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NEW ESTIMATES FOR THE FIRST EIGENVALUE OF THE JACOBI OPERATOR ON CLOSED HYPERSURFACES IN RIEMANNIAN SPACE FORMS

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Abstract. We study the first eigenvalue of the Jacobi operator on closed hypersurfaces with constant mean curvature in non-flat Riemannian space forms. Under an appropriate constraint on the totally umbilical tensor of the hypersurfaces and following Meléndez's ideas in J. Meléndez (2014) we obtain a new sharp upper bound of the first eigenvalue of the Jacobi operator.

Keywords: Jacobi operator; first eigenvalue; closed hypersurface

MSC 2020: 53C50

1. Introduction

Let us denote by $\mathbb{Q}^{n+1}(c)$ the standard model of an $(n+1)$-dimensional Riemannian space form with constant sectional curvature c, $c \in \{1, -1\}$. This is the Euclidean sphere $\mathbb{S}^{n+1}$ when $c = 1$ and the hyperbolic space $\mathbb{H}^{n+1}$ when $c = -1$.

The problem of characterizing hypersurfaces immersed in the Riemannian space form $\mathbb{Q}^{n+1}(c)$ with constant mean curvature constitutes a classical topic in the theory of isometric immersions, which was widely approached by many authors. In this direction, the investigation concerning the behavior of the spectrum of the Schrödinger operators (of the form $-\Delta + q$ with Δ being the Laplacian operator on a Riemannian manifold M^n and q a continuous function on M^n) constitutes an interesting and fruitful research topic in geometric analysis, see [7] and [17].

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In the case that M^n is a closed hypersurface immersed in the Riemannian space form $\mathbb{Q}^{n+1}(c)$ with constant mean curvature, an important Schrödinger operator is the so-called Jacobi operator or stability operator which is defined by

$$J = -\Delta - S - nc,$$

where S denotes the squared norm of the second fundamental form of M^n. We note that J is just the Jacobi operator established by Alías in [2] in order to study the problem of minimizing the area functional for volume-preserving variations.

From the mathematical point of view, this is mostly due to the fact that such hypersurfaces exhibit nice gap theorems. For example, Simons in [22] studied the first eigenvalue λ_1^J of a minimal closed hypersurface M^n immersed in $\mathbb{S}^{n+1}$ and proved that either $\lambda_1^J = -n$ and M^n is a totally geodesic sphere, or $\lambda_1^J \leqslant -2n$ otherwise. Furthermore, Wu in [23] characterized the equality $\lambda_1^J = -2n$ by showing that it holds only for the minimal Clifford torus of the form $\mathbb{S}^k(\sqrt{k/n}) \times \mathbb{S}^{n-k}(\sqrt{(n-k)/n})$ with $k \in \{1, \dots, n-1\}$. On the other hand, Perdomo in [21] provided a new proof of this spectral characterization by the value of λ_1^J.

Later on, Alías, Barros and Brasil Jr. in [3] extended these results to the case of constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$, characterizing some Clifford torus of the form $\mathbb{S}^1(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ with $r \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$ with $r \in (0, (n-1)/n)$ via the value of their first stability eigenvalue λ_1^J. Shortly thereafter, Chen and Cheng in [9] obtained the upper bound for λ_1^J of nontotally umbilical compact hypersurfaces with constant mean curvature, which depends only on the mean curvature H and the dimension n. We also refer the readers to the recent article [6], in which the authors Aquino, de Lima, dos Santos and Velásquez obtained upper bounds for λ_1^J of a closed hypersurfaces with constant mean curvature immersed either in the Euclidean space $\mathbb{R}^{n+1}$ or in the hyperbolic space $\mathbb{H}^{n+1}$ that are either $\lambda_1^J = -n(H^2 + c)$ and M^n is totally umbilical, or $\lambda_1^J < -2n(H^2 + c) + nC(n, 1)|H| \max |\varphi|$.

Very recently, the authors of [16] offered a comprehensive and nice presentation of a new upper bound for the first eigenvalue λ_1^J on a closed constant mean curvature hypersurface in a Riemannian space form. We observe that the upper bound of the first eigenvalue λ_1^J does not depend only on the mean curvature H and the dimension n, but also depends on the immersion.

