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COMPLEX SYMMETRIC WEIGHTED COMPOSITION OPERATORS ON THE HARDY SPACE

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Abstract. This paper identifies a class of complex symmetric weighted composition operators on $H^2(\mathbb{D})$ that includes both the unitary and the Hermitian weighted composition operators, as well as a class of normal weighted composition operators identified by Bourdon and Narayan. A characterization of algebraic weighted composition operators with degree no more than two is provided to illustrate that the weight function of a complex symmetric weighted composition operator is not necessarily linear fractional.

Keywords: complex symmetry; weighted composition operator; Hardy space

MSC 2020: 47B33, 47B38

1. Preliminaries

1.1. Complex symmetry. Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{L}(\mathcal{H})$ the collection of all continuous linear operators on $\mathcal{H}$. A map $\mathcal{C}\colon \mathcal{H} \to \mathcal{H}$ is called a *conjugation* over $\mathcal{H}$ if it is

- ▷ anti-linear: $\mathcal{C}(ax+by) = \bar{a}\mathcal{C}(x) + \bar{b}\mathcal{C}(y)$, $x, y \in \mathcal{H}$, $a, b \in \mathbb{C}$;
- ▷ isometric: $\|\mathcal{C}x\| = \|x\|$, $x \in \mathcal{H}$;
- ▷ involutive: $\mathcal{C}^2 = I$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *complex symmetric* if

$$\mathcal{C}T = T^*\mathcal{C}$$

for some conjugation $\mathcal{C}$ and in this case we say T is *complex symmetric with conjugation $\mathcal{C}$*.

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The general study of complex symmetric operators was started by Garcia, Putinar and Wogen in [7], [8], [9], [10]. The class of complex symmetric operators turns to be quite diverse, see [7], [8], [10]. In this paper, we focus on two classes: normal operators and algebraic operators with degree no more than two. The spectral theorem states that if a normal operator is unitarily equivalent to some multiplier $M_\phi\colon L^2(X, \mathrm{d}v) \to L^2(X, \mathrm{d}v)$, then it is easy to check that M_ϕ is complex symmetric with the usual conjugation $Cf(z) = \overline{f(z)}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *algebraic* if it is annihilated by some nonzero polynomial p and the minimal degree of p is called the *degree* of T. Garcia and Wogen in [10] proved that an algebraic operator with degree no more than two is complex symmetric.

1.2. Hardy space. The classical Hardy-Hilbert space is defined by

$$H^2(\mathbb{D}) = \left\{ f \in H(\mathbb{D});\ \|f\|^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^2 \,\mathrm{d}\theta < \infty \right\},$$

where $\mathbb{D}$ denotes the open unit disk in the complex plane. The space $H^2(\mathbb{D})$ is a reproducing kernel Hilbert space (RKHS); i.e., for any point $w \in \mathbb{D}$ there exists a unique function $K_w \in H^2(\mathbb{D})$ such that

$$f(w) = \langle f, K_w \rangle,$$

where $K_w = 1/(1 - \overline{w}z)$ is called the *reproducing kernel* at w.

1.3. Weighted composition operator. Let $H(\mathbb{D})$ denote the set of all holomorphic functions over the unit disk and $S(\mathbb{D})$ the set of all holomorphic selfmaps of the unit disk. For any $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, the associated *weighted composition operator* is defined by

$$W_{\psi,\varphi}(f) = \psi \cdot (f \circ \varphi).$$

When $u \equiv 1$, we write $W_{1,\varphi} = C_\varphi$ which is called a *composition operator*. The study of composition operators and weighted composition operators over various holomorphic function spaces has undergone a rapid development over the past four decades. For general background, see [4] and [14].

Composition operators on $H^2(\mathbb{D})$ are relatively well understood. For example, every normal composition operator C_φ on $H^2(\mathbb{D})$ must be induced by a dilation $\varphi(z) = az$, $|a| \leqslant 1$. However, the case for weighted composition operators is far more complicated. There exist many nontrivial normal weighted composition operators, and even nontrivial Hermitian weighted composition operators, on $H^2(\mathbb{D})$, see [1], [3]. We will discuss such examples later.

The study of complex symmetric weighted composition operators on $H^2(\mathbb{D})$ was initiated independently by Garcia and Hammond in [6] and Jung et al. in [11]. Their idea is to consider the conjugation

$$\mathcal{J}\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} \bar{a}_n z^n$$

and to characterize when $W_{\psi,\varphi}$ is complex symmetric with conjugation $\mathcal{J}$. Their main result is the following theorem:

Theorem A. *Let $\varphi \in S(\mathbb{D})$ and $\psi \in H^\infty(\mathbb{D})$. Then $W_{\psi,\varphi}$ is complex symmetric with conjugation $\mathcal{J}$ if and only if*

$$\psi(z) = \frac{b}{1-a_0 z} \quad \textit{and} \quad \varphi(z) = a_0 + \frac{a_1 z}{1-a_0 z}.$$

By making unitary transformations, Jung et al. in [11] provided some more examples. For more work on complex symmetric composition operators, see [2], [5], [13], [15], [16].

