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STABILITY OF UNIQUE PSEUDO ALMOST PERIODIC SOLUTIONS WITH MEASURE

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Abstract. By means of the fixed-point methods and the properties of the μ-pseudo almost periodic functions, we prove the existence, uniqueness, and exponential stability of the μ-pseudo almost periodic solutions for some models of recurrent neural networks with mixed delays and time-varying coefficients, where μ is a positive measure. A numerical example is given to illustrate our main results.

Keywords: measure pseudo almost periodic solution; recurrent neural networks; mixed delays

MSC 2020: 34C27, 34K14, 35B15, 37B25, 92C20

1. Introduction

The qualitative theory of differential equations, involving almost periodicity, has been an attractive topic because of its significance and applications in areas such as physics, mathematical biology, and control theory. The concept of almost periodicity was first introduced in the literature by Bohr in 1923. For more details about this topic we refer the reader to the recent book of N'Guérékata [22] where the author gave an important overview about the theory of almost periodic functions and their applications to differential equations. The notion of μ-pseudo almost periodicity, which was introduced and developed in [4], [7], [12], [19], [20], and [21], is a generalization of the almost periodicity and pseudo almost periodicity introduced by Zhang [27], [28]; it is also a generalization of weighted pseudo almost periodicity first introduced by Diagana [11].

During the past few years, the problem of dynamics of different classes of recurrent neural networks (RNNs) has been one of the most active areas of research and has attracted the attention of many researchers; we refer to [1], [2], [5], [6], [23], [25], [26].

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Cellular neural networks were introduced by Chua and Yang [8], [9] in 1988. They have found important applications in signal processing, especially in static image treatment. Processing of moving images requires the introduction of delay in the signals transmitted among the cells [24].

Many phenomena exhibit great regularity without being periodic. This is modeled using the notion of μ-pseudo almost periodic and related functions, which allow complex repetitive phenomena to be represented as an almost periodic process plus an ergodic component.

Recently, dynamic behaviors of neural networks with time delays have been extensively investigated and numerous important results have been reported [3], [5]. In this work, we are concerned with the μ-pseudo almost periodic solution of the following model:

$$(1)\qquad \begin{aligned} x_i'(t) = & -a_i(t)x_i(t) + \sum_{j=1}^{n} \alpha_{ij}(t) f_j(t, x_j(t)) + \sum_{j=1}^{n} \beta_{ij}(t) g_j(t, x_j(t-\tau_{ij})) \\ & + \sum_{j=1}^{n} \gamma_{ij}(t) \int_{-\infty}^{t} K_{ij}(t-s) h_j(s, x_j(s))\,\mathrm{d}s + J_i(t), \quad 1 \leqslant i \leqslant n. \end{aligned}$$

Here n is the number of the neurons in the neural network, $x_i(t)$ denotes the state of the ith neuron at time t, f_j, g_j, and h_j are the activation functions of jth neuron. The functions $\alpha_{ij}(t)$, $\beta_{ij}(t)$, and $\gamma_{ij}(t)$ denote respectively the connection weights, the discretely delayed connection weights, and the distributively delayed connection weights of the jth neuron on the ith neuron. Furthermore, $J_i(t)$ is the external bias on the ith neuron, K_{ij} correspond to the transmission delay kernels, $a_i(t) > 0$ denotes the rate with which the ith neuron will reset its potential to the resting state in isolation when disconnected from the network and external inputs, and $\tau_{ij} > 0$ is the constant discrete time delay. The mixed delays include time-varying delays and unbounded distributed delays.

This model (1) has been the subject of intensive analysis by numerous authors in recent years. In particular, there have been extensive results on the problem of the existence and stability of periodic and almost periodic solutions of RNNs in the literature ([15], [29] and the references therein). Since the space of pseudo almost periodic functions contains strictly the space of almost periodic functions and of periodic functions, in [25], Alimi et al. studied the problem of pseudo almost periodic solutions. Many authors worked on this category of solutions (see [13], [16], [17]). In [18], the authors prove the existence and the global exponential stability of the unique weighted pseudo almost periodic solution with mixed time-varying delays comprising different discrete and distributed time delays. Notice that in [25], the delay $\tau(t)$ is a continuously almost periodic function on $\mathbb{R}$, while in this paper we will consider a constant delay. Moreover, what is important and new here in our system (1) compared to other works is that the activation functions f_j, g_j, and h_j of the jth neuron are of two variables. This paper is concerned with the existence and uniqueness of μ-pseudo almost periodic solutions to recurrent delayed neural networks. Several conditions guaranteeing the existence and uniqueness of such solutions are obtained in a suitable convex domain. Furthermore, several methods are applied to establish sufficient criteria for the globally exponential stability of this system (1).

The rest of this paper is organized as follows. In Section 2, we recall briefly some basic definitions and properties of the measure pseudo almost periodic functions. The existence, the uniqueness and the exponential stability of measure pseudo almost periodic solutions of (1) in the suitable convex set are discussed in Section 3. Finally, in Section 4, we give an example to illustrate our abstract results.

2. Measure pseudo almost periodic functions

Definition 1 ([10]). Let n be a nonzero natural number. A continuous function $f\colon \mathbb{R} \to \mathbb{R}^n$ is said to be *almost periodic* if for every $\varepsilon > 0$, there exists a positive number $l(\varepsilon)$, such that every interval of length $l(\varepsilon)$ contains a number τ such that

$$\|f(t+\tau) - f(t)\| < \varepsilon \quad \forall\, t \in \mathbb{R}.$$

Let $AP(\mathbb{R}, \mathbb{R}^n)$ be the set of all almost periodic functions from $\mathbb{R}$ to $\mathbb{R}^n$. Then $(AP(\mathbb{R}, \mathbb{R}^n), \|\cdot\|_\infty)$ is a Banach space with supremum norm given by

$$\|u\|_\infty = \sup_{t\in\mathbb{R}} \|u(t)\|.$$

Definition 2 ([14]). A continuous function $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}^n$ is said to be *almost periodic in t uniformly for $y \in \mathbb{R}$* if for every $\varepsilon > 0$, and any compact subset K of $\mathbb{R}$, there exists a positive number $l(\varepsilon)$ such that every interval of length $l(\varepsilon)$ contains a number τ such that

$$\|f(t+\tau, y) - f(t, y)\| < \varepsilon \quad \forall\, (t, y) \in \mathbb{R}\times K.$$

We denote the set of such functions as $APU(\mathbb{R}\times\mathbb{R}, \mathbb{R}^n)$.

Theorem 1 ([4]). *If $F \in APU(\mathbb{R}\times\mathbb{R}, \mathbb{R}^n)$ and $x \in AP(\mathbb{R}, \mathbb{R})$, then $t \mapsto F(t, x(t)) \in AP(\mathbb{R}, \mathbb{R}^n)$.*

In the next section, we give the new concept of the ergodic functions developed in [4], and generalize the ergodicity given in [11]. Let $BC(\mathbb{R},\mathbb{R}^n)$ be the space of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$. We denote by $\mathbf{B}$ the Lebesque σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathbf{B}$ satisfying $\mu(\mathbb{R})=\infty$ and $\mu([a,b])<\infty$, for all $a,b\in\mathbb{R}$ $(a\leqslant b)$.

Definition 3 ([4]). Let $\mu\in\mathcal{M}$. A function $f\in BC(\mathbb{R},\mathbb{R}^n)$ is said to be *μ-ergodic* if

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\|f(s)\|\,\mathrm{d}\mu(s)=0.$$

We denote the space of all such functions by $\mathcal{E}(\mathbb{R},\mathbb{R}^n,\mu)$.

Definition 4 ([4]). Let $\mu\in\mathcal{M}$. A continuous function $f\colon\ \mathbb{R}\to\mathbb{R}^n$ is said to be *μ-pseudo almost periodic* if it is written in the form

$$f=g+h,$$

where $g\in AP(\mathbb{R},\mathbb{R}^n)$ and $h\in\mathcal{E}(\mathbb{R},\mathbb{R}^n,\mu)$. The collection of such functions is denoted by $PAP(\mathbb{R},\mathbb{R}^n,\mu)$. Then we have

$$AP(\mathbb{R},\mathbb{R}^n)\subset PAP(\mathbb{R},\mathbb{R}^n,\mu)\subset BC(\mathbb{R},\mathbb{R}^n).$$

Theorem 2 ([4]). *Let $\mu\in\mathcal{M}$. Then $(\mathcal{E}(\mathbb{R},\mathbb{R}^n,\mu),\|\cdot\|_\infty)$ is a Banach space.*

We formulate the following hypothesis that we take from [4].

(M1) For any $\tau\in\mathbb{R}$ there exists $\beta>0$ and a bounded interval I such that

$$\mu_\tau(A):=\mu(\{a+\tau\colon\ a\in A\})\leqslant\beta\mu(A)\quad\text{when } A\in\mathbf{B}\text{ satisfies } A\cap I=\emptyset.$$

Theorem 3 ([4]). *Let $\mu\in\mathcal{M}$ satisfy* (M1). *Then the decomposition of a μ-pseudo almost periodic function in the form $f=g+h$, where $g\in AP(\mathbb{R},\mathbb{R}^n)$ and $h\in\mathcal{E}(\mathbb{R},\mathbb{R}^n,\mu)$, is unique.*

Theorem 4 ([4]). *Let $\mu\in\mathcal{M}$ satisfy* (M1). *Then $(PAP(\mathbb{R},\mathbb{R}^n,\mu),\|\cdot\|_\infty)$ is a Banach space.*

Theorem 5 ([4]). *Suppose that assumptions* (M1) *hold. If $f\in PAP(\mathbb{R},\mathbb{R}^n,\mu)$, then $f_\tau\in PAP(\mathbb{R},\mathbb{R}^n,\mu)$ for all $\tau\in\mathbb{R}$.*

Definition 5 ([4]). Let $\mu \in \mathcal{M}$. A continuous function $f\colon \mathbb{R} \times Y \to \mathbb{R}^n$ is said to be *μ-ergodic in t uniformly with respect to $y \in Y$* if the following conditions are true:

(i) For all $y \in Y$, $f(\cdot, y) \in \mathcal{E}(\mathbb{R}, \mathbb{R}^n, \mu)$.

(ii) f is uniformly continuous on each compact set K in Y with respect to the second variable y.

The collection of such functions is denoted by $\mathcal{E}U(\mathbb{R} \times \mathbb{R}, \mathbb{R}^n, \mu)$.

Definition 6 ([4]). Let $\mu \in \mathcal{M}$. A continuous function $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ is said to be *μ-pseudo almost periodic* if is written in the form

$$f = g + h,$$

where $g \in APU(\mathbb{R} \times \mathbb{R}, \mathbb{R}^n)$ and $h \in \mathcal{E}U(\mathbb{R} \times \mathbb{R}, \mathbb{R}^n, \mu)$. The collection of such functions is denoted by $PAPU(\mathbb{R} \times \mathbb{R}, \mathbb{R}^n, \mu)$.