In this paper, under constraints on the total umbilicity tensor of M^n and following the approach introduced in [19], our objective is to extend de Lima's results in [16] to the case of nonzero constant mean curvature hypersurfaces in a Riemannian space form $\mathbb{Q}^{n+1}(c)$.

Theorem 1.1. *Let M^n $(n \geqslant 5)$ be a closed hypersurface in $\mathbb{S}^{n+1}$ with nonzero constant mean curvature H. Let λ_1^J stand for the first eigenvalue of the Jacobi operator J. Let its total umbilicity operator φ satisfy*

$$|\mathrm{tr}(\varphi^3)| \leqslant C(n,k)|\varphi|^3, \tag{1.1}$$

where $C(n,k) = (n-2k)/\sqrt{nk(n-k)}$ for a given integer k. When M^n is totally umbilical, then $\lambda_1^J = -n(H^2+1)$. When M^n is not totally umbilical.

(a) *If $n^2H^2 < 16k(n-k)/((n-2k)^2-4)$ and $k < \frac{1}{2}(n-2)$, then*

$$\lambda_1^J \leqslant -n(H^2+1) - \frac{n}{4k(n-k)}\big(\sqrt{4k(n-k)+n^2H^2} - (n-2k)|H|\big)^2.$$

Moreover, the equality holds if and only if M^n is a product of spheres $\mathbb{S}^{n-k}(r) \times \mathbb{S}^k(\sqrt{1-r^2})$ with $r^2 < 1-k/n$.

(b) *If $n^2H^2 \geqslant 16k(n-k)/((n-2k)^2-4)$ and $k < \frac{1}{2}(n-2)$, then*

$$\lambda_1^J \leqslant -2(n-1)(H^2+1) + \frac{(n-2)^2(n-2k)^2H^2}{8k(n-k)}.$$

Moreover, the equality holds if and only if M^n is a product of spheres $\mathbb{S}^{n-k}(r) \times \mathbb{S}^k(\sqrt{1-r^2})$ with $r^2 < 1-k/n$.

Theorem 1.2. *Let M^n $(n \geqslant 5)$ be a closed hypersurface in a hyperbolic space $\mathbb{H}^{n+1}(-1)$ with nonzero constant mean curvature H. Let λ_1^J stand for the first eigenvalue of the Jacobi operator J. Let its total umbilicity operator φ satisfy*

$$|\mathrm{tr}(\varphi^3)| \leqslant C(n,k)|\varphi|^3,$$

where $C(n,k) = (n-2k)/\sqrt{nk(n-k)}$ for a given integer k. When M^n is totally umbilical, then $\lambda_1^J = -n(H^2-1)$. When M^n is not totally umbilical, then:

(a) *If $4k(n-k) < n^2H^2 < 16k(n-k)/(4-(n-2k)^2)$ and $k > \frac{1}{2}(n-2)$, then*

$$\lambda_1^J \leqslant -n(H^2-1) - \frac{n}{4k(n-k)}\big(\sqrt{n^2H^2-4k(n-k)} - (n-2k)|H|\big)^2.$$

(b) *If $n^2H^2 \geqslant 16k(n-k)/(4-(n-2k)^2)$ and $k > \frac{1}{2}(n-2)$, then*

$$\lambda_1^J \leqslant -2(n-1)(H^2-1) + \frac{(n-2)^2(n-2k)^2H^2}{8k(n-k)}.$$

Remark 1.1. Theorem 1.1 above is a gap theorem that extends well-known results on minimal hypersurfaces in the Euclidean sphere to the case of nonzero constant mean curvature, see [14], [18], [22]. In particular, for $k = 1$, it follows from the classical Okumura type condition (see [20]) that (1.1) holds automatically, so Theorem 1.1 contains Chen and Cheng's theorem in [9] as its special case.