In this paper, we will make a slight generalization of Jung's result. As a corollary, we will give the conjugation with which a unitary weighted composition operator or a Hermitian weighted composition operator is complex symmetry. It is a surprise to us that the normal subclass of our complex symmetric weighted composition operators coincides with that identified by Bourdon and Narayan, see [1]. We do not know whether this is an accident or some hint that there may be no more normal weighted composition operators. It is still an open question to find a complete characterization of normal weight composition operators on $H^2(\mathbb{D})$. Besides, we will also try to characterize the algebraic weighted composition operators of degree no more than two, since by doing so we will show that the weight function of a complex symmetric weighted composition operator is not necessarily linear fractional.

2. Main results

2.1. The conjugation. The main difficulty in constructing complex symmetric weighted composition operators is to find a conjugation which is amenable to calculation. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two conjugations, then $\mathcal{C}_1\mathcal{C}_2$ is a unitary operator. So any two conjugations are related by some unitary operator, i.e. $\mathcal{C}_2 = \mathcal{C}_1(\mathcal{C}_1\mathcal{C}_2)$. Following this idea, we would get a new conjugation if we composite a unitary operator with the conjugation $\mathcal{J}$. Our choice of the unitary operator is a unitary weighted composition operator of the following lemma.

Lemma 1 ([1], Theorem 6). *A weighted composition operator $W_{\psi,\varphi}$ is unitary on $H^2(\mathbb{D})$ if and only if φ is an automorphism of the unit disk and*

$$\psi = \mu\frac{K_p}{\|K_p\|} = \mu\frac{\sqrt{1-|p|^2}}{1-\bar{p}z},$$

where $p = \varphi^{-1}(0)$ and $|\mu| = 1$.

Lemma 2. *Suppose that*

$$\sigma(z) = \lambda\frac{p-z}{1-\bar{p}z} \in \mathrm{Aut}(\mathbb{D}) \quad \textit{and} \quad k_p(z) = \frac{\sqrt{1-|p|^2}}{1-\bar{p}z},$$

where $p \in \mathbb{D}$ and $|\lambda| = 1$. Then $\mathcal{J}W_{k_p,\sigma}$ defines a conjugation on $H^2(\mathbb{D})$ if and only if $\bar{p} = \lambda p$.

Proof. By Lemma 1, $\mathcal{J}W_{k_p,\sigma}$ is anti-linear and isometric, so it remains to analyze when $\mathcal{J}W_{k_p,\sigma}$ is involutive.

For any $f \in H^2(\mathbb{D})$, we have

$$\mathcal{J}W_{k_p,\sigma}f(z) = \mathcal{J}(k_p(z)f(\sigma(z))) = \overline{k_p(\overline{z})f(\sigma(\overline{z}))},$$

and then

$$(\mathcal{J}W_{k_p,\sigma})^2 f(z) = \mathcal{J}W_{k_p,\sigma}(\overline{k_p(\overline{z})f(\sigma(\overline{z}))}) = \overline{k_p(\overline{z})}k_p(\overline{\sigma(\overline{z})})f(\sigma(\overline{\sigma(\overline{z})})).$$

If $\mathcal{J}W_{k_p,\sigma}$ is involutive, then by taking $f \equiv 1$ we have

$$1 = \overline{k_p(\overline{z})}k_p(\overline{\sigma(\overline{z})}) = \frac{\sqrt{1-|p|^2}}{1-pz}\,\frac{\sqrt{1-|p|^2}}{1-\bar{p}\bar{\lambda}(\bar{p}-z)/(1-pz)} = \frac{1-|p|^2}{1-\overline{\lambda p^2}-(p-\overline{p\lambda})z},$$

which implies $\lambda p = \bar{p}$.

Conversely, if we assume that $\lambda p = \bar{p}$, then

$$\overline{k_p(\overline{z})}k_p(\overline{\sigma(\overline{z})}) = 1,$$

and

$$\begin{aligned}\sigma(\overline{\sigma(\overline{z})}) &= \lambda\frac{p-\bar{\lambda}(\bar{p}-z)/(1-pz)}{1-\bar{p}\bar{\lambda}(\bar{p}-z)/(1-pz)} = \frac{\lambda p-\lambda p^2 z-\bar{p}+z}{1-pz-\bar{\lambda}\bar{p}^2+\bar{\lambda}\bar{p}z}\\ &= \frac{z-|p|^2 z}{1-|p|^2} = z.\end{aligned}$$

So $(\mathcal{J}W_{k_p,\sigma})^2 = I$; that is, $\mathcal{J}W_{k_p,\sigma}$ is a conjugation. □

Remark 1. Throughout the paper, the symbol $\mathcal{J}W_{k_p,\sigma}$ is always referred to the conjugation constructed above; that is, $\lambda = \bar{p}/p$ if $p \neq 0$ and λ is any unimodular number if $p = 0$.