3. Main result

In this section, we present some results for the existence, uniqueness, and global exponential stability of the μ-pseudo almost periodic solution of the system (1). For simplicity, we introduce the following notations:

$$\overline{\alpha}_{ij} = \sup_{t\in\mathbb{R}} |\alpha_{ij}(t)|, \quad \overline{\beta}_{ij} = \sup_{t\in\mathbb{R}} |\beta_{ij}(t)|,$$
$$\overline{\gamma}_{ij} = \sup_{t\in\mathbb{R}} |\gamma_{ij}(t)|, \quad \overline{J}_{i} = \sup_{t\in\mathbb{R}} |J_{i}(t)|.$$

To study the existence and uniqueness of μ-pseudo almost periodic solutions to (1), we require the following assumption:

(H1) For $p > 1$ and for all $1 \leqslant j \leqslant n$, the functions f_j, g_j, and h_j are μ-pseudo almost periodic and there exist positive continuous functions $L_j^g, L_j^h, L_j^f \in L^p(\mathbb{R}, \mathrm{d}\mu) \cap L^p(\mathbb{R}, \mathrm{d}x)$ such that for all $t, x, y \in \mathbb{R}$,

$$|f_j(t,x) - f_j(t,y)| < L_j^f(t)|x-y|,$$
$$|g_j(t,x) - g_j(t,y)| < L_j^g(t)|x-y|,$$
$$|h_j(t,x) - h_j(t,y)| < L_j^h(t)|x-y|.$$

In addition, we suppose as well that for all $j \in \{1, 2, \ldots, n\}$, $g_j(t,0) = h_j(t,0) = f_j(t,0) = 0$ for all $t \in \mathbb{R}$.

(H2) For all $1 \leqslant i,\ j \leqslant n$, the delay kernels $K_{ij}\colon\ [0,\infty) \to [0,\infty)$ satisfy

$$\int_0^\infty K_{ij}(s)\,\mathrm{d}s = K_{ij}^{+}.$$

(H3) For all $1 \leqslant i,j \leqslant n$, the functions $t \mapsto \alpha_{ij}(t)$, $t \mapsto \beta_{ij}(t)$, $t \mapsto \gamma_{ij}(t)$, and $t \mapsto J_i(t)$ are μ-pseudo almost periodic.

(H4) For all $1 \leqslant i \leqslant n$, the functions $t \mapsto a_i(t)$ are almost periodic with $\inf\limits_{t\in\mathbb{R}} a_i(t) = a_{i*} > 0$.

(H5) Assume that there exists a nonnegative constant r_0 such that

$$r_0 = \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} \frac{\overline{\alpha}_{ij}\|L_j^{f_j}\|_p + \overline{\beta}_{ij}\|L_j^{g_j}\|_p + \overline{\gamma}_{ij}K_{ij}^{+}\|L_j^{h_j}\|_p}{a_{i*}^{1-1/p}} < \frac{1}{2}.$$

(H6) There exists a constant $\lambda_0 > 0$ such that for all $1 \leqslant i,j \leqslant n$

$$\int_0^\infty K_{ij}(s)\mathrm{e}^{\lambda_0 s}\,\mathrm{d}s < \infty.$$

Lemma 1. *If $\varphi,\psi \in PAP(\mathbb{R},\mathbb{R},\mu)$, then $\varphi\cdot\psi \in PAP(\mathbb{R},\mathbb{R},\mu)$.*

P r o o f. By definition, we can write $\varphi = \varphi_1 + \varphi_2$ and $\psi = \psi_1 + \psi_2$, where $\varphi_1,\psi_1 \in AP(\mathbb{R},\mathbb{R})$ and $\varphi_2,\psi_2 \in \mathcal{E}(\mathbb{R},\mathbb{R},\mu)$. Then we have

$$\varphi\cdot\psi = \varphi_1\cdot\psi_1 + \varphi_1\cdot\psi_2 + \varphi_2\cdot\psi_1 + \varphi_2\cdot\psi_2.$$

This shows that $\varphi_1\cdot\psi_1 \in AP(\mathbb{R},\mathbb{R})$. If we take $\varphi_1 = \psi_1 \in AP(\mathbb{R},\mathbb{R})$, then $\varphi_1^2 \in AP(\mathbb{R},\mathbb{R})$, since

$$\begin{aligned}\|\varphi_1^2(t+\tau) - \varphi_1^2(t)\| &= \|\varphi_1(t+\tau) + \varphi_1(t)\|\cdot\|\varphi_1(t+\tau) - \varphi_1(t)\| \\ &\leqslant \varepsilon\|\varphi_1(t+\tau) + \varphi_1(t)\| \\ &\leqslant \varepsilon(\|\varphi_1(t+\tau) - \varphi_1(t)\| + 2\|\varphi_1(t)\|) \\ &\leqslant 2(M+\varepsilon)\varepsilon \leqslant \varepsilon',\end{aligned}$$

where $M = \|\varphi_1\|_\infty < \infty$.

On the other hand, we have

$$\varphi_1\cdot\psi_1 = \frac{1}{4}((\varphi_1+\psi_1)^2 - (\varphi_1-\psi_1)^2).$$

Since $(\varphi_1+\psi_1)^2 \in AP(\mathbb{R},\mathbb{R})$ and $(\varphi_1-\psi_1)^2 \in AP(\mathbb{R},\mathbb{R})$, it follows that $\varphi_1\cdot\psi_1 \in AP(\mathbb{R},\mathbb{R})$. In addition, for the function $(\varphi_1\cdot\psi_2+\varphi_2\cdot\psi_1+\varphi_2\cdot\psi_2)$, one has that:

For all $t\in\mathbb{R}$,

$$\begin{aligned}(2)\qquad &|\varphi_1(t)\cdot\psi_2(t)+\varphi_2(t)\cdot\psi_1(t)+\varphi_2(t)\cdot\psi_2(t)|\\ &\quad\leqslant \|\varphi_1\|_\infty|\psi_2(t)|+|\varphi_2(t)|\|\psi_1\|_\infty+\|\varphi_2\|_\infty|\psi_2(t)|.\end{aligned}$$

Using the properties of the functions φ_2,ψ_2, we obtain

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}(\|\varphi_1\|_\infty|\psi_2(t)|+|\varphi_2(t)|\|\psi_1\|_\infty+\|\varphi_2\|_\infty|\psi_2(t)|)\,\mathrm{d}\mu(t)=0.$$

Then, (2) yields

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}(|(\varphi_1\cdot\psi_2+\varphi_2\cdot\psi_1+\varphi_2\cdot\psi_2)(t)|)\,\mathrm{d}\mu(t)=0.$$

This proves our lemma. □

Lemma 2. *Let $\mu\in\mathcal{M}$, $F\in PAPU(\mathbb{R}\times\mathbb{R},\mathbb{R},\mu)$ satisfy $|F(t,x)-F(t,y)|\leqslant l(t)|x-y|$, where $l\in L^p(\mathbb{R},\mathrm{d}\mu)$, $1\leqslant p\leqslant\infty$. If $x\in PAP(\mathbb{R},\mathbb{R},\mu)$, then $t\mapsto F(t,x(t))\in PAP(\mathbb{R},\mathbb{R},\mu)$.*

P r o o f. Let $F(t,x)=F_1(t,x)+F_2(t,x)$, where F_1 is the almost periodic component and F_2 is the ergodic perturbation. From the results on the composition of almost periodic functions and Theorem 1, it is enough to show that the result remains valid in the case of measure ergodicity. Let us consider the quantity for F_2, which is measure ergodic in t uniformly with respect to the second variable, then we have

$$F_2(t,x(t))=F_2(t,0)+(F_2(t,x(t))-F_2(t,0)).$$

Since $[t\mapsto F_2(t,0)]\in\mathcal{E}(\mathbb{R},\mathbb{R},\mu)$, it is enough to show that $[t\mapsto F_2(t,x(t))-F_2(t,0)]\in\mathcal{E}(\mathbb{R},\mathbb{R},\mu)$. One has

$$|F_2(t,x(t))-F_2(t,0)|\leqslant l(t)|x(t)|.$$

If $1\leqslant p<\infty$, then

$$|F_2(t,x(t))-F_2(t,0)|\leqslant l(t)\|x\|_\infty.$$

Since $l\in L^p(\mathbb{R},d\mu)\subset\mathcal{E}(\mathbb{R},\mathbb{R},\mu)$, then

$$[t\mapsto F_2(t,x(t))-F_2(t,0)]\in\mathcal{E}(\mathbb{R},\mathbb{R},\mu).$$

If $p=\infty$, then

$$|F_2(t,x(t))-F_2(t,0)|\leqslant|x(t)|\|l\|_\infty.$$

Finally $t\mapsto F(t,x(t))\in PAP(\mathbb{R},\mathbb{R},\mu)$. □

Lemma 3. *Let assumptions* (M1), (H1), (H2) *hold and for all* $1 \leqslant j \leqslant n$, $x_j \in PAP(\mathbb{R}, \mathbb{R}, \mu)$. *Then for all* $1 \leqslant i, j \leqslant n$ *the function*

$$\phi_{ij}\colon\ t \mapsto \int_{-\infty}^{t} K_{ij}(t-s)h_j(s, x_j(s))\,\mathrm{d}s$$

belongs to $PAP(\mathbb{R}, \mathbb{R}, \mu)$

Proof. To prove this lemma, we should generalize the proof giving in [18] to our setting. To begin, observe that

$$|\phi_{ij}(t)| \leqslant \int_{-\infty}^{t} K_{ij}(t-s)|h(s, x_j(s))|\,\mathrm{d}s.$$

By Lemma 2 and assumption (H1), the map $t \mapsto h_j(t, x_t(s)) \in PAP(\mathbb{R}, \mathbb{R}, \mu)$. Therefore, it is bounded and there exists a constant $M^{h_j} > 0$ such that for all $t \in \mathbb{R}$, we have

$$|h_j(t, x_j(t))| \leqslant M^{h_j}.$$

It follows that the function ϕ_{ij} is bounded and satisfies

$$|\phi_{ij}(t)| \leqslant \int_{-\infty}^{t} K_{ij}(t-s)M^{h_j}\,\mathrm{d}s = K_{ij}^{+}M^{h_j}.$$

By the same arguments given in [18], we prove that ϕ_{ij} is continuous. Let us now prove that $\phi_{ij} \in PAP(\mathbb{R}, \mathbb{R}, \mu)$. Using respectively the composition theorem of μ-pseudo almost periodic functions, one has that $[s \mapsto h(s, x_j(s))] \in PAP(\mathbb{R}, \mathbb{R}, \mu)$ for all $1 \leqslant j \leqslant n$. By the decomposition theorem of μ-pseudo almost periodic functions, there exist two functions $u_j \in AP(\mathbb{R}, \mathbb{R})$ and $v_j \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$ such that

$$h_j(s, x_j(s)) = u_j(s) + v_j(s).$$

As a consequence, we have

$$\begin{aligned}\phi_{ij}(t) &= \int_{-\infty}^{t} K_{ij}(t-s)[u_j(s) + v_j(s)]\,\mathrm{d}s\\ &= \int_{-\infty}^{t} K_{ij}(t-s)u_j(s)\,\mathrm{d}s + \int_{-\infty}^{t} K_{ij}(t-s)v_j(s)\,\mathrm{d}s = \phi_{ij}^{1}(t) + \phi_{ij}^{2}(t).\end{aligned}$$