2. Preliminaries

Let M^n be an n-dimensional connected hypersurface immersed in the Riemannian space form $\mathbb{Q}^{n+1}(c)$, $c \in \{1, -1\}$. We choose a local orthonormal frame field $\{e_1, \ldots, e_n, e_{n+1}\}$ and its dual coframe $\{\omega_1, \ldots, \omega_n, \omega_{n+1}\}$ such that $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M^n. Hence, the second fundamental form II and the mean curvature H of M^n are defined by

$$\mathrm{II} = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j \otimes e_{n+1}, \quad H = \frac{1}{n}\sum_i h_{ii},$$

respectively. As is well-known, the Gauss equation of M^n is given by

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}).$$

Denoting by $S = \sum\limits_{i,j} h_{ij}^2$ the squared norm of the second fundamental form and by R the normalized scalar curvature of M^n, we have, from the Gauss equation, the well-known relation

$$n(n-1)R = n(n-1)c + n^2H^2 - S.$$

The Codazzi equation and Ricci identity on M^n are given by

$$h_{ijk} = h_{ikj},$$
$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{mi}R_{mjkl},$$

respectively. For any C^2-function f on M^n, we define its gradient and Hessian by

$$\mathrm{d}f = \sum_i f_i\omega_i, \quad \sum_j f_{ij}\omega_j = \mathrm{d}f_i + \sum_j f_j\omega_{ji}.$$

Then the Laplace-Beltrami operator Δ acting on f is given by $\Delta f = \sum\limits_i f_{ii}$. Taking a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n such that $h_{ij} = \lambda_i\delta_{ij}$ and using the Gauss equation, Codazzi equation and Ricci identity, the Simons type formula, see [10], [11] or [22]

$$\frac{1}{2}\Delta S=\sum_{i,j,k}h_{ijk}^2+\sum_{i,j}\lambda_i(nH)_{ii}+ncS-cn^2H^2-S^2+nH\sum_i\lambda_i^3 \tag{2.1}$$

follows. Set $\varphi_{ij}=h_{ij}-H\delta_{ij}$, then the symmetric tensor $\varphi=\sum\limits_{ij}\varphi_{ij}\omega_i\omega_j$ is called the total umbilicity (or traceless) tensor of M^n and satisfies the Codazzi equation. Let $|\varphi|^2=\sum\limits_{i,j}\varphi_{ij}^2$ be the squared length of φ, then it is easy to check that

$$|\varphi|^2=S-nH^2. \tag{2.2}$$

In order to prove our main theorems, we need the following two lemmas.

Lemma 2.1 ([19], [20]). *Let μ_i be real numbers such that $\sum\limits_i\mu_i=0$ and $\sum\limits_i\mu_i^2=\beta^2$, where $\beta\geqslant 0$. Then*

$$\sum_i\mu_i^3=\frac{n-2k}{\sqrt{nk(n-k)}}\beta^3\quad\Bigl(\sum_i\mu_i^3=-\frac{n-2k}{\sqrt{nk(n-k)}}\beta^3\Bigr)$$

holds if and only if k of the μ_i's are nonnegative (or nonpositive) and equal and the rest $n-k$ of the μ_i's are nonpositive (or nonnegative) and equal.

Lemma 2.2 ([15]). *Let M^n be a Riemannian manifold isometrically imbedded into a Riemannian manifold $\mathbb{N}^{n+p}$. Consider a traceless symmetric tensor*

$$\psi=\sum_{\alpha,i,j}\psi_{ij}^\alpha\omega_i\otimes\omega_j\otimes e_\alpha$$

satisfying the Codazzi equation. Then the inequality

$$|\nabla|\psi|^2|^2\leqslant\frac{4n}{n+2}|\nabla\psi|^2|\psi|^2$$

holds, where $|\psi|^2=\sum\limits_{\alpha,i,j}(\psi_{ij}^\alpha)^2$ and $|\nabla\psi|^2=\sum\limits_{\alpha,i,j,k}(\psi_{ijk}^\alpha)^2$.

3. Proofs of Theorems 1.1 and 1.2

P r o o f. From now we assume $H\neq 0$. Since H is constant, we can assume $H>0$. It follows from Simons' formula (2.1) together with (1.1) and (2.2) that

$$\frac{1}{2}\Delta|\varphi|^2\geqslant|\nabla\varphi|^2-|\varphi|^2\Bigl(|\varphi|^2-n(c+H^2)+\frac{nH(n-2k)|\varphi|}{\sqrt{nk(n-k)}}\Bigr). \tag{3.1}$$

When M^n is totally umbilical, then $|\varphi|^2 = 0$ and $J = -\Delta - n(c + H^2)$, so $\lambda_1^J = \lambda_1^{-\Delta} - n(c + H^2) = -n(c + H^2)$, whose corresponding eigenfunctions are constant functions.