2.2. Complex symmetry of $W_{\psi,\varphi}$. Now we will investigate when a weighted composition operator $W_{\psi,\varphi}$ is complex symmetric with conjugation $\mathcal{J}W_{k_p,\sigma}$. Having the result of Jung et al. at hand, this is not a difficult question to answer. We state the result in the following theorem.

Theorem 1. *Let $\varphi \in S(\mathbb{D})$, $\psi \in H^\infty(\mathbb{D})$. Then $W_{\psi,\varphi}$ is complex symmetric with conjugation $\mathcal{J}W_{k_p,\sigma}$, where*

$$\sigma(z) = \lambda\frac{p-z}{1-\bar{p}z} \quad \text{and} \quad k_p(z) = \frac{\sqrt{1-|p|^2}}{1-\bar{p}z}$$

with $p \in \mathbb{D}$, $|\lambda| = 1$ and $\lambda p = \bar{p}$, if and only if

$$\psi(z) = \frac{c}{1-a_0p-(p-\bar{\lambda}a_0)z} \quad \text{and} \quad \varphi(z) = a_0 + \frac{a_1(p-\bar{\lambda}z)}{1-a_0p-(p-\bar{\lambda}a_0)z}.$$

Proof. By definition, $W_{\psi,\varphi}$ is complex symmetric with conjugation $\mathcal{J}W_{k_p,\sigma}$ if and only if

$$\mathcal{J}W_{k_p,\sigma}W_{\psi,\varphi} = W^*_{\psi,\varphi}\mathcal{J}W_{k_p,\sigma}. \tag{2.1}$$

Note that $\mathcal{J}W_{k_p,\sigma}$ is a conjugation and $W_{k_p,\sigma}$ is unitary, so

$$\mathcal{J}W_{k_p,\sigma}\mathcal{J} = W^{-1}_{k_p,\sigma} = W^*_{k_p,\sigma},$$

and then (2.1) is equivalent to

$$W_{k_p\cdot\psi\circ\sigma,\varphi\circ\sigma}\mathcal{J} = \mathcal{J}W^*_{k_p\cdot\psi\circ\sigma,\varphi\circ\sigma}.$$

It follows from Theorem A that $W_{k_p\cdot\psi\circ\sigma,\varphi\circ\sigma}$ is complex symmetric with conjugation $\mathcal{J}$ if and only if

$$k_p(z)\cdot\psi(\sigma(z)) = \frac{b}{1-a_0z} \quad \text{and} \quad \varphi(\sigma(z)) = a_0 + \frac{a_1z}{1-a_0z}.$$

Consequently, we have

$$\psi(z) = \frac{k_p\cdot\psi\circ\sigma}{k_p}\circ\sigma^{-1}(z) = \frac{b\sqrt{1-|p|^2}}{1-a_0p-(p-\bar{\lambda}a_0)z} = \frac{c}{1-a_0p-(p-\bar{\lambda}a_0)z}$$

and

$$\varphi(z) = \varphi\circ\sigma\circ\sigma^{-1}(z) = a_0 + \frac{a_1(p-\bar{\lambda}z)}{1-a_0p-(p-\bar{\lambda}a_0)z}.$$

This completes the proof. $\square$

2.3. Relation with the normal class. We have mentioned that the class of complex symmetric operators includes the normal operators, particularly the unitary operators and the Hermitian operators. In this subsection, we will apply Theorem 1 to give a conjugation with which a unitary (Hermitian, normal) weighted composition operator is complex symmetric.

We first consider the unitary class and Hermitian class. For the unitary weighted composition operator listed in Lemma 1, the conjugation is given by the following theorem.

Theorem 2. *A unitary weighted composition operator $W_{\psi,\varphi}$, where*

$$\varphi(z) = \mu_1 \frac{q-z}{1-\bar{q}z} \quad \text{and} \quad \psi(z) = \mu_2 \frac{\sqrt{1-|q|^2}}{1-\bar{q}z}$$

with $q \in \mathbb{D}$ and $|\mu_1| = |\mu_2| = 1$, is complex symmetric with conjugation $\mathcal{J}W_{k_{\bar{q}},\sigma}$.