Using the same arguments given in [18], we can prove that $\phi_{ij}^1 \in AP(\mathbb{R},\mathbb{R})$. To achieve our proof, we must show that $\phi_{ij}^2 \in \mathcal{E}(\mathbb{R},\mathbb{R},\mu)$. Indeed,

$$\begin{aligned}\frac{1}{\mu([-r,r])}\int_{-r}^{r}|\phi_{ij}^2(t)|\,\mathrm{d}\mu(t) &= \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left|\int_{-\infty}^{t}K_{ij}(t-s)v_j(s)\,\mathrm{d}s\right|\mathrm{d}\mu(t)\\ &= \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left|\int_{0}^{\infty}K_{ij}(s)v_j(t-s)\,\mathrm{d}s\right|\mathrm{d}\mu(t)\\ &\leqslant \frac{1}{\mu([-r,r])}\int_{-r}^{r}\left(\int_{0}^{\infty}K_{ij}(s)|v_j(t-s)|\,\mathrm{d}s\right)\mathrm{d}\mu(t)\\ &= \int_{0}^{\infty}K_{ij}(s)\left(\frac{1}{\mu([-r,r])}\int_{-r}^{r}|v_j(t-s)|\,\mathrm{d}\mu(t)\right)\mathrm{d}s.\end{aligned}$$

Since μ satisfies (M1), from Theorem 5, we have $t\mapsto v_j(t-s)\in\mathcal{E}(\mathbb{R},\mathbb{R},\mu)$ for every $s\in\mathbb{R}$. By Lebesgue's dominated convergence theorem, we have

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}|\phi_{ij}^2(t)|\,\mathrm{d}\mu(t)=0.$$

This proves that $\phi_{ij}^2 \in \mathcal{E}(\mathbb{R},\mathbb{R},\mu)$. As consequence, $\phi_{ij}\in PAP(\mathbb{R},\mathbb{R},\mu)$ for all $1\leqslant i,j\leqslant n$. $\square$

Lemma 4. *Let $p>1$ and $\Phi\in\mathcal{C}(\mathbb{R}\times\mathbb{R},\mathbb{R})$ such that for all $t,x,y\in\mathbb{R}$*

$$|\Phi(t,x)-\Phi(t,y)|<l(t)|x-y|,$$

where $l\in L^p(\mathbb{R},\mathrm{d}\mu)$. If $\varphi\in PAP(\mathbb{R},\mathbb{R},\mu)$ and $\theta\in\mathbb{R}$, then

$$[s\mapsto\Phi(s,\varphi(s-\theta))]\in PAP(\mathbb{R},\mathbb{R},\mu).$$

Proof. Pose $\varphi=\varphi_1+\varphi_2$, where $\varphi_1\in AP(\mathbb{R},\mathbb{R})$ and $\varphi_2\in\mathcal{E}(\mathbb{R},\mathbb{R},\mu)$. Let us consider the following function:

$$\Theta(t)=\Phi(t,\varphi_1(t-\theta))+[\Phi(t,\varphi_1(t-\theta)+\varphi_2(t-\theta))-\Phi(t,\varphi_1(t-\theta))]=\Theta_1(t)+\Theta_2(t),$$

where $\Theta_1(t)=\Phi(t,\varphi_1(t-\theta))$ and $\Theta_2(t)=\Phi(t,\varphi_1(t-\theta)+\varphi_2(t-\theta))-\Phi(t,\varphi_1(t-\theta))$.

First, it follows from Theorem 2.11 in [18] that $\Theta_1\in AP(\mathbb{R},\mathbb{R})$. Let us show that $\Theta_2\in\mathcal{E}(\mathbb{R},\mathbb{R},\mu)$. We have

$$\begin{aligned}&\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}|\Theta_2(t)|\,\mathrm{d}\mu(t)\\ &\quad=\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}|\Phi(t,\varphi_1(t-\theta)+\varphi_2(t-\theta))-\Phi(t,\varphi_1(t-\theta))|\,\mathrm{d}\mu(t)\\ &\quad\leqslant\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}l(t)|\varphi_2(t-\theta)|\,\mathrm{d}\mu(t).\end{aligned}$$

Since $\varphi_2 \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$, we have

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{-r}^{r} |\Theta_2(t)| \,\mathrm{d}\mu(t) \\
&\quad \leqslant \lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{-r}^{r} l(t)|\varphi_2(t-\theta)| \,\mathrm{d}\mu(t) \\
&\quad \leqslant \lim_{r\to\infty} \frac{\|\varphi_2\|_\infty}{\mu([-r,r])} \int_{-r}^{r} l(t) \,\mathrm{d}\mu(t) \\
&\quad \leqslant \lim_{r\to\infty} \frac{\|\varphi_2\|_\infty}{\mu([-r,r])} \left[\int_{-r}^{r} (l(t))^p \,\mathrm{d}\mu(t)\right]^{1/p} \left[\int_{-r}^{r} \mathrm{d}\mu(t)\right]^{1/q} \quad \left(\text{where } \frac{1}{p} + \frac{1}{q} = 1\right) \\
&\quad \leqslant \lim_{r\to\infty} \lim \frac{\|\varphi_2\|_\infty}{\mu([-r,r])^{1/q}} \left[\int_{-r}^{r} (l(t))^p \,\mathrm{d}\mu(t)\right]^{1/p} \\
&\quad \leqslant \lim_{r\to\infty} \frac{\|\varphi_2\|_\infty \|l\|_p}{\mu([-r,r])^{1/q}} = 0.
\end{aligned}$$

Therefore, $[t \mapsto \Theta_2(t)] \in \mathcal{E}(\mathbb{R}, \mathbb{R}, \mu)$. Finally, we conclude that $[s \mapsto \Phi(s, \varphi(s-\theta))] \in PAP(\mathbb{R}, \mathbb{R}, \mu)$, and the proof is finished. $\square$

Lemma 5. *Suppose that assumptions* (M1) *and* (H1)–(H4) *hold. Define the nonlinear operator* Γ *as follows:*

For each $\varphi = (\varphi_1, \ldots, \varphi_n) \in PAP(\mathbb{R}, \mathbb{R}^n, \mu)$,

$$\Gamma\varphi = x_\varphi = (x_\varphi^1, x_\varphi^2, \ldots, x_\varphi^n)$$

such that for all $i \in \{1, 2, \ldots, n\}$, *for all* $t \in \mathbb{R}$,

$$x_\varphi^i(t) = \int_{-\infty}^{t} \mathrm{e}^{-\int_s^t a_i(u)\,\mathrm{d}u} F_i(s) \,\mathrm{d}s$$

and for all $i \in \{1, 2, \ldots, n\}$ *the function* F_i *is given by*

$$\begin{aligned}
(3) \qquad F_i(s) = {}& \sum_{j=1}^{n} \alpha_{ij}(s) f_j(s, \varphi_j(s)) + \sum_{j=1}^{n} \beta_{ij}(s) g_j(s, \varphi_j(s-\tau_{ij})) \\
&+ \sum_{j=1}^{n} \gamma_{ij}(s) \int_0^\infty K_{ij}(u) h_j(s, \varphi_j(s-u)) \,\mathrm{d}u + J_i(s), \quad 1 \leqslant i \leqslant n.
\end{aligned}$$

Then, Γ *maps* $PAP(\mathbb{R}, \mathbb{R}^n, \mu)$ *onto itself.*

P r o o f. First note that, from Lemmas 1 and 3, for all $1 \leqslant i \leqslant n$, the function

$$\Big[s \mapsto \gamma_{ij}(s)\int_0^\infty K_{ij}(u)h_j(s,\varphi_j(s-u))\,\mathrm{d}u\Big] \in PAP(\mathbb{R},\mathbb{R}^n,\mu).$$

Then, the function $s \mapsto F_i(s)$ is μ-pseudo almost periodic by using Lemmas 1–5. Consequently, for all $1 \leqslant i,j \leqslant n$, there exist two functions $F_i^1 \in AP(\mathbb{R},\mathbb{R})$ and $F_i^2 \in \mathcal{E}(\mathbb{R},\mathbb{R},\mu)$ such that

$$F_i = F_i^1 + F_i^2.$$

It follows that

$$\begin{aligned}(\Gamma_i\varphi)(t) &= \int_{-\infty}^t \mathrm{e}^{-\int_t^s a_i(u)\,\mathrm{d}u} F_i(s)\,\mathrm{d}s\\ &= \int_{-\infty}^t \mathrm{e}^{-\int_s^t a_i(u)\,\mathrm{d}u} F_i^1(s)\,\mathrm{d}s + \int_{-\infty}^t \mathrm{e}^{-\int_s^t a_i(u)\,\mathrm{d}u} F_i^2(s)\,\mathrm{d}s\\ &= G_i^1(t) + G_i^2(t).\end{aligned}$$

By the same arguments given in [18], we can prove that $G_i^1 \in AP(\mathbb{R},\mathbb{R})$. To complete the proof of this lemma, it remains to show that $G_i^2 \in \mathcal{E}(\mathbb{R},\mathbb{R},\mu)$. Indeed,

$$\begin{aligned}&\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^r |G_i^2(t)|\,\mathrm{d}\mu(t)\\ &\qquad\leqslant \lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^r\int_{-\infty}^t \mathrm{e}^{-(t-s)a_{i*}}|F_i^2(s)|\,\mathrm{d}s\,\mathrm{d}\mu(t) = I.\end{aligned}$$

Pose $y = t-s$. Then, by Fubini's theorem one has

$$\begin{aligned}I &= \lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^r\int_{-\infty}^t \mathrm{e}^{-(t-s)a_{i*}}|F_i^2(s)|\,\mathrm{d}s\,\mathrm{d}\mu(t)\\ &= \lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^r\int_0^\infty \mathrm{e}^{-ya_{i*}}|F_i^2(t-y)|\,\mathrm{d}y\,\mathrm{d}\mu(t)\\ &\leqslant \lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^r\int_0^\infty \mathrm{e}^{-ya_{i*}}|F_i^2(t-y)|\,\mathrm{d}y\,\mathrm{d}\mu(t)\\ &= \int_0^\infty \mathrm{e}^{-ya_{i*}}\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^r |F_i^2(t-y)|\,\mathrm{d}\mu(t)\,\mathrm{d}y.\end{aligned}$$

Since the function $t \mapsto F_i^2(t-y) \in \mathcal{E}(\mathbb{R},\mathbb{R},\mu)$ from Lemma 5, by the Lebesgue dominated convergence theorem, we obtain that $I = 0$. It follows that,

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^r\left|\int_{-\infty}^t \mathrm{e}^{-(t-s)a_{i*}}F_i^2(s)\,\mathrm{d}s\right|\mathrm{d}\mu(t) = 0.$$

This shows that $G_i^2 \in \mathcal{E}(\mathbb{R},\mathbb{R},\mu)$. So, for all $1 \leqslant i \leqslant n$, $\Gamma_i\varphi$ belongs to $PAP(\mathbb{R},\mathbb{R},\mu)$ and consequently $\Gamma\varphi$ belongs to $PAP(\mathbb{R},\mathbb{R}^n,\mu)$. □

Theorem 6. *Suppose that conditions* (M1) *and* (H1)–(H5) *hold. Then, the delayed RNNs* (1) *has a unique μ-pseudo almost periodic solution in the region*

$$\mathbb{B} = \Big\{\varphi \in PAP(\mathbb{R}, \mathbb{R}^n, \mu), \|\varphi - \varphi_0\|_\infty \leqslant \frac{r_0\beta}{1 - r_0}\Big\},$$

where

$$\beta = \max_{1\leqslant i\leqslant n}\Big\{\frac{\overline{J}_i}{a_{i*}}\Big\} \quad \text{and} \quad \varphi_0(t) = \begin{pmatrix} \int_{-\infty}^t \mathrm{e}^{-\int_s^t a_1(u)\,\mathrm{d}u} J_1(s)\,\mathrm{d}s \\ \vdots \\ \int_{-\infty}^t \mathrm{e}^{-\int_s^t a_i(u)\,\mathrm{d}u} J_i(s)\,\mathrm{d}s \\ \vdots \\ \int_{-\infty}^t \mathrm{e}^{-\int_s^t a_n(u)\,\mathrm{d}u} J_n(s)\,\mathrm{d}s \end{pmatrix}.$$