In the following, we assume that M^n is not totally umbilical. For this case, following the ideas of [9] or [19], consider the positive smooth test function $f_\varepsilon \in C^\infty(M)$ given by

$$f_\varepsilon = (\varepsilon + |\varphi|^2)^\alpha > 0$$

for any arbitrary $\alpha > 0$ and $\varepsilon > 0$. By a straightforward computation, we obtain

$$\Delta f_\varepsilon = \alpha(\alpha - 1)\left(\varepsilon + |\varphi|^2\right)^{\alpha-2} |\nabla|\varphi|^2|^2 + \alpha\left(\varepsilon + |\varphi|^2\right)^{\alpha-1} \Delta|\varphi|^2. \tag{3.2}$$

Thus, it follows from (3.1) and (3.2) that

$$\begin{aligned} f_\varepsilon \Delta f_\varepsilon \geqslant{}& \alpha(\alpha - 1)\left(\varepsilon + |\varphi|^2\right)^{2\alpha-2} |\nabla|\varphi|^2|^2 + 2\alpha\left(\varepsilon + |\varphi|^2\right)^{2\alpha-2} \\ &\times (\varepsilon + |\varphi|^2)|\nabla\varphi|^2 - 2\alpha(\varepsilon + |\varphi|^2)^{2\alpha-2}(\varepsilon + |\varphi|^2)|\varphi|^2 \\ &\times \left(|\varphi|^2 - n(c + H^2) + \frac{nH(n-2k)|\varphi|}{\sqrt{nk(n-k)}}\right). \end{aligned} \tag{3.3}$$

Applying Lemma 2.2 to φ, we have $|\nabla|\varphi|^2|^2 \leqslant (4n/(n+2))|\nabla\varphi|^2|\varphi|^2$. Therefore,

$$\begin{aligned} &\alpha(\alpha - 1)(\varepsilon + |\varphi|^2)^{2\alpha-2}|\nabla|\varphi|^2|^2 + 2\alpha(\varepsilon + |\varphi|^2)^{2\alpha-2}(\varepsilon + |\varphi|^2)|\nabla\varphi|^2 \\ &\geqslant -\frac{4n\alpha(1-\alpha)}{n+2}(\varepsilon + |\varphi|^2)^{2\alpha-2}|\varphi|^2|\nabla\varphi|^2 + 2\alpha(\varepsilon + |\varphi|^2)^{2\alpha-2}(\varepsilon + |\varphi|^2)|\nabla\varphi|^2 \\ &= \alpha(\varepsilon + |\varphi|^2)^{2\alpha-2}\left(2 - \frac{4n(1-\alpha)}{n+2}\right)|\varphi|^2|\nabla\varphi|^2. \end{aligned} \tag{3.4}$$

Plugging (3.4) into (3.3), we have

$$\begin{aligned} f_\varepsilon \Delta f_\varepsilon \geqslant{}& -2\alpha(\varepsilon + |\varphi|^2)^{2\alpha-2}(\varepsilon + |\varphi|^2)|\varphi|^2 \\ &\times \left(|\varphi|^2 - n(c + H^2) + \frac{nH(n-2k)|\varphi|}{\sqrt{nk(n-k)}}\right) + \alpha(\varepsilon + |\varphi|^2)^{2\alpha-2} \\ &\times \left(2 - \frac{4n(1-\alpha)}{n+2}\right)|\varphi|^2|\nabla\varphi|^2. \end{aligned} \tag{3.5}$$