Proof. Let $a_0 = 0$, $p = \bar{q}$, $a_1 = \lambda\mu_1$ and $c = \mu_2\sqrt{1-|q|^2}$ in Theorem 1, the result then follows. □

The class of Hermitian weighted composition operators on $H^2(\mathbb{D})$ is completely characterized by Cowen and Ko, see [3]. For this, we have:

Theorem 3. *A Hermitian weighted composition operator $W_{\psi,\varphi}$, where*

$$\varphi(z) = b_0 + \frac{b_1 z}{1-\bar{b}_0 z} \quad \text{and} \quad \psi(z) = \frac{b_2}{1-\bar{b}_0 z}$$

with $b_0 \in \mathbb{D}$ and $b_1, b_2 \in \mathbb{R}$, is complex symmetric with conjugation $\mathcal{J}C_{\lambda z}$, where $\lambda\bar{b}_0 + b_0 = 0$.

Proof. Let $a_0 = b_0$, $c = b_2$, $p = 0$ and λ, a_1 be such that $\lambda\bar{b}_0 = -b_0,\ a_1\bar{\lambda} = -b_1$ in Theorem 1, then the result follows. □

The case of the normal class is a bit more complicated. Bourdon and Narayan in [1] studied the normality of weighted composition operators on $H^2(\mathbb{D})$. Their main results are the following two theorems.

Theorem B ([1], Theorem 10)**.** *Suppose that $\varphi \in S(\mathbb{D})$ has a fixed point $p \in \mathbb{D}$. Then $W_{\psi,\varphi}$ acting on $H^2(\mathbb{D})$ is normal if and only if*

$$\psi = \gamma \frac{K_p}{K_p \circ \varphi} \quad \text{and} \quad \varphi = \alpha_p \circ (\delta\alpha_p),$$

where $\alpha_p(z) = (p-z)/(1-\bar{p}z)$, γ, δ are constants with δ satisfying $|\delta| \leqslant 1$.

Theorem C ([1], Proposition 12)**.** *Suppose that*

$$\varphi(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0$$

is a linear fractional selfmap of the unit disk and $\psi = K_{\varphi^*(0)}$*, where* $\varphi^*(z) = (\bar{a}z - \bar{c})/(-\bar{b}z + \bar{d})$*. Then* $W_{\psi,\varphi}$ *acting on* $H^2(\mathbb{D})$ *is normal if and only if*

$$\frac{|d|^2}{|d|^2 - |b|^2 - (\bar{b}a - \bar{d}c)z} C_{\varphi^* \circ \varphi} = \frac{|d|^2}{|d|^2 - |c|^2 - (\bar{b}d - c\bar{a})z} C_{\varphi \circ \varphi^*}. \tag{2.2}$$

The above two theorems list all the normal weighted composition operators known up to now: Theorem B gives a complete characterization in the case when φ has an interior fixed point; Theorem C gives some partial results in the case when φ is a linear fractional transformation. In the remainder of this section, we will give a conjugation with which a normal weighted composition operator in the above theorems is complex symmetric.

For Theorem B, checking its proof, one can find that $W_{\psi,\varphi}$ is actually unitarily equivalent to the composition operator $C_{\delta z}$ via the formula

$$W_{\psi,\varphi} = W_{k_p,\alpha_p} \circ C_{\delta z} \circ W^{-1}_{k_p,\alpha_p} = W_{k_p,\alpha_p} \circ C_{\delta z} \circ W_{1/(k_p \circ \alpha_p),\alpha_p}.$$

Hence it is easy to give a conjugation for $W_{\psi,\varphi}$ in this case.

Theorem 4. *A normal weighted composition operator* $W_{\psi,\varphi}$*, where* φ *has an interior fixed point* p*, i.e.,*

$$\psi = \gamma \frac{K_p}{K_p \circ \varphi} \quad \text{and} \quad \varphi = \alpha_p \circ (\delta \alpha_p),$$

where $\alpha_p(z) = (p - z)/(1 - \bar{p}z)$*,* γ*,* δ *are constants with* δ *satisfying* $|\delta| \leqslant 1$*, is complex symmetric with conjugation* $\mathcal{J} W_{k_q,\sigma}$*, where* $q = (p - \bar{p})/(p^2 - 1)$*.*

P r o o f. It is obvious that $C_{\delta z}$ is complex symmetric with the classical conjugation $\mathcal{J}$. So $W_{\psi,\varphi}$ is complex symmetric with conjugation $W_{k_p,\alpha_p} \circ \mathcal{J} \circ W_{1/(k_p \circ \alpha_p),\alpha_p}$. Elementary calculation shows that

$$W_{k_p,\alpha_p} \circ \mathcal{J} \circ W_{1/(k_p \circ \alpha_p),\alpha_p} = \mu \mathcal{J} W_{k_q,\sigma},$$

where $q = (p - \bar{p})/(p^2 - 1)$ and $\mu = |1 - p^2|/(1 - p^2)$. □

For Theorem C, we will only consider the case when φ admits a boundary fixed point since the case when φ admits an interior fixed point is already answered in Theorem 4. Another reason is the following.