P r o o f. First, it is easy to see that the function $\varphi_0 \in L^\infty(\mathbb{R}, \mathbb{R})$ and satisfies the following estimation:

$$\|\varphi_0\|_\infty \leqslant \beta.$$

Second, let consider the set

$$\mathbb{B} = \Big\{\varphi \in PAP(\mathbb{R}, \mathbb{R}^n, \mu),\ \|\varphi - \varphi_0\|_\infty \leqslant \frac{r_0\beta}{1 - r_0}\Big\}.$$

Clearly, $\mathbb{B}$ is a closed convex subset of $PAP(\mathbb{R}, \mathbb{R}^n, \mu)$. Moreover, for any $\varphi \in \mathbb{B}$, we have

$$\begin{aligned}
&\|\Gamma_\varphi - \varphi_0\|_\infty \\
&= \max_{1\leqslant i\leqslant n}\sup_{t\in\mathbb{R}}\Big|\int_{-\infty}^t \mathrm{e}^{-\int_s^t a_i(u)\,\mathrm{d}u} F_i(s)\,\mathrm{d}s - \int_{-\infty}^t \mathrm{e}^{-\int_s^t a_i(u)\,\mathrm{d}u} J_i(s)\,\mathrm{d}s\Big| \\
&\leqslant \max_{1\leqslant i\leqslant n}\sup_{t\in\mathbb{R}}\Big|\int_{-\infty}^t \mathrm{e}^{-\int_s^t a_i(u)\,\mathrm{d}u}(F_i(s) - J_i(s))\,\mathrm{d}s\Big| \\
&= \max_{1\leqslant i\leqslant n}\sup_{t\in\mathbb{R}}\Big|\int_{-\infty}^t \mathrm{e}^{-\int_s^t a_i(u)\,\mathrm{d}u}\Big(\sum_{j=1}^n \alpha_{ij}(s) f_j(s, \varphi_j(s)) \\
&\quad + \sum_{j=1}^n \beta_{ij}(s) g_j(s, \varphi_j(s - \tau_{ij})) + \sum_{j=1}^n \gamma_{ij}(s)\int_0^\infty K_{ij}(u) h_j(s, \varphi_j(s - u))\,\mathrm{d}u\Big)\,\mathrm{d}s\Big| \\
&\leqslant \max_{1\leqslant i\leqslant n}\sup_{t\in\mathbb{R}}\Big\{\sum_{j=1}^n\Big(\int_{-\infty}^t \mathrm{e}^{-\int_s^t a_i(u)\,\mathrm{d}u}|\alpha_{ij}(s)||f_j(s, \varphi_j(s))|\,\mathrm{d}s \\
&\quad + \int_{-\infty}^t \mathrm{e}^{-\int_s^t a_i(u)\,\mathrm{d}u}|\beta_{ij}(s)||g_j(s, \varphi_j(s - \tau_{ij}))|\,\mathrm{d}s \\
&\quad + \int_{-\infty}^t \mathrm{e}^{-\int_s^t a_i(u)\,\mathrm{d}u}|\gamma_{ij}(s)|\int_0^\infty K_{ij}(u)|h_j(s, \varphi_j(s - u))|\,\mathrm{d}u\Big)\,\mathrm{d}s\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \max_{1\leqslant i\leqslant n}\sup_{t\in\mathbb{R}}\sum_{j=1}^{n}\biggl(\int_{-\infty}^{t}\mathrm{e}^{-(t-s)a_{i*}}\overline{\alpha}_{ij}L_j^{f_j}(s)|\varphi_j(s)|\,\mathrm{d}s\\
&\quad+\int_{-\infty}^{t}\mathrm{e}^{-(t-s)a_{i*}}\overline{\beta}_{ij}L_j^{g_j}(s)|\varphi_j(s-\tau_{ij})|\,\mathrm{d}s\\
&\quad+\int_{-\infty}^{t}\mathrm{e}^{-(t-s)a_{i*}}\overline{\gamma}_{ij}\int_0^{\infty}K_{ij}(u)L_j^{h_j}(s)|\varphi_j(s-u)|\,\mathrm{d}u\biggr)\,\mathrm{d}s.
\end{aligned}
$$

By using the assumption (H1), we get the following:

$$
\begin{aligned}
&\|\Gamma_\varphi-\varphi_0\|_\infty\\
&\leqslant \|\varphi\|_\infty\max_{1\leqslant i\leqslant n}\sup_{t\in\mathbb{R}}\sum_{j=1}^{n}\biggl(\int_{-\infty}^{t}\mathrm{e}^{-(t-s)a_{i*}}\overline{\alpha}_{ij}L_j^{f_j}(s)\,\mathrm{d}s+\int_{-\infty}^{t}\mathrm{e}^{-(t-s)a_{i*}}\overline{\beta}_{ij}L_j^{g_j}(s)\,\mathrm{d}s\\
&\quad+\int_{-\infty}^{t}\mathrm{e}^{-(t-s)a_{i*}}\overline{\gamma}_{ij}\int_0^{\infty}K_{ij}(u)L_j^{h_j}(s)\,\mathrm{d}u\biggr)\,\mathrm{d}s\\
&\leqslant \|\varphi\|_\infty\Bigl(\frac{p-1}{p}\Bigr)^{1-1/p}\max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}\frac{\overline{\alpha}_{ij}\|L_j^{f_j}\|_p+\overline{\beta}_{ij}\|L_j^{g_j}\|_p+\overline{\gamma}_{ij}K_{ij}^{+}\|L_j^{h_j}\|_p}{a_{i*}^{1-1/p}}\\
&\leqslant \|\varphi\|_\infty\Bigl(1-1/p\Bigr)^{1-1/p}\max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}\frac{\overline{\alpha}_{ij}\|L_j^{f_j}\|_p+\overline{\beta}_{ij}\|L_j^{g_j}\|_p+\overline{\gamma}_{ij}K_{ij}^{+}\|L_j^{h_j}\|_p}{a_{i*}^{1-1/p}}\\
&\leqslant \|\varphi\|_\infty\max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}\frac{\overline{\alpha}_{ij}\|L_j^{f_j}\|_p+\overline{\beta}_{ij}\|L_j^{g_j}\|_p+\overline{\gamma}_{ij}K_{ij}^{+}\|L_j^{h_j}\|_p}{a_{i*}^{1-1/p}}=\|\varphi\|_\infty r_0\leqslant\frac{r_0\beta}{1-r_0}.
\end{aligned}
$$

From (H1), for any $\varphi,\psi\in\mathbb{B}$, we have

$$
\begin{aligned}
&\|\Gamma_\varphi-\Gamma_\psi\|_\infty\\
&\leqslant \|\varphi-\psi\|_\infty\max_{(1\leqslant i\leqslant n)}\sup_{t\in\mathbb{R}}\sum_{j=1}^{n}\int_{-\infty}^{t}\mathrm{e}^{-(t-s)a_{i*}}[\alpha_{ij}L_j^{f}(s)+\beta_{ij}L_j^{g}(s)+\gamma_{ij}L_j^{h}(s))]\,\mathrm{d}s\\
&\leqslant \Bigl(\frac{p-1}{p}\Bigr)^{1-1/p}\max_{1\leqslant i\leqslant n}\biggl\{\sum_{j=1}^{n}\frac{\overline{\alpha}_{ij}\|L_j^{f_j}\|_p+\overline{\beta}_{ij}\|L_j^{g_j}\|_p+\overline{\gamma}_{ij}K_{ij}^{+}\|L_j^{h_j}\|_p}{a_{i*}^{1-1/p}}\biggr\}\|\varphi-\psi\|_\infty\\
&\leqslant \Bigl(1-\frac{1}{p}\Bigr)^{1-1/p}\max_{1\leqslant i\leqslant n}\biggl\{\sum_{j=1}^{n}\frac{\overline{\alpha}_{ij}\|L_j^{f_j}\|_p+\overline{\beta}_{ij}\|L_j^{g_j}\|_p+\overline{\gamma}_{ij}K_{ij}^{+}\|L_j^{h_j}\|_p}{a_{i*}^{1-1/p}}\biggr\}\|\varphi-\psi\|_\infty\\
&\leqslant \max_{1\leqslant i\leqslant n}\biggl\{\sum_{j=1}^{n}\frac{\overline{\alpha}_{ij}\|L_j^{f_j}\|_p+\overline{\beta}_{ij}\|L_j^{g_j}\|_p+\overline{\gamma}_{ij}K_{ij}^{+}\|L_j^{h_j}\|_p}{a_{i*}^{1-1/p}}\biggr\}\|\varphi-\psi\|_\infty\\
&= r_0\|\varphi-\psi\|_\infty.
\end{aligned}
$$

Then, we prove that Γ is contraction mapping from the region $\mathbb{B}$ into itself. By virtue of the Banach fixed-point theorem, Γ has a unique fixed point which corresponds to the solution of (1) in $\mathbb{B}\subset PAP(\mathbb{R},\mathbb{R}^n,\mu)$. □

We introduce the phase space $C((-\infty,0],\mathbb{R}^n)$ as a Banach space of continuous mappings from $(-\infty,0]$ to $\mathbb{R}^n$ equipped with the norm defined by

$$\|\varphi\|_\infty = \max_{1\leqslant i\leqslant n,-\infty\leqslant t\leqslant 0}\sup |\varphi_i(t)|,$$

for all $\varphi = [\varphi_1,\varphi_2,\dots,\varphi_n]^\top \in C((-\infty,0],\mathbb{R}^n)$. The initial conditions associated with (1) are of the form

$$x_i(s) = \varphi_i(s), \quad s \in (-\infty,0],\ i = 1,2,\dots,n,$$

where $\varphi = [\varphi_1,\varphi_2,\dots,\varphi_n]^\top \in C((-\infty,0],\mathbb{R}^n)$.