Now, we recall that λ_1^J has the min-max characterization, see [8], [13]

$$\lambda_1^J = \min\left\{\frac{\int_{M^n} fJf\,\mathrm{d}M}{\int_{M^n} f^2\,\mathrm{d}M} :\ f \in C^\infty(M^n),\ f \neq 0\right\},$$

where $\mathrm{d}M$ stands for the volume element with respect to the induced metric of M^n. Hence, from (3.5), we infer

$$\begin{aligned}
(3.6)\quad \lambda_1^J \int_{M^n} f_\varepsilon^2 \,\mathrm{d}M &\leqslant \int_{M^n} f_\varepsilon J(f_\varepsilon)\,\mathrm{d}M \\
&= -\int_{M^n} f_\varepsilon \Delta f_\varepsilon \,\mathrm{d}M - \int_{M^n} (S+nc) f_\varepsilon^2 \,\mathrm{d}M \\
&\leqslant \int_{M^n} \frac{f_\varepsilon^2 |\varphi|^2}{\varepsilon + |\varphi|^2}\biggl((2\alpha-1)|\varphi|^2 + \frac{2\alpha n(n-2k)H|\varphi|}{\sqrt{nk(n-k)}} - \varepsilon\biggr)\,\mathrm{d}M \\
&\quad - 2\alpha n(H^2+c)\int_{M^n} \frac{f_\varepsilon^2|\varphi|^2}{\varepsilon+|\varphi|^2}\,\mathrm{d}M - n(H^2+c)\int_{M^n} f_\varepsilon^2\,\mathrm{d}M \\
&\quad - \int_{M^n} \alpha(\varepsilon+|\varphi|^2)^{-2}\biggl(2 - \frac{4n(1-\alpha)}{n+2}\biggr)|\varphi|^2|\nabla\varphi|^2 f_\varepsilon^2\,\mathrm{d}M.
\end{aligned}$$

Assuming that $2\alpha - 1 < 0$ and using the inequality $-a^2 + 2ab \leqslant b^2$, we have

$$(3.7)\qquad (2\alpha-1)|\varphi|^2 + \frac{2\alpha n(n-2k)H|\varphi|}{\sqrt{nk(n-k)}} \leqslant \frac{\alpha^2(n-2k)^2(nH)^2}{(1-2\alpha)nk(n-k)}.$$

In view of (3.7), (3.6) then becomes

$$\begin{aligned}
(3.8)\quad \lambda_1^J \int_{M^n} f_\varepsilon^2 \,\mathrm{d}M &\leqslant \int_{M^n} \frac{f_\varepsilon^2|\varphi|^2}{\varepsilon+|\varphi|^2}\biggl(\frac{\alpha^2(n-2k)^2}{(1-2\alpha)nk(n-k)}(nH)^2 - \varepsilon\biggr)\,\mathrm{d}M \\
&\quad - 2n\alpha(H^2+c)\int_{M^n} \frac{f_\varepsilon^2|\varphi|^2}{\varepsilon+|\varphi|^2}\,\mathrm{d}M - n(H^2+c)\int_{M^n} f_\varepsilon^2\,\mathrm{d}M \\
&\quad - \alpha\biggl(2 - \frac{4n(1-\alpha)}{n+2}\biggr)\int_{M^n} \frac{|\varphi|^2|\nabla\varphi|^2}{(\varepsilon+|\varphi|^2)^2} f_\varepsilon^2\,\mathrm{d}M.
\end{aligned}$$

Putting

$$(3.9)\qquad \alpha = \begin{cases} \dfrac{1}{2}\biggl(1 - \sqrt{\dfrac{(n-2k)^2H^2}{n^2H^2 + 4k(n-k)c}}\biggr), & \text{if } n^2H^2 < \dfrac{16k(n-k)c}{(n-2k)^2-4}, \\[2ex] \dfrac{n-2}{2n}, & \text{if } n^2H^2 \geqslant \dfrac{16k(n-k)c}{(n-2k)^2-4}. \end{cases}$$

Then $2\alpha - 1 < 0$ and $2 - 4n(1-\alpha)/(n+2) \geqslant 0$. We get from (3.8) that

$$\begin{aligned}
(3.10)\quad \lambda_1^J \int_{M^n} f_\varepsilon^2 \,\mathrm{d}M &\leqslant \int_{M^n} \frac{f_\varepsilon^2|\varphi|^2}{\varepsilon+|\varphi|^2}\biggl(\frac{\alpha^2(n-2k)^2}{(1-2\alpha)nk(n-k)}(nH)^2 - \varepsilon\biggr)\,\mathrm{d}M \\
&\quad - 2n\alpha(H^2+c)\int_{M^n} \frac{f_\varepsilon^2|\varphi|^2}{\varepsilon+|\varphi|^2}\,\mathrm{d}M - n(H^2+c)\int_{M^n} f_\varepsilon^2\,\mathrm{d}M.
\end{aligned}$$