Lemma 3. *Suppose that*

$$\varphi(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$

is a linear fractional selfmap of the unit disk, which admits a boundary fixed point $\eta \in \mathbb{T}$, and $\psi = K_{\varphi^*(0)}$, where $\varphi^*(z) = (\bar{a}z - \bar{c})/(-\bar{b}z + \bar{d})$. Then $W_{\psi,\varphi}$ acting on $H^2(\mathbb{D})$ is normal if and only if $|b| = |c|$.

P r o o f. We need to show that equation (2.2) is equivalent to $|b| = |c|$. If (2.2) holds, then by taking the test function $f \equiv 1$, we get

$$\frac{|d|^2}{|d|^2 - |b|^2 - (\bar{b}a - \bar{d}c)z} = \frac{|d|^2}{|d|^2 - |c|^2 - (\bar{b}d - c\bar{a})z}.$$

Since $d \neq 0$, we have $|b| = |c|$.

If $|b| = |c|$, since

$$\varphi(\eta) = \frac{a\eta + b}{c\eta + d} = \eta,$$

we get

$$a - d = c\eta - b\overline{\eta},$$

and then

$$(\bar{b}a - \bar{d}c) - (\bar{b}d - c\bar{a}) = \bar{b}(a-d) + c(\bar{a} - \bar{d}) = \bar{b}(c\eta - b\overline{\eta}) + c(\bar{c}\overline{\eta} - \bar{b}\eta) = (|c|^2 - |b|^2)\overline{\eta} = 0,$$

Hence

$$\frac{|d|^2}{|d|^2 - |b|^2 - (\bar{b}a - \bar{d}c)z} = \frac{|d|^2}{|d|^2 - |c|^2 - (\bar{b}d - c\bar{a})z}.$$

Elementary calculation shows that

$$\varphi^*(\varphi(z)) = \frac{(|a|^2 - |c|^2)z + b\bar{a} - d\bar{c}}{(\bar{d}c - \bar{b}a)z + |d|^2 - |b|^2}, \quad \varphi(\varphi^*(z)) = \frac{(|a|^2 - |b|^2)z + b\bar{d} - a\bar{c}}{(\bar{a}c - \bar{b}d)z + |d|^2 - |c|^2},$$

so we also have $\varphi^* \circ \varphi = \varphi \circ \varphi^*$. Therefore, equation (2.2) holds.

The proof is complete. $\square$

Theorem 5. *Suppose that*

$$\varphi(z)=\frac{az+b}{cz+d},\quad ad-bc\neq 0,\ |b|=|c|$$

is a linear fractional selfmap of the unit disk which admits a boundary fixed point $\eta\in\mathbb{T}$*, and* $\psi=K_{\varphi^*(0)}$*, where* $\varphi^*(z)=(\bar{a}z-\bar{c})/(-\bar{b}z+\bar{d})$*. Then* $W_{\psi,\varphi}$ *acting on* $H^2(\mathbb{D})$ *is complex symmetric with conjugation* $\mathcal{J}W_{k_{p\overline{\eta}},\sigma}$*, where* p *is any nonzero number such that* $bp(\bar{p}-1)=c\eta^2\bar{p}(p-1)$.

P r o o f. From Lemma 3, we know $W_{\psi,\varphi}$ is normal and hence is complex symmetric. If we denote

$$\varphi_1(z)=\overline{\eta}\varphi(\eta z)=\frac{a_1z+b_1}{z+d_1},$$

where $a_1=a\overline{\eta}/c$, $b_1=b\overline{\eta}^2/c$ and $d_1=d\overline{\eta}/c$, then $\varphi_1(1)=1$, $|b_1|=1$ and it suffices to prove that $C_{\eta z}W_{\psi,\varphi}C_{\overline{\eta}z}$ is complex symmetric with conjugation $C_{\eta z}\mathcal{J}W_{k_{p\overline{\eta}},\sigma}C_{\overline{\eta}z}$. Direct calculation shows that

$$C_{\eta z}W_{\psi,\varphi}C_{\overline{\eta}z}=W_{\psi_1,\varphi_1}\quad\text{and}\quad C_{\eta z}\mathcal{J}W_{k_{p\overline{\eta}},\sigma}C_{\overline{\eta}z}=\mathcal{J}W_{k_p,\sigma_1},$$

where $\psi_1(z)=d_1/(z+d_1)$ and $\sigma_1(z)=(|p|^2-\bar{p}z)/(p-|p|^2z)$.