Definition 7. Let $x^*(t) = [x_1^*(t),x_2^*(t),\dots,x_n^*(t)]^\top$ be a μ-pseudo almost periodic solution of system (1) with initial value

$$\varphi^*(t) = [\varphi_1^*(t),\varphi_2^*(t),\dots,\varphi_n^*(t)]^\top \in C((-\infty,0],\mathbb{R}^n).$$

We say $x^*(t)$ is globally exponentially stable if there exist constants $\lambda > 0$ and $M(\varphi) > 1$ such that for every solution

$$x(t) = [x_1(t),x_2(t),\dots,x_n(t)]^\top$$

of system (1) with any initial value

$$\varphi(t) = [\varphi_1(t),\varphi_2(t),\dots,\varphi_n(t)]^\top$$

satisfying that:

$$\|x(t)-x^*(t)\|_\infty \leqslant M(\varphi)\|\varphi-\varphi^*\|_\infty e^{-\lambda t} \quad \forall\, t>0,$$

where

$$\|x(t)-x^*(t)\|_\infty = \max_{1\leqslant i\leqslant n}|x_i(t)-x_i^*(t)| \text{ and } \|\varphi-\varphi^*\|_\infty = \max_{1\leqslant i\leqslant n}\sup_{-\infty\leqslant t\leqslant 0}|\varphi_i(t)-\varphi_i^*(t)|.$$

Theorem 7. *Suppose that conditions* (M1) *and* (H1)–(H6) *hold. Then the system* (1) *has a unique μ-pseudo almost periodic solution $z(t)$ which is globally exponentially stable.*

P r o o f. Let

$$z(t) = [z_1(t), z_2(t), \ldots, z_n(t)]^\top$$

be the unique μ-pseudo almost periodic solution of system (1) with initial value

$$u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^\top.$$

Let $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^\top$ be an arbitrary solution of system (1) with initial value $\varphi^*(t) = [\varphi_1^*(t), \varphi_2^*(t), \ldots, \varphi_n^*(t)]^\top$. Let $y_i(t) = x_i(t) - z_i(t)$, $\varphi_i(t) = \varphi_i^*(t) - u_i(t)$. Then, we obtain the following:

$$\begin{aligned} (4)\qquad & y_i'(t) + a_i(t)y_i(t) \\ &= \sum_{j=1}^n \alpha_{ij}(t)[f_j(t, y_j(t) + z_j(t)) - f_j(t, z_j(t))] \\ &\quad + \sum_{j=1}^n \beta_{ij}(t)[g_j(t, y_j(t-\tau_{ij}) + z_j(t-\tau_{ij})) - g_j(t, z_j(t-\tau_{ij}))] \\ &\quad + \sum_{j=1}^n \gamma_{ij}(t) \int_{-\infty}^t K_{ij}(t-s)[h_j(t, y_j(s) + z_j(s)) - h_j(t, z_j(s))]\,\mathrm{d}s. \end{aligned}$$

For all $i \in \{1, 2, \ldots, n\}$, let G_i be defined by

$$\begin{aligned} G_i(z) = a_{i*} - z - 2a_{i*}^{1/p} \sum_{j=1}^n \biggl(& \overline{\alpha}_{ij} \|L_j^{f_j}\|_p + \overline{\beta}_{ij} \mathrm{e}^{z\tau_{ij}} \|L_j^{g_j}\|_p \\ & + \overline{\gamma}_{ij} \|L_j^{h_j}\|_p \int_0^\infty K_{ij}(m) \mathrm{e}^{zm}\,\mathrm{d}m \biggr), \end{aligned}$$

where $z \in [0, \infty)$. By using the fact that $r_0 < \frac{1}{2}$ and the assumption (H2), then we obtain, for all $i \in \{1, 2, \ldots, n\}$

$$G_i(0) = a_{i*} - 2a_{i*}^{1/p} \sum_{j=1}^n (\overline{\alpha}_{ij} \|L_j^{f_j}\|_p + \overline{\beta}_{ij} \|L_j^{g_j}\|_p + \overline{\gamma}_{ij} \|L_j^{h_j}\|_p K_{ij}^+) > 0.$$

In addition, for $i \in \{1, 2, \ldots, n\}$, G_i is a continuous function on $[0, \infty)$ and

$$\lim_{z \to \infty} G_i(z) = -\infty.$$

Then, there exist $\eta_i^* > 0$ such that $G_i(\eta_i^*) = 0$ and $G_i(\eta_i) > 0$ for $\eta_i \in]0, \eta_i^*[$. Taking

$$\eta = \min\{\eta_1^*, \eta_2^*, \ldots, \eta_n^*\},$$

we obtain $G_i(\eta) \geqslant 0$. Let us consider the positive constant λ such that

$$0 < \lambda < \min\{\eta, a_{1*}, a_{2*}, \ldots, a_{n*}, \lambda_0\} \quad \text{and} \quad G_i(\lambda) > 0,$$

where λ_0 is such that $\int_0^{\infty} K_{ij}(s) \mathrm{e}^{\lambda_0 s} \, \mathrm{d}s < \infty$. Therefore, for all $i \in \{1, 2, \ldots, n\}$, we have the following

$$(5)\quad \frac{2a_{i*}^{1/p}}{a_{i*} - \lambda} \sum_{j=1}^{n} \Big(\overline{\alpha}_{ij} \| L_j^{f_j} \|_p + \overline{\beta}_{ij} \mathrm{e}^{\lambda \tau_{ij}} \| L_j^{g_j} \|_p + \overline{\gamma}_{ij} \| L_j^{h_j} \|_p \int_0^{\infty} K_{ij}(m) \mathrm{e}^{\lambda m} \, \mathrm{d}m \Big) < 1.$$

Multiplying (4) by $\mathrm{e}^{\int_0^s a_i(u) \, \mathrm{d}u}$ and integrating over $[0, t]$, we get

$$\begin{aligned}
y_i(t) &= \varphi_i(0) \mathrm{e}^{-\int_0^t a_i(u) \, \mathrm{d}u} \\
&\quad + \int_0^t \mathrm{e}^{-\int_s^t a_i(u) \, \mathrm{d}u} \sum_{j=1}^{n} \alpha_{ij}(s) [f_j(s, y_j(s) + z_j(s)) - f_j(s, z_j(s))] \, \mathrm{d}s \\
&\quad + \int_0^t \mathrm{e}^{-\int_s^t a_i(u) \, \mathrm{d}u} \sum_{j=1}^{n} \beta_{ij}(s) [g_j(s, y_j(s - \tau_{ij}) + z_j(s - \tau_{ij})) - g_j(s, z_j(s - \tau_{ij}))] \, \mathrm{d}s \\
&\quad + \int_0^t \mathrm{e}^{-\int_s^t a_i(u) \, \mathrm{d}u} \sum_{j=1}^{n} \gamma_{ij}(s) \int_{-\infty}^{s} K_{ij}(s - \tau) [h_j(s, y_j(\tau) + z_j(\tau)) - h_j(s, z_j(\tau))] \, \mathrm{d}\tau \, \mathrm{d}s \\
&= \varphi_i(0) \mathrm{e}^{-\int_0^t a_i(u) \, \mathrm{d}u} \\
&\quad + \sum_{j=1}^{n} \int_0^t \mathrm{e}^{-\int_s^t a_i(u) \, \mathrm{d}u} \alpha_{ij}(s) [f_j(s, y_j(s) + z_j(s)) - f_j(s, z_j(s))] \, \mathrm{d}s \\
&\quad + \sum_{j=1}^{n} \int_0^t \mathrm{e}^{-\int_s^t a_i(u) \, \mathrm{d}u} \beta_{ij}(s) [g_j(s, y_j(s - \tau_{ij}) + z_j(s - \tau_{ij})) - g_j(s, z_j(s - \tau_{ij}))] \, \mathrm{d}s \\
&\quad + \sum_{j=1}^{n} \int_0^t \mathrm{e}^{-\int_s^t a_i(u) \, \mathrm{d}u} \gamma_{ij}(s) \int_{-\infty}^{s} K_{ij}(s - \tau) [h_j(s, y_j(\tau) + z_j(\tau)) - h_j(s, z_j(\tau))] \, \mathrm{d}\tau \, \mathrm{d}s \\
&\leqslant |\varphi_i(0)| \mathrm{e}^{-\int_0^t a_i(u) \, \mathrm{d}u} + \sum_{j=1}^{n} \int_0^t \mathrm{e}^{-(t-s) a_{i*}} \overline{\alpha}_{ij} |f_j(s, y_j(s) + z_j(s)) - f_j(s, z_j(s))| \, \mathrm{d}s \\
&\quad + \sum_{j=1}^{n} \int_0^t \mathrm{e}^{-(t-s) a_{i*}} \overline{\beta}_{ij} |g_j(s, y_j(s - \tau_{ij}) + z_j(s - \tau_{ij})) - g_j(s, z_j(s - \tau_{ij}))| \, \mathrm{d}s \\
&\quad + \sum_{j=1}^{n} \int_0^t \mathrm{e}^{-(t-s) a_{i*}} \overline{\gamma}_{ij} \int_{-\infty}^{s} K_{ij}(s - \tau) |h_j(s, y_j(\tau) + z_j(\tau)) - h_j(s, z_j(\tau))| \, \mathrm{d}\tau \, \mathrm{d}s.
\end{aligned}$$

Then, for all $i \in \{1, 2, \ldots, n\}$ we obtain

$$
\begin{aligned}
(6) \qquad |y_i(t)| \leqslant |\varphi_i(0)| \mathrm{e}^{-\int_0^t a_i(u)\,\mathrm{d}u} &+ \sum_{j=1}^{n} \int_0^t \mathrm{e}^{-(t-s)a_{i*}} \overline{\alpha}_{ij} L_j^{f_j}(s) |y_j(s)| \,\mathrm{d}s \\
&+ \sum_{j=1}^{n} \int_0^t \mathrm{e}^{-(t-s)a_{i*}} \overline{\beta}_{ij} L_j^{g_j}(s) |y_j(s - \tau_{ij})| \,\mathrm{d}s \\
&+ \sum_{j=1}^{n} \int_0^t \mathrm{e}^{-(t-s)a_{i*}} \overline{\gamma}_{ij} \int_{-\infty}^{s} K_{ij}(s - \tau) L_j^{h_j}(s) |y_j(\tau)| \,\mathrm{d}\tau \,\mathrm{d}s
\end{aligned}
$$

Let

$$M = \max_{1 \leqslant i \leqslant n} \biggl(a_{i*}^{1-1/p} \Big/ \sum_{j=1}^{n} \Bigl(\overline{\alpha}_{ij} \|L_j^{f_j}\|_p + \overline{\beta}_{ij} \|L_j^{g_j}\|_p + \overline{\gamma}_{ij} \|L_j^{h_j}\|_p \int_0^\infty K_{ij}(m) \mathrm{e}^{\lambda m} \,\mathrm{d}m \Bigr) \biggr).$$

It is easy to see that $M > 2$ and

$$
\begin{aligned}
(7) \qquad \frac{1}{M} - \frac{a_{i*}^{1/p}}{a_{i*} - \lambda} \sum_{j=1}^{n} \biggl(\overline{\alpha}_{ij} \|L_j^{f_j}\|_p &+ \overline{\beta}_{ij} \mathrm{e}^{\lambda \tau_{ij}} \|L_j^{g_j}\|_p \\
&+ \overline{\gamma}_{ij} \|L_j^{h_j}\|_p \int_0^\infty K_{ij}(m) \mathrm{e}^{\lambda m} \,\mathrm{d}m \biggr) \leqslant 0.
\end{aligned}
$$

It is easy to see that for all $t \in (-\infty, 0]$, we have

$$\|y(t)\|_\infty = \|\varphi(t)\|_\infty \leqslant \|\varphi\|_\infty \leqslant M \|\varphi\|_\infty \mathrm{e}^{-\lambda t},$$

where $\|y(t)\|_\infty = \max_{1 \leqslant i \leqslant n} |y_i(t)|$. In the following, we will prove that

$$(8) \qquad \|y(t)\|_\infty \leqslant M \|\varphi\|_\infty \mathrm{e}^{-\lambda t} \quad \forall\, t > 0.$$

To prove (8), we first show for any $\sigma > 1$, the following inequality holds:

$$\|y(t)\|_\infty < \sigma M \|\varphi\|_\infty \mathrm{e}^{-\lambda t}, \quad t > 0.$$