As M^n is not total umbilicity, it follows that

$$\lim_{\varepsilon\to 0}\int_{M^n} f_\varepsilon^2\,\mathrm{d}M = \int_{M^n} |\varphi|^{4\alpha}\,\mathrm{d}M > 0. \tag{3.11}$$

Hence, letting $\varepsilon \to 0$ in (3.10) and using (3.11), we have

$$\lambda_1^J \leqslant \frac{\alpha^2(n-2k)^2}{(1-2\alpha)nk(n-k)}(nH)^2 - 2n\alpha(H^2+c) - n(H^2+c). \tag{3.12}$$

Substituting the value of α defined by (3.9) into (3.12), we arrive at:

(a) When $c = 1$ and $k < \frac{1}{2}(n-2)$.

▷ If $n^2H^2 < 16k(n-k)/((n-2k)^2-4)$, then

$$\lambda_1^J \leqslant -n(H^2+1) - \frac{n}{4k(n-k)}\big(\sqrt{4k(n-k)+n^2H^2} - (n-2k)|H|\big)^2. \tag{3.13}$$

▷ If $n^2H^2 \geqslant 16k(n-k)/((n-2k)^2-4)$, then

$$\lambda_1^J \leqslant -2(n-1)(H^2+1) + \frac{(n-2)^2(n-2k)^2H^2}{8k(n-k)}. \tag{3.14}$$

(b) When $c = -1$ and $k > \frac{1}{2}(n-2)$.

▷ If $4k(n-k) < n^2H^2 < 16k(n-k)/(4-(n-2k)^2)$, then

$$\lambda_1^J \leqslant -n(H^2-1) - \frac{n}{4k(n-k)}\big(\sqrt{n^2H^2-4k(n-k)} - (n-2k)|H|\big)^2.$$

▷ If $n^2H^2 \geqslant 16k(n-k)/(4-(n-2k)^2)$, then

$$\lambda_1^J \leqslant -2(n-1)(H^2-1) + \frac{(n-2)^2(n-2k)^2H^2}{8k(n-k)}.$$

Now, let us suppose that the equality in (3.13) holds. Then, all the inequalities along this proof must be equalities. Hence, we know that the second fundamental form is parallel and S is constant. Using Lemma 2.1 once more we obtain that M^n has exactly two constant principal curvatures with multiplicities $(n-k)$ and k. Then, by the classical results on isoparametric hypersurfaces of Riemannian space forms (see [1], [4], [5] or [12]), we know that when $c = 1$, M^n is a product of spheres $\mathbb{S}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1-r^2})$ with $r^2 < 1 - k/n$.

If the equality in (3.14) holds, then all inequalities along this proof must be equalities. In particular, the equality occurs in (3.7) which implies

$$|\varphi| = \frac{\alpha(n-2k)nH}{(1-2\alpha)\sqrt{nk(n-k)}}. \tag{3.15}$$

Inserting $\alpha = (n-2)/2n$ into (3.15) we get

$$|\varphi|^2 = \frac{(n-2)^2(n-2k)^2(nH)^2}{16nk(n-k)}.$$

Furthermore, from $J = -\Delta - (|\varphi|^2 + n(H^2+c))$ it follows that

$$\lambda_1^{-\Delta} - (|\varphi|^2 + n(H^2+c)) = -2(n-1)(H^2+c) + \frac{(n-2)^2(n-2k)^2H^2}{8k(n-k)}.$$

Thus, since M^n is closed, we obtain

$$0 = \lambda_1^{-\Delta} = (|\varphi|^2 + n(H^2+c)) - 2(n-1)(H^2+c) + \frac{(n-2)^2(n-2k)^2H^2}{8k(n-k)}$$

and, hence,

$$n^2H^2 = \frac{16k(n-k)c}{(n-2k)^2-4}.$$

From [9], we know that when $c=1$, M^n is a product of spheres $\mathbb{S}^{n-k}(r)\times\mathbb{S}^k(\sqrt{1-r^2})$ with $r^2 < 1-k/n$. This finishes the proofs of Theorems 1.1 and 1.2. □

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