Now, we will prove that

$$\mathcal{J}W_{k_p,\sigma_1}W_{\psi_1,\varphi_1}=W^*_{\psi_1,\varphi_1}\mathcal{J}W_{k_p,\sigma_1}$$

which is equivalent to

$$W_{k_p,\sigma_1}W_{\psi_1,\varphi_1}\mathcal{J}W_{k_p,\sigma_1}=\mathcal{J}W^*_{\psi_1,\varphi_1}.$$

Since the span of reproducing kernels forms a dense subset of $H^2(\mathbb{D})$, it suffices to check that

$$W_{k_p,\sigma_1}W_{\psi_1,\varphi_1}\mathcal{J}W_{k_p,\sigma_1}(K_w)=\mathcal{J}W^*_{\psi_1,\varphi_1}(K_w)\tag{2.3}$$

for all $w\in\mathbb{D}$.

For the right-hand side of (2.3), we have

$$\begin{aligned}\mathcal{J}W^*_{\psi_1,\varphi_1}(K_w)(z)&=\mathcal{J}(\overline{\psi_1(w)}K_{\varphi_1(w)})(z)=\psi_1(w)K_{\overline{\varphi_1(w)}}(z)\\&=\frac{d_1}{w+d_1}\cdot\frac{1}{1-(a_1w+b_1)z/(w+d_1)}=\frac{d_1}{w+d_1-(a_1w+b_1)z}.\end{aligned}$$

For the left-hand side of (2.3), we have

$$\begin{aligned}
&W_{k_p,\sigma_1}W_{\psi_1,\varphi_1}\mathcal{J}W_{k_p,\sigma_1}(K_w)(z)\\
&=W_{k_p,\sigma_1}W_{\psi_1,\varphi_1}\mathcal{J}\biggl(\frac{\sqrt{1-|p|^2}}{1-\bar{p}z}\cdot\frac{1}{1-\bar{p}\overline{w}p^{-1}(p-z)(1-\bar{p}z)^{-1}}\biggr)\\
&=W_{k_p,\sigma_1}W_{\psi_1,\varphi_1}\mathcal{J}\biggl(\frac{p\sqrt{1-|p|^2}}{(p-|p|^2\overline{w})+(\bar{p}\overline{w}-|p|^2)z}\biggr)\\
&=W_{k_p,\sigma_1}W_{\psi_1,\varphi_1}\biggl(\frac{\bar{p}\sqrt{1-|p|^2}}{(\bar{p}-|p|^2w)+(pw-|p|^2)z}\biggr)\\
&=W_{k_p,\sigma_1}\biggl(\frac{d_1}{z+d_1}\frac{\bar{p}\sqrt{1-|p|^2}}{(\bar{p}-|p|^2w)+(pw-|p|^2)(a_1z+b_1)(z+d_1)^{-1}}\biggr)\\
&=W_{k_p,\sigma_1}\biggl(\frac{d_1\bar{p}\sqrt{1-|p|^2}}{(d_1\bar{p}-b_1|p|^2-d_1|p|^2w+b_1pw)+(a_1pw-|p|^2w-a_1|p|^2+\bar{p})z}\biggr)\\
&=\frac{\sqrt{1-|p|^2}}{1-\bar{p}z}d_1\bar{p}\sqrt{1-|p|^2}\\
&\quad\times\biggl(d_1\bar{p}-b_1|p|^2-d_1|p|^2w+b_1pw)+(a_1pw-|p|^2w-a_1|p|^2+\bar{p})\frac{(|p|^2-\bar{p}z)}{(p-|p|^2z)}\biggr)^{-1}\\
&=\frac{d_1|p|^2(1-|p|^2)}{(d_1\bar{p}-b_1|p|^2-d_1|p|^2w+b_1pw)(p-|p|^2z)+(a_1pw-|p|^2w-a_1|p|^2+\bar{p})(|p|^2-\bar{p}z)}\\
&=\frac{d_1|p|^2(1-|p|^2)}{(Aw+B)-(Cw+D)z},
\end{aligned}$$

where

$$\begin{aligned}
A&=p^2(b_1+a_1\bar{p}-\bar{p}^2-d_1\bar{p}), &\quad B&=|p|^2(d_1+\bar{p}-a_1|p|^2-b_1p),\\
C&=|p|^2(a_1+b_1p-\bar{p}-d_1|p|^2), &\quad D&=\bar{p}^2(d_1p+1-a_1p-b_1p^2).
\end{aligned}$$

Using the conditions $\varphi_1(1)=1$ and $b_1p(\bar{p}-1)=\bar{p}(p-1)$, one can easily verify that (2.3) holds. This completes the proof. □