If it is false, then there must be some $t_1 > 0$ and some $i \in \{1, 2, \ldots, n\}$, such that

$$(9) \qquad \|y(t_1)\|_\infty = \|y_i(t_1)\|_\infty = \sigma M \|\varphi\|_\infty \mathrm{e}^{-\lambda t_1}$$

and

$$(10) \qquad \|y(t)\|_\infty < \sigma M \|\varphi\|_\infty \mathrm{e}^{-\lambda t} \quad \forall\, t \in (-\infty, t_1).$$

By using (5), (6), (7), (10) and the assumption (H1), the function y_i satisfies

$$\begin{aligned}
|y_i(t_1)| &\leqslant |\varphi_i(0)|\mathrm{e}^{-t_1 a_{i*}} + \sum_{j=1}^{n} \int_0^{t_1} \mathrm{e}^{-(t_1-s)a_{i*}} \overline{\alpha}_{ij} L_j^{f_j}(s)|y_j(s)|\,\mathrm{d}s \\
&\quad + \sum_{j=1}^{n} \int_0^{t_1} \mathrm{e}^{-(t_1-s)a_{i*}} \overline{\beta}_{ij} L_j^{g_j}(s)|y_j(s-\tau_{ij})|\,\mathrm{d}s \\
&\quad + \sum_{j=1}^{n} \int_0^{t_1} \mathrm{e}^{-(t_1-s)a_{i*}} \overline{\gamma}_{ij} \int_{-\infty}^{s} K_{ij}(s-\tau) L_j^{h_j}(s)|y_j(\tau)|\,\mathrm{d}\tau\,\mathrm{d}s \\
&\leqslant \|\varphi\|_\infty \mathrm{e}^{-t_1 a_{i*}} + \sum_{j=1}^{n} \int_0^{t_1} \mathrm{e}^{-(t_1-s)a_{i*}} \overline{\alpha}_{ij} L_j^{f_j}(s)\|y_j(s)\|_\infty\,\mathrm{d}s \\
&\quad + \sum_{j=1}^{n} \int_0^{t_1} \mathrm{e}^{-(t_1-s)a_{i*}} \overline{\beta}_{ij} L_j^{g_j}(s)\|y_j(s-\tau_{ij})\|_\infty\,\mathrm{d}s \\
&\quad + \sum_{j=1}^{n} \int_0^{t_1} \mathrm{e}^{-(t_1-s)a_{i*}} \overline{\gamma}_{ij} \int_{-\infty}^{s} K_{ij}(s-\tau) L_j^{h_j}(s)\|y_j(\tau)\|_\infty\,\mathrm{d}\tau\,\mathrm{d}s.
\end{aligned}$$

Then, we have

$$\begin{aligned}
|y_i(t_1)| &\leqslant \|\varphi\|_\infty \mathrm{e}^{-t_1 a_{i*}} + \mathrm{e}^{-t_1 a_{i*}} \sum_{j=1}^{n} \overline{\alpha}_{ij} \int_0^{t_1} \mathrm{e}^{(a_{i*}-\lambda)s} L_j^{f_j}(s)\sigma M\|\varphi\|_\infty\,\mathrm{d}s \\
&\quad + \mathrm{e}^{-t_1 a_{i*}} \sum_{j=1}^{n} \overline{\beta}_{ij} \mathrm{e}^{\lambda\tau_{ij}} \int_0^{t_1} \mathrm{e}^{(a_{i*}-\lambda)s} L_j^{g_j}(s)\sigma M\|\varphi\|_\infty\,\mathrm{d}s \\
&\quad + \mathrm{e}^{-t_1 a_{i*}} \sum_{j=1}^{n} \overline{\gamma}_{ij} \int_0^{t_1} L_j^{h_j}(s)\mathrm{e}^{(a_{i*}-\lambda)s}\,\mathrm{d}s \int_0^{\infty} K_{ij}(m)\sigma M\|\varphi\|_\infty \mathrm{e}^{\lambda m}\,\mathrm{d}m \\
&\leqslant \sigma M\|\varphi\|_\infty \mathrm{e}^{-\lambda t_1} \biggl[\frac{\mathrm{e}^{(\lambda-a_{i*})t_1}}{\sigma M} + \mathrm{e}^{(\lambda-a_{i*})t_1} \sum_{j=1}^{n} \overline{\alpha}_{ij} \int_0^{t_1} \mathrm{e}^{(a_{i*}-\lambda)s} L_j^{f_j}(s)\,\mathrm{d}s \\
&\quad + \mathrm{e}^{(\lambda-a_{i*})t_1} \sum_{j=1}^{n} \overline{\beta}_{ij} \mathrm{e}^{\lambda\tau_{ij}} \int_0^{t_1} \mathrm{e}^{(a_{i*}-\lambda)s} L_j^{g_j}(s)\,\mathrm{d}s \\
&\quad + \mathrm{e}^{(\lambda-a_{i*})t_1} \sum_{j=1}^{n} \overline{\gamma}_{ij} \int_0^{t_1} L_j^{h_j}(s)\mathrm{e}^{(a_{i*}-\lambda)s}\,\mathrm{d}s \int_0^{\infty} K_{ij}(m)\mathrm{e}^{\lambda m}\,\mathrm{d}m \biggr].
\end{aligned}$$

Let $q > 1$ such that $1/p + 1/q = 1$. It follows that for all $i \in \{1, 2, \ldots, n\}$:

$$\begin{aligned}
&|y_i(t_1)| \leqslant \sigma M\|\varphi\|_\infty \mathrm{e}^{-\lambda t_1} \\
&\quad \times \biggl[\frac{\mathrm{e}^{(\lambda-a_{i*})t_1}}{\sigma M} + \mathrm{e}^{(\lambda-a_{i*})t_1} \sum_{j=1}^{n} \overline{\alpha}_{ij} \|L_j^{f_j}\|_p \biggl(\int_0^{t_1} \mathrm{e}^{q(a_{i*}-\lambda)s}\,\mathrm{d}s \biggr)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&+ \mathrm{e}^{(\lambda-a_{i*})t_1}\sum_{j=1}^{n}\overline{\beta}_{ij}\mathrm{e}^{\lambda\tau_{ij}}\|L_j^{g_j}\|_p\biggl(\int_0^{t_1}\mathrm{e}^{q(a_{i*}-\lambda)s}\,\mathrm{d}s\biggr)^{1/q}\\
&+ \mathrm{e}^{(\lambda-a_{i*})t_1}\sum_{j=1}^{n}\overline{\gamma}_{ij}\|L_j^{h_j}\|_p\biggl(\int_0^{t_1}\mathrm{e}^{q(a_{i*}-\lambda)s}\,\mathrm{d}s\biggr)^{1/q}\int_0^{\infty}K_{ij}(m)\mathrm{e}^{\lambda m}\,\mathrm{d}m\biggr]\\
&= \sigma M\|\varphi\|_\infty \mathrm{e}^{-\lambda t_1}\\
&\times\biggl[\frac{\mathrm{e}^{(\lambda-a_{i*})t_1}}{\sigma M}+\frac{1}{[q(a_{i*}-\lambda)]^{1/q}}\sum_{j=1}^{n}\overline{\alpha}_{ij}\|L_j^{f_j}\|_p\mathrm{e}^{(\lambda-a_{i*})t_1}(\mathrm{e}^{q(a_{i*}-\lambda)t_1}-1)^{1/q}\\
&+\frac{1}{[q(a_{i*}-\lambda)]^{1/q}}\sum_{j=1}^{n}\overline{\beta}_{ij}\mathrm{e}^{\lambda\tau_{ij}}\|L_j^{g_j}\|_p\mathrm{e}^{(\lambda-a_{i*})t_1}(\mathrm{e}^{q(a_{i*}-\lambda)t_1}-1)^{1/q}\\
&+\frac{1}{[q(a_{i*}-\lambda)]^{1/q}}\sum_{j=1}^{n}\overline{\gamma}_{ij}\|L_j^{h_j}\|_p\mathrm{e}^{(\lambda-a_{i*})t_1}(\mathrm{e}^{q(a_{i*}-\lambda)t_1}-1)^{1/q}\int_0^{\infty}K_{ij}(m)\mathrm{e}^{\lambda m}\,\mathrm{d}m\biggr].
\end{aligned}
$$

Then, we obtain the following:

$$
\begin{aligned}
|y_i(t_1)| &\leqslant \sigma M\|\varphi\|_\infty\mathrm{e}^{-\lambda t_1}\biggl[\frac{\mathrm{e}^{(\lambda-a_{i*})t_1)}}{\sigma M}+\frac{1}{[q(a_{i*}-\lambda)]^{1/q}}\sum_{j=1}^{n}\overline{\alpha}_{ij}\|L_j^{f_j}\|_p(1-\mathrm{e}^{q(\lambda-a_{i*}t_1)})^{1/q}\\
&+\frac{1}{[q(a_{i*}-\lambda)]^{1/q}}\sum_{j=1}^{n}\overline{\beta}_{ij}\mathrm{e}^{\lambda\tau_{ij}}\|L_j^{g_j}\|_p(1-\mathrm{e}^{q(\lambda-a_{i*}t_1)})^{1/q}\\
&+\frac{1}{[q(a_{i*}-\lambda)]^{1/q}}\sum_{j=1}^{n}\overline{\gamma}_{ij}\|L_j^{h_j}\|_p(1-\mathrm{e}^{q(\lambda-a_{i*}t_1)})^{1/q}\int_0^{\infty}K_{ij}(m)\mathrm{e}^{\lambda m}\,\mathrm{d}m\biggr]\\
&\leqslant \sigma M\|\varphi\|_\infty\mathrm{e}^{-\lambda t_1}\biggl[\frac{\mathrm{e}^{(\lambda-a_{i*})t_1}}{\sigma M}+\frac{(a_{i*}-\lambda)^{1/p}}{a_{i*}-\lambda}\biggl(\sum_{j=1}^{n}\overline{\alpha}_{ij}\|L_j^{f_j}\|_p+\sum_{j=1}^{n}\overline{\beta}_{ij}\mathrm{e}^{\lambda\tau_{ij}}\|L_j^{g_j}\|_p\\
&+\sum_{j=1}^{n}\overline{\gamma}_{ij}\|L_j^{h_j}\|_p\int_0^{\infty}K_{ij}(m)\mathrm{e}^{\lambda m}\,\mathrm{d}m\biggr)(2-\mathrm{e}^{(\lambda-a_{i*})t_1})\biggr]\\
&\leqslant \sigma M\|\varphi\|_\infty\mathrm{e}^{-\lambda t_1}\biggl[\frac{\mathrm{e}^{(\lambda-a_{i*})t_1}}{\sigma M}+\frac{a_{i*}^{1/p}}{a_{i*}-\lambda}\biggl(\sum_{j=1}^{n}\overline{\alpha}_{ij}\|L_j^{f_j}\|_p+\sum_{j=1}^{n}\overline{\beta}_{ij}\mathrm{e}^{\lambda\tau_{ij}}\|L_j^{g_j}\|_p\\
&+\sum_{j=1}^{n}\overline{\gamma}_{ij}\|L_j^{h_j}\|_p\int_0^{\infty}K_{ij}(m)\mathrm{e}^{\lambda m}\,\mathrm{d}m\biggr)(2-\mathrm{e}^{(\lambda-a_{i*})t_1})\biggr]\\
&\leqslant \sigma M\|\varphi\|_\infty\mathrm{e}^{-\lambda t_1}\biggl[\mathrm{e}^{(\lambda-a_{i*})t_1}\frac{1}{M}-\frac{a_{i*}^{1/p}}{a_{i*}-\lambda}\biggl(\sum_{j=1}^{n}\overline{\alpha}_{ij}\|L_j^{f_j}\|_p+\sum_{j=1}^{n}\overline{\beta}_{ij}\mathrm{e}^{\lambda\tau_{ij}}\|L_j^{g_j}\|_p\\
&+\sum_{j=1}^{n}\overline{\gamma}_{ij}\|L_j^{h_j}\|_p\int_0^{\infty}K_{ij}(m)\mathrm{e}^{\lambda m}\,\mathrm{d}m\biggr)
\end{aligned}
$$

