+t)\,\mathrm{d}t < \infty, \tag{3.6}$$

$$\lim_{t\to\infty} \frac{1}{1+t}\int_0^t sf(s,1+s^2,1+s)\,\mathrm{d}s = 0. \tag{3.7}$$

Then problem (1.1) *has at least one* $C^2[0,\infty)\cap C^3(0,\infty)$ *positive solution.*

P r o o f. Suppose that (3.6) and (3.7) hold. According to the proof of Theorem 3.2, there exists a $C^3(0,\infty)$ positive solution $\widetilde{x}$ to problem (1.1) such that $r < \|\widetilde{x}\| < R$. Let $0 < c_6 \leqslant 1$ be a constant such that $c_6\max\{M_2,1\}\|\widetilde{x}\| \leqslant 1$. We have

$$|(\widetilde{x})'''| = f(t,\widetilde{x}(t),\widetilde{x}'(t)) \leqslant c_6^{-\mu_1-\mu_2} f(t,1+t^2,1+t).$$

Then $|\widetilde{x}'''|$ is absolutely integrable on $[0,\infty)$, which implies that $\widetilde{x}\in C^2[0,\infty)$; so $\widetilde{x}$ is a $C^2[0,\infty)$ positive solution of problem (1.1). □

E x a m p l e 3.2. Consider the following boundary value problem:

$$\begin{aligned} x(0)-\frac{1}{2}x\Big(\frac{1}{2}\Big) &= 0,\\ x'(0)-\frac{1}{3}x'\Big(\frac{1}{3}\Big) &= 0\\ \lim_{t\to\infty} x''(t) &= 0,\\ x'''(t)+\frac{t^p x(t)^\lambda x'(t)^\mu}{t(1+t^2)^{\lambda+1}} &= 0, \quad t\geqslant 0, \end{aligned} \tag{3.8}$$

where $\lambda>0$, $0\leqslant\mu<1$, $\lambda+\mu>1$. By Theorem 3.4, problem (3.8) has at least one $C^2[0,\infty)\cap C^3(0,\infty)$ positive solution whenever $p>0$ and $p+\mu<1$.

R e m a r k 3.1 (Concluding remarks). Examples 3.1 and 3.2 show that in this work existence of solutions was obtained under sub-linear growth in the second argument, the derivative x' of the solution, of the nonlinearity f. In this case, one can take any power of x provided that f has a global joint super-linear growth in the space arguments. The time singularity in Example 3.1, which has a killing effect, has order $1/t$ in the vicinity of the origin. However, for a more regular solution to exist, say of class $C^2[0,\infty)$, we required in Example 3.2 that f behaves as t^{p-1} with exponent $0 < p < 1 - \mu \leqslant 0$, in the vicinity of the time origin.

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