Remark 2. The equation

$$b_1p(\bar{p}-1)=\bar{p}(p-1),\quad |b_1|=1$$

has many nonzero solutions in the unit disk. In fact, if we write $p=r\mathrm{e}^{\mathrm{i}\theta}\neq 0$, then

$$b_1=\frac{\bar{p}(p-1)}{p(\bar{p}-1)}=\frac{r-\mathrm{e}^{-\mathrm{i}\theta}}{r-\mathrm{e}^{\mathrm{i}\theta}}=\mathrm{e}^{2\mathrm{i}\arg(r-\mathrm{e}^{-\mathrm{i}\theta})},$$

which has exactly two solutions θ for any fixed $r\in(0,1)$.

2.4. Algebraic $W_{\psi,\varphi}$ of degree $\leqslant$ 2. So far, all complex symmetric weighted composition operators have had linear fractional symbols. In this section, we will show that the weight function ψ need not be linear fractional, even when the composition symbol φ is. The counterexample is a weighted composition operator that is algebraic of degree two. Recall that each algebraic operator of degree no more than two is complex symmetric. The following theorem characterizes when a weighted composition operator is algebraic with degree no more than two.

Theorem 6. *Suppose that $\varphi \in S(\mathbb{D})$ and $\psi \in H(\mathbb{D})$ is not identically zero. Then $W_{\psi,\varphi}$ is algebraic with degree $\leqslant 2$ exactly when one of the following holds:*
(1) *φ is a constant function,*
(2) *φ is the identity map and ψ is a constant function,*
(3) *$\varphi(z) = \alpha_p(-\alpha_p(z))$ with $p \in \mathbb{D}$ and*

$$\psi \circ \alpha_p(z) = c \exp\Bigg\{\sum_{k=0}^{\infty} a_{2k+1} z^{2k+1}\Bigg\} \in H(\mathbb{D}).$$

P r o o f. Suppose that $W_{\psi,\varphi}$ satisfies the equation

$$AW_{\psi,\varphi}^2 - BW_{\psi,\varphi} - C = 0$$

with $|A|^2 + |B|^2 + |C|^2 \neq 0$.

Obviously, the degree of $W_{\psi,\varphi}$ is at least one.

If the degree is one, then we have $BW_{\psi,\varphi} + C = 0$; thus $\psi \equiv -C/B$ and φ is the identity map.

If the degree of $W_{\psi,\varphi}$ is two, we may assume without loss of generality that $A = 1$ and then divide the proof into three cases.

Case I: If $B \neq 0$, $C = 0$, that is

$$W_{\psi,\varphi}^2 - BW_{\psi,\varphi} = 0.$$

Taking test functions $f \equiv 1$ and $g(z) = z$, we get

$$\begin{cases} \psi \cdot \psi \circ \varphi - B\psi = 0, \\ \psi \cdot \psi \circ \varphi \cdot \varphi \circ \varphi - B\psi \cdot \varphi = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} \psi \circ \varphi - B = 0, \\ \varphi \circ \varphi - \varphi = 0. \end{cases}$$

If φ is constant, then ψ is any holomorphic function such that $\psi(\varphi(0)) = B$; otherwise, $\psi \equiv B$ and φ is the identity map since $\varphi(\mathbb{D})$ is a nonempty open set.

Case II: If $B = 0$, $C \neq 0$, that is

$$W_{\psi,\varphi}^2 = C.$$

Taking test functions $f \equiv 1$ and $g(z) = z$, we get

$$\begin{cases} \psi \cdot \psi \circ \varphi = C, \\ \varphi \circ \varphi = \mathrm{id}. \end{cases}$$

If φ is the identity map, then $\psi = \pm\sqrt{C}$ is constant.

If φ is not the identity map, we first consider the simple case when the fixed point of φ is zero. In this case, $\varphi(z) = -z$ and $\psi(z) \cdot \psi(-z) = C \neq 0$. Setting $\psi = \mathrm{e}^h$ yields

$$h(z) + h(-z) = \log C,$$

which implies that the even terms in the series expansion of $h - \frac{1}{2}\log C$ vanish. Consequently,

$$\psi(z) = \sqrt{C}\exp\Big(\sum_{k=0}^{\infty} a_{2k+1}z^{2k+1}\Big).$$

For the case where $\varphi(p) = p \neq 0$, set $\widetilde{\psi} = \psi \circ \alpha_p$ and $\widetilde{\varphi} = \alpha_p \circ \varphi \circ \alpha_p$, then $W_{\psi,\varphi}$ is similar to $W_{\widetilde{\psi},\widetilde{\varphi}}$ and hence $P(W_{\widetilde{\psi},\widetilde{\varphi}}) = 0$. So we have

$$\widetilde{\psi}(z) = \sqrt{C}\exp\Big(\sum_{k=0}^{\infty} a_{2k+1}z^{2k+1}\Big).$$