$$+\frac{2a_{i*}^{1/p}}{a_{i*}-\lambda}\Big(\sum_{j=1}^{n}\overline{\alpha}_{ij}\|L_j^{f_j}\|_p+\sum_{j=1}^{n}\overline{\beta}_{ij}\mathrm{e}^{\lambda\tau_{ij}}\|L_j^{g_j}\|_p+\sum_{j=1}^{n}\overline{\gamma}_{ij}\|L_j^{h_j}\|_p\int_0^{\infty}K_{ij}(m)\mathrm{e}^{\lambda m}\,\mathrm{d}m\Big)\Big]$$

which implies that for all $i\in\{1,2,\dots,n\}$, the function y_i satisfies the estimate

$$|y_i(t_1)|\leqslant\sigma M\|\varphi\|_{\infty}\mathrm{e}^{-\lambda t_1}\frac{2a_{i*}^{1/p}}{a_{i*}-\lambda}\Big(\sum_{j=1}^{n}\overline{\alpha}_{ij}\|L_j^{f_j}\|_p+\sum_{j=1}^{n}\overline{\beta}_{ij}\mathrm{e}^{\lambda\tau_{ij}}\|L_j^{g_j}\|_p$$
$$+\sum_{j=1}^{n}\overline{\gamma}_{ij}\|L_j^{h_j}\|_p\int_0^{\infty}K_{ij}(m)\mathrm{e}^{\lambda m}\,\mathrm{d}m\Big)<\sigma M\|\varphi\|_{\infty}\mathrm{e}^{-\lambda t_1}$$

which contradicts the equality (9). Then, for any $\sigma>1$, we have

$$\|y(t)\|_{\infty}<\sigma M\|\varphi\|_{\infty}\mathrm{e}^{-\lambda t},\quad t>0.$$

If we let $\sigma\to1$, then (8) holds. Hence, the μ-pseudo almost periodic solution of system (1) is globally exponentially stable. We complete the proof. □

4. Application

Let us consider the following recurrent neural networks (RNNs):

$$\begin{aligned}x_i'(t)=&-a_i(t)x_i(t)+\sum_{j=1}^{2}\alpha_{ij}(t)f_j(t,x_j(t))+\sum_{j=1}^{2}\beta_{ij}(t)g_j(t,x_j(t-\tau_{ij}))\\&+\sum_{j=1}^{2}\gamma_{ij}(t)\int_{-\infty}^{t}K_{ij}(t-s)h_j(t,x_j(s))\,\mathrm{d}s+J_i(t),\quad 1\leqslant i\leqslant2.\end{aligned}\tag{11}$$

where $K_{ij}(t)=\mathrm{e}^{-t}$, which implies that $K_{ij}^{+}=1$, $\tau_{11}=\tau_{12}=\tau_{21}=\tau_{22}=1$ and $a_1(t)=2.6+0.1\cos(t)$, $a_2(t)=1.7+0.1\sin(t)$, then $a_{1*}=2.5$ and $a_{2*}=1.6$. For all $x\in\mathbb{R}$, $j=1,2$, we pose

$$f_j(t,x)=g_j(t,x)=h_j(t,x)=\mathrm{e}^{-|t|}\sin(x).$$

Then, we have

$$|f_j(t,x)-f_j(t,y)|\leqslant\mathrm{e}^{-|t|}|x-y|,$$
$$|g_j(t,x)-g_j(t,y)|\leqslant\mathrm{e}^{-|t|}|x-y|,$$

and

$$|h_j(t,x)-h_j(t,y)|\leqslant\mathrm{e}^{-|t|}|x-y|.$$

This gives that

$$l(t)=L_j^g(t)=L_j^h(t)=L_j^f(t)=\mathrm{e}^{-|t|}.$$

Since $\|L_j^f\|_{\mathcal{L}^2(\mathbb{R},\,\mathrm{d}x)}=\|L_j^g\|_{\mathcal{L}^2(\mathbb{R},\,\mathrm{d}x)}=\|L_j^h\|_{\mathcal{L}^2(\mathbb{R},\,\mathrm{d}x)}=1$ for all $j=1,2$, then $l\in\mathcal{L}^2(\mathbb{R},\,\mathrm{d}x)$.

Now, we consider the measure μ, where its Radon-Nikodym derivative is $\varrho(t)=\mathrm{e}^t$ for all $t\in\mathbb{R}$. Then $\mu\in\mathcal{M}$. If $\varrho(t)>0$, then from [4], the hypothesis (M1) is equivalent to

$$\forall\,\tau\in\mathbb{R}\limsup_{|t|\to\infty}\frac{\varrho(t+\tau)}{\varrho(t)}<\infty.$$

Then μ satisfies hypothesis (M1) and $l\in L^2(\mathbb{R},\,\mathrm{d}\mu)$, since

$$\int_{-\infty}^{\infty}l^2(t)\,\mathrm{d}\mu(t)=\int_{-\infty}^{\infty}\mathrm{e}^{t-2|t|}\,\mathrm{d}t=\frac{4}{3}.$$

Let

$$\begin{aligned}
(\alpha_{ij}(t))_{1\leqslant i,j\leqslant 2}&=\begin{pmatrix}0.05\cos(\sqrt{3}t)+0.01\mathrm{e}^{-t^2} & 0.01\mathrm{e}^{-t^2}\\ 0.02\sin(\sqrt{3}t) & 0.02\cos(\sqrt{3}t)+0.01\mathrm{e}^{-t^2}\end{pmatrix}\\
&\Rightarrow(\overline{\alpha}_{ij})_{1\leqslant i,j\leqslant 2}=\begin{pmatrix}0.06 & 0.01\\ 0.02 & 0.03\end{pmatrix}.\\
(\beta_{ij}(t))_{1\leqslant i,j\leqslant 2}&=\begin{pmatrix}0.05\sin(t) & 0.02\cos(t)+0.01\mathrm{e}^{-t^2}\\ 0.05\sin(t) & 0.02\cos(t)+0.01\mathrm{e}^{-t^2}\end{pmatrix}\\
&\Rightarrow(\overline{\beta}_{ij})_{1\leqslant i,j\leqslant 2}=\begin{pmatrix}0.05 & 0.03\\ 0.05 & 0.03\end{pmatrix}.\\
(\gamma_{ij}(t))_{1\leqslant i,j\leqslant 2}&=\begin{pmatrix}0.03\cos(\sqrt{3}t)+0.01\mathrm{e}^{-t^2} & 0.01\sin(\sqrt{3}t)+0.01\mathrm{e}^{-t^2}\\ 0.01\cos(t)+0.01\mathrm{e}^{-t^2} & 0.03\cos(t)\end{pmatrix}\\
&\Rightarrow(\overline{\gamma}_{ij})_{1\leqslant i,j\leqslant 2}=\begin{pmatrix}0.04 & 0.02\\ 0.02 & 0.03\end{pmatrix}.\\
(J_i(t))_{1\leqslant i\leqslant 2}&=\begin{pmatrix}0.5\cos(\sqrt{5}t)+0.1\mathrm{e}^{-t^2}\\ 0.7\cos(\sqrt{5}t)+0.1\mathrm{e}^{-t^2}\end{pmatrix}\Rightarrow(\overline{J}_i)_{1\leqslant i\leqslant 2}=\begin{pmatrix}0.6\\ 0.8\end{pmatrix}.
\end{aligned}$$

Remark 1 ([4]). Let $n\in\mathbb{N}^*$. A continuous function $f\colon\,\mathbb{R}\to\mathbb{R}^n$ satisfying

$$\lim_{|t|\to\infty}f(t)=0$$

is μ-ergodic, for all $\mu\in\mathcal{M}$.

From Remark 1, we have $[t\to\mathrm{e}^{-t^2}]\in\mathcal{E}(\mathbb{R},\mathbb{R},\mu)$, for all $\mu\in\mathcal{M}$. Since $\lim\limits_{|t|\to\infty}\mathrm{e}^{-t^2}=0$, it follows that (H3) holds. Then

$$r_0=\max_{1\leqslant i\leqslant 2}\frac{\sum_{j=1}^{2}\overline{\alpha}_{ij}+\overline{\beta}_{ij}+\overline{\gamma}_{ij}}{\sqrt{a_{i*}}}=\max\Big(\frac{21}{50\sqrt{10}},\frac{9}{20\sqrt{10}}\Big)=\frac{9}{20\sqrt{10}}<\frac{1}{2},$$

and

$$\beta = \max_{1\leqslant i\leqslant 2} \frac{\overline{J}_i}{a_{i*}} = \max\left(\frac{6}{25}, \frac{1}{2}\right) = \frac{1}{2}.$$

Since for all $\lambda_0 \in]0,1[$, we have

$$\int_0^{\infty} K_{ij}(s)\mathrm{e}^{\lambda_0 s}\,\mathrm{d}s = \int_0^{\infty} \mathrm{e}^{-(1-\lambda_0)s}\,\mathrm{d}s = 1 < \infty,$$

it follows that (H6) holds. All conditions from Theorems 6 and 7 are satisfied, so the delayed recurrent neural networks (11) have a unique μ-pseudo almost periodic solution that is globally exponentially stable (see Fig. 1) in the region

$$\mathbb{B}_1 = \left\{\varphi \in PAP(\mathbb{R}, \mathbb{R}^2, \mu),\ \|\varphi - \varphi_0\| \leqslant \frac{9}{40\sqrt{10} - 18}\right\}.$$

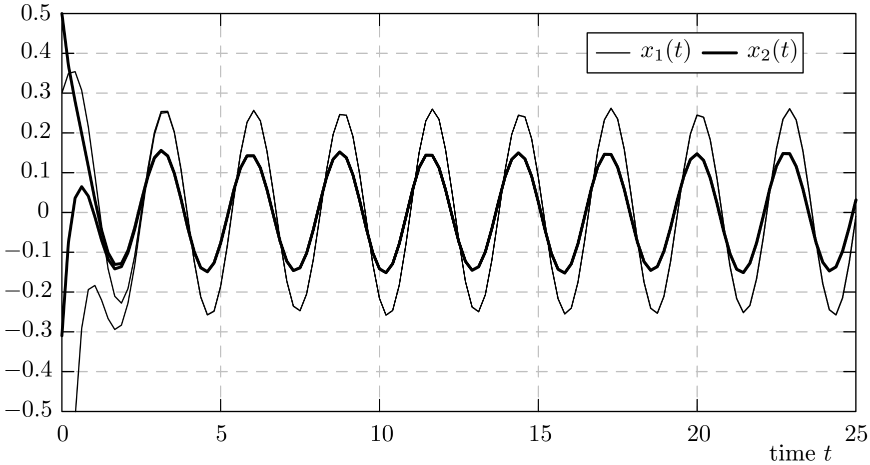

Figure 1. Curve of the μ-pseudo almost periodic solution for RNNs from the model of system (11) in the case $f_j(t,x) = \mathrm{e}^{-|t|}\sin(x)$ and a_1, a_2 periodic.

If in our application, we choose the expression of $a_1(t)$ and $a_2(t)$ as follows:

$$a_1(t) = 2.25 + \sin(t) + \cos\left(\sqrt{3}t\right) \text{ and } a_2(t) = 2.16 + \sin(t) + \cos\left(\sqrt{7}t\right),$$

then in this case we have

$$r_0 = \max_{1\leqslant i\leqslant 2} \frac{\sum_{j=1}^{2} \overline{\alpha}_{ij} + \overline{\beta}_{ij} + \overline{\gamma}_{ij}}{\sqrt{a_{i*}}} = \max\left(\frac{21}{50}, \frac{9}{20}\right) = \frac{9}{20} < \frac{1}{2}$$

and

$$\beta = \max_{1\leqslant i\leqslant 2} \frac{\overline{J}_i}{a_{i*}} = \max\left(\frac{12}{5}, 5\right) = 5.$$

Let

$$\mathbb{B}_2 = \Big\{\varphi \in PAP(\mathbb{R}, \mathbb{R}^2, \mu),\ \|\varphi - \varphi_0\| \leqslant \frac{45}{11}\Big\}.$$

Then the unique μ-pseudo almost periodic solution for the system (11) in the region $\mathbb{B}_2$, which is globally exponentially stable, admits the following two figures (see Figs. 2, 3), which were represented by making a change of $f_j(t, x)$.

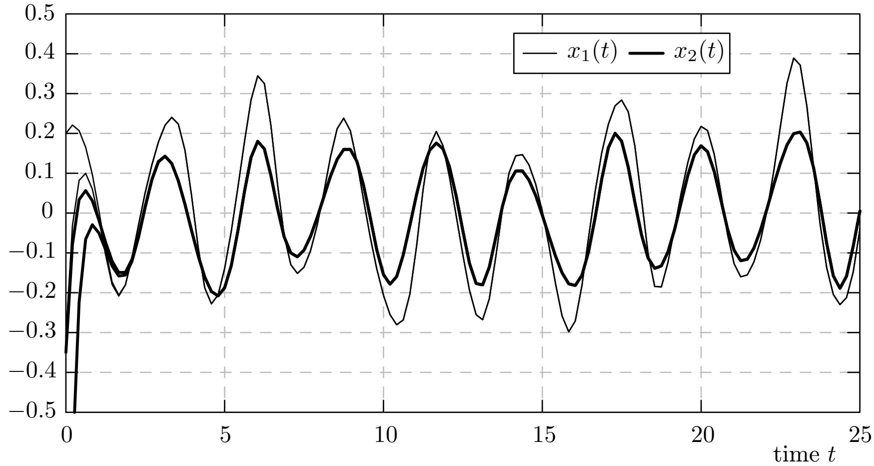

Figure 2. Curve of the μ-pseudo almost periodic solution for RNNs from the model of system (11) in the case $f_j(t, x) = \mathrm{e}^{-|t|}\sin(x)$ and a_1, a_2 almost periodic.

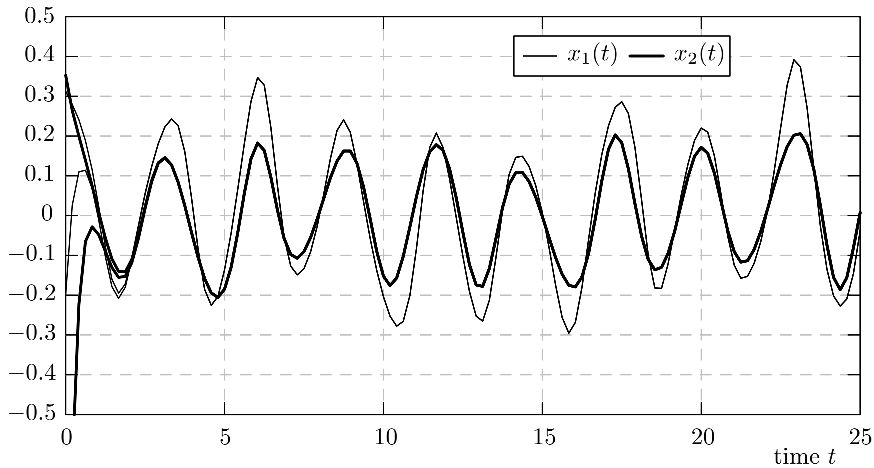

Figure 3. Curve of the μ-pseudo almost periodic solution for RNNs from the model of system (11) in the case $f_j(t, x) = \mathrm{e}^{-|t|}\tanh(x)$.

5. Conclusion

In nature, there is no phenomenon that is purely periodic, which allows one to consider the measure pseudo almost periodic oscillation. In this paper, the recurrent neural networks with mixed delays and time-varying coefficient have been studied. By employing the fixed-point theorem and some properties of the measure pseudo almost periodic functions, some sufficient conditions for the existence, uniqueness and global exponential stability of the measure pseudo almost periodic solutions have been established. To the best of our knowledge, this is the first paper to study the measure pseudo almost periodic solution for recurrent neural networks with mixed delays and time-varying coefficient. Finally, an illustrative example is given to demonstrate the effectiveness of the obtained results.

References

[1] *S. Abbas, L. Mahto, M. Hafayed, A. M. Alimi*: Asymptotic almost automorphic solutions of impulsive neural network with almost automorphic coefficients. Neurocomputing *142* (2014), 326–334. doi

[2] *B. Ammar, F. Chérif, A. M. Alimi*: Existence and uniqueness of pseudo almost-periodic solution recurrent neural networks with time varying coefficient and mixed delays. IEEE Trans. Neural Netw. Learn Syst. *23* (2012), 109–118. doi

[3] *C. Bai*: Existence and stability of almost periodic solutions of Hopfield neural networks with continuously distributed delays. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *71* (2009), 5850–5859. zbl MR doi

[4] *J. Blot, P. Cieutat, K. Ezzinbi*: New approach for weighted pseudo-almost periodic functions under the light of measure theory, basic results and applications. Appl. Anal. *92* (2013), 493–526. zbl MR doi

[5] *J. Cao, J. Liang, J. Lam*: Exponential stability of high-order bidirectional associative memory neural networks with time delays. Physica D *199* (2004), 425–436. zbl MR doi

[6] *F. Chérif*: Sufficient conditions for global stability and existence of almost automorphic solution of a class of RNNs. Differ. Equ. Dyn. Syst. *22* (2014), 191–207. zbl MR doi

[7] *F. Chérif, M. Miraoui*: New results for a Lasota-Wazewska model. Int. J. Biomath. *12* (2019), Article ID 1950019, 20 pages. zbl MR doi

[8] *L. O. Chua, L. Yang*: Cellular neural networks: Theory. IEEE Trans. Circuits Syst. *35* (1988), 1257–1272. zbl MR doi

[9] *L. O. Chua, L. Yang*: Cellular neural networks: Applications. IEEE Trans. Circuits Syst. *35* (1988), 1273–1290. MR doi

[10] *C. Corduneanu, N. Georghiu, V. Barbu*: Almost Periodic Functions. Interscience Tracts in Pure and Applied Mathematics 22, Interscience Publishers, New York, 1968. zbl MR

[11] *T. Diagana*: Weighted pseudo-almost periodic solutions to some differential equations. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *68* (2008), 2250–2260. zbl MR doi

[12] *T. Diagana, K. Ezzinbi, M. Miraoui*: Pseudo-almost periodic and pseudo-almost automorphic solutions to some evolution equations involving theoretical measure theory. Cubo *16* (2014), 1–31. zbl MR doi

[13] *L. Duan*: Existence and global exponential stability of pseudo almost periodic solutions of a general delayed BAM neural networks. J. Syst. Sci. Complex. *31* (2018), 608–620. zbl MR doi

[14] *M. Fréchet*: Sur le théorème ergodique de Birkhoff. C. R. Acad. Sci., Paris *213* (1941), 607–609. (In French.) zbl MR

[15] *H. Huang, J. Cao, J. Wang*: Global exponential stability and periodic solutions of recurrent neural networks with delays. Phys. Lett., A *298* (2002), 393–404. zbl MR doi

[16] *Y. Li, X. Meng, L. Xiong*: Pseudo almost periodic solutions for neutral type high-order Hopfield neural networks with mixed time-varying delays and leakage delays on time scales. Int. J. Mach. Learn. Cyb. *8* (2017), 1915–1927. doi

[17] *X. Meng, Y. Li*: Pseudo almost periodic solutions for quaternion-valued cellular neural networks with discrete and distributed delays. J. Inequal. Appl. *2018* (2018), Article ID 245, 17 pages. MR doi

[18] *M. S. M'hamdi, C. Aouiti, A. Touati, A. M. Alimi, V. Snasel*: Weighted pseudo almost-periodic solutions of shunting inhibitory cellular neural networks with mixed delays. Acta Math. Sci., Ser. B, Engl. Ed. *36* (2016), 1662–1682. zbl MR doi

[19] *M. Miraoui*: Existence of μ-pseudo almost periodic solutions to some evolution equations. Math. Methods Appl. Sci. *40* (2017), 4716–4726. zbl MR doi

[20] *M. Miraoui*: μ-pseudo-almost automorphic solutions for some differential equations with reflection of the argument. Numer. Funct. Anal. Optim. *38* (2017), 376–394. zbl MR doi

[21] *M. Miraoui, N. Yaakoubi*: Measure pseudo almost periodic solutions of shunting inhibitory cellular neural networks with mixed delays. Numer. Funct. Anal. Optim. *40* (2019), 571–585. zbl MR doi

[22] *G. M. N'Guérékata*: Almost Automorphic and Almost Periodic Functions in Abstract Spaces. Kluwer Academic/Plenum Publishers, New York, 2001. zbl MR doi

[23] *F. Qiu, B. Cui, W. Wu*: Global exponential stability of high order recurrent neural network with time-varying delays. Appl. Math. Modelling *33* (2009), 198–210. zbl MR doi

[24] *T. Roska, L. O. Chua*: Cellular neural networks with non-linear and delay-type template elements and non-uniform grids. Int. J. Circuit Theory Appl. *20* (1992), 469–481. zbl doi

[25] *H. Xiang, J. Cao*: Almost periodic solutions of recurrent neural networks with continuously distributed delays. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *71* (2009), 6097–6108. zbl MR doi

[26] *Y. Yu, M. Cai*: Existence and exponential stability of almost-periodic solutions for high-order Hopfield neural networks. Math. Comput. Modelling *47* (2008), 943–951. zbl MR doi

[27] *C. Zhang*: Pseudo-almost-periodic solutions of some differential equations. J. Math. Anal. Appl. *181* (1994), 62–76. zbl MR doi

[28] *C. Zhang*: Pseudo almost periodic solutions of some differential equations. II. J. Math. Anal. Appl. *192* (1995), 543–561. zbl MR doi

[29] *H. Zhao*: Existence and global attractivity of almost periodic solution for cellular neural network with distributed delays. Appl. Math. Comput. *154* (2004), 683–695. zbl MR doi

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