Case III: If $B \neq 0$, $C \neq 0$, that is

$$W_{\psi,\varphi}^2 - BW_{\psi,\varphi} - C = 0.$$

Taking the monomials $\{z^n\}_{n=0}^{\infty}$ as test functions, we get

$$\psi \cdot \psi \circ \varphi \cdot (\varphi \circ \varphi)^n = B\psi \cdot \varphi^n + Cz^n, \quad n \geqslant 0. \tag{2.4}$$

For $n = 0$, if we let $z = 0$, then we get

$$\psi(0) \cdot \psi(\varphi(0)) = B\psi(0) + C,$$

thus $\psi(0) \neq 0$.

For $n \geqslant 1$, if we let $z = 0$, then we get

$$\psi(\varphi(0))[\varphi(\varphi(0))]^n = B\varphi(0)^n,$$

that is,

$$\Big[\frac{\varphi(\varphi(0))}{\varphi(0)}\Big]^n = \frac{B}{\psi(\varphi(0))}, \quad n \geqslant 1.$$

So we must have $\varphi(\varphi(0)) = \varphi(0)$ and $B = \psi(\varphi(0))$. Evaluating (2.4) at $\varphi(0)$ yields

$$\psi(\varphi(0))^2\varphi^n(\varphi(0)) = B\psi(\varphi(0))\varphi^n(\varphi(0)) + C\varphi(0)^n = \psi(\varphi(0))^2\varphi^n(\varphi(0)) + C\varphi(0)^n,$$

so we get $\varphi(0) = 0$.

It follows from (2.4) that

$$\psi(z) \cdot \psi(\varphi(z))\Big[\frac{\varphi(\varphi(z))}{z}\Big]^n = B\psi(z)\Big(\frac{\varphi(z)}{z}\Big)^n + C, \quad n \geqslant 0. \tag{2.5}$$

Since $\varphi(0) = 0$, equation (2.5), along with the Schwartz lemma, imply that $\varphi(z) = \lambda z$ for some $|\lambda| = 1$.

Now, equation (2.4) is equivalent to

$$\psi(z)\psi(\lambda z)\lambda^{2n} = B\psi(z)\lambda^n + C, \quad n \geqslant 0. \tag{2.6}$$

If $\lambda = 1$, then

$$\psi^2(z) = B\psi(z) + C,$$

where ψ has to be constant and then $W_{\psi,\varphi}$ is a multiple of the identity map.

If $\lambda = -1$, then

$$\psi(z)\psi(\lambda z) - C = B\psi(z) = -B\psi(z).$$

Since $B \neq 0$ and ψ is not identically zero, this is impossible.

If λ is rational and $\lambda^N = 1$ with N minimal and $N \geqslant 3$, then summing (2.6) with respect to n, we get

$$\psi(z)\psi(\lambda z)\sum_{n=0}^{N-1}\lambda^{2n} = B\psi(z)\sum_{n=0}^{N-1}\lambda^n + CN.$$

Since

$$\lambda^N - 1 = (\lambda - 1)(1 + \lambda + \lambda^2 + \ldots + \lambda^{N-1}) = 0,$$
$$\lambda^{2N} - 1 = (\lambda^2 - 1)(1 + \lambda^2 + \lambda^4 + \ldots + \lambda^{2(N-1)}) = 0,$$

we have $C = 0$, which is a contradiction.

If λ is irrational, then

$$\psi(z)\psi(\lambda z)w^2 = B\psi(z)w + C, \quad z \in \mathbb{D},\ w \in \mathbb{C}.$$

This is obviously impossible.

The proof is complete. $\square$

Example 1. Let $\varphi(z) = -z$ and $\psi_1(z) = \mathrm{e}^z$, $\psi_2(z) = \mathrm{e}^{\sin z}$; then $W_{\psi_1,\varphi}$ and $W_{\psi_2,\varphi}$ are complex symmetric on $H^2(\mathbb{D})$.

2.5. Open problem. We have shown that the weight symbol ψ of a complex symmetric $W_{\psi,\varphi}$ need not to be a linear fractional map. We also want to know:

Problem 1. Is it necessary for a complex symmetric weighted composition operator $W_{\psi,\varphi}$ to have its composition symbol φ being linear fractional?

We are also interested in a similar problem concerning normality, see also [12].

Problem 2. For a normal weighted composition operator $W_{\psi,\varphi}$ with φ a linear fractional map, must ψ be also linear fractional? Ultimately, can we obtain a complete characterization of normal weighted composition operators on $H^2(\mathbb{D})$?

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