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CRITERION OF THE REALITY OF ZEROS IN A POLYNOMIAL SEQUENCE SATISFYING A THREE-TERM RECURRENCE RELATION

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Abstract. This paper establishes the necessary and sufficient conditions for the reality of all the zeros in a polynomial sequence $\{P_i\}_{i=1}^{\infty}$ generated by a three-term recurrence relation $P_i(x) + Q_1(x)P_{i-1}(x) + Q_2(x)P_{i-2}(x) = 0$ with the standard initial conditions $P_0(x) = 1, P_{-1}(x) = 0$, where $Q_1(x)$ and $Q_2(x)$ are arbitrary real polynomials.

Keywords: recurrence relation; polynomial sequence; support; real zeros

MSC 2020: 12D10, 26C10, 30C15

1. Introduction

Asymptotic root distributions for sequences of univariate polynomials have been a topic of study in analysis for many decades, see [8]. In particular, sequences of polynomials with all zeros real are important in many branches of mathematics. Such polynomials possess several nice properties. For example, if a polynomial $P(x) = \sum_{i=0}^{n} b_i x^i$ is real-rooted and has nonnegative coefficients, then the sequence $\{b_i\}_{i=0}^{n}$ is log-concave, i.e. $b_i^2 \geqslant b_{i+1}b_{i-1}$ for all $1 \leqslant i < n$, see [3]. This log-concavity implies that $\{b_i\}_{i=0}^{n}$ is unimodal, whereby the sequence increases to the greatest value (or possibly two consecutive equal values) and then decreases, see [3]. In addition, polynomials with real zeros are closed with respect to differentiation and the zeros of the derivative interlace with the zeros of the polynomial. A good source of information about real-rooted polynomials is a book by Kostov, see [6].

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In this paper, we discuss some cases of the problem under which conditions the polynomials satisfying a finite linear recurrence relation have all real roots. Our general set-up is as follows. Fix complex-valued polynomials $Q_1(x), \dots, Q_k(x)$ and consider a finite linear recurrence relation of the form

$$P_i(x)+Q_1(x)P_{i-1}(x)+Q_2(x)P_{i-2}(x)+\ldots+Q_k(x)P_{i-k}(x)=0, \quad i=1,2,\ldots \tag{1.1}$$

with the standard initial conditions

$$P_0(x)=1, \quad P_{-1}(x)=P_{-2}(x)=\ldots=P_{-k+1}(x)=0. \tag{1.2}$$

The generating function for the polynomial sequence $\{P_i\}_{i=1}^{\infty}$ of this recurrence is given by

$$\sum_{i=0}^{\infty} P_i(x)t^i = \frac{1}{1+Q_1(x)t+Q_2(x)t^2+\ldots+Q_k(x)t^k}.$$

One of the well-known results in this area is a description of the accumulation set for the zeros of $P_i(x)$ when $i \to \infty$ provided by the theorem by Beraha, Kahane and Weiss, see [1]. It asserts that the support of the limiting root-counting measure coincides with the following set. Let $Q_1, \dots, Q_k$ be complex polynomials as given in equation (1.1) above. Define a curve $\Gamma_Q \subset \mathbb{C}$ consisting of all the values of x such that the characteristic equation

$$1+Q_1(x)t+Q_2(x)t^2+\ldots+Q_k(x)t^k=0 \tag{1.3}$$

has at least two roots t_1, t_2 for which
(a) $|t_1|=|t_2|$,
(b) $|t_1|$ is the minimum among the absolute values of all roots.

Theorem 1 ([1]). *Suppose that $\{P_i(x)\}$ satisfies* (1.1), (1.2) *and* (1.3). *Suppose further that $\{P_i(x)\}$ satisfies no recursion of order less than k and that there is no constant $\omega \in \mathbb{C}$ of unit modulus for which $t_r=\omega t_s$ for some $r \neq s$. Then the zeros of $P_i(x)$ accumulate along the curve Γ_Q as $i \to \infty$.*

This result provides a description of the asymptotic behaviour of the roots of $P_i(x)$. However, recently Tran in [9] has found a number of cases when the zeros of $P_i(x)$ actually lie on the limiting curve Γ_Q for all or for all sufficiently large i. In particular, he has proven the following results.

Theorem 2 ([9]). *Let $\{P_i(x)\}$ be a polynomial sequence whose generating function is*

$$\sum_{i=0}^{\infty} P_i(x)t^i = \frac{1}{1+Q_1(x)t+Q_2(x)t^2},$$

where $Q_1(x)$ and $Q_2(x)$ are polynomials in x with complex coefficients. All the zeros of every polynomial in the sequence $\{P_i(x)\}$ which satisfy $Q_2(x) \neq 0$ lie on the curve Γ_Q defined by

$$\operatorname{Im}\Big(\frac{Q_1^2(x)}{Q_2(x)}\Big) = 0 \quad \textit{and} \quad 0 \leqslant \operatorname{Re}\Big(\frac{Q_1^2(x)}{Q_2(x)}\Big) \leqslant 4. \tag{1.4}$$

Moreover, these zeros become dense in Γ_Q when $i \to \infty$.

Theorem 2 covers the special case of (1.1) and (1.2) for the polynomials generated by the recurrence

$$P_i(x) + Q_1(x)P_{i-1}(x) + Q_2(x)P_{i-2}(x) = 0$$

with the standard initial conditions $P_0(x) = 1$ and $P_{-1}(x) = 0$.

A more general result of Tran is as follows.

Theorem 3 ([10]). *Let $\{P_i(x)\}$ be a polynomial sequence with the generating function*

$$\sum_{i=0}^{\infty} P_i(x)t^i = \frac{1}{1 + Q_1(x)t + Q_2(x)t^k},$$

where $Q_1(x)$ and $Q_2(x)$ are polynomials in x with complex coefficients. Then there exists a constant $C = C(k)$ such that all the zeros of $P_i(x)$ which satisfy $Q_2(x) \neq 0$ lie for all $i > C$ on the curve Γ_Q defined by

$$\operatorname{Im}\Big(\frac{Q_1^k(x)}{Q_2(x)}\Big) = 0 \quad \textit{and} \quad 0 \leqslant (-1)^k \operatorname{Re}\Big(\frac{Q_1^k(x)}{Q_2(x)}\Big) \leqslant \frac{k^k}{(k-1)^{k-1}}.$$

Moreover, these zeros become dense in Γ_Q when $i \to \infty$.

In the present paper, we want to characterize a situation as above that gives rise to polynomial sequences with only real zeros.

Problem 1. *In the above notation, consider the recurrence relation*

$$P_i(x) + Q_1(x)P_{i-1}(x) + Q_2(x)P_{i-2}(x) = 0$$

with the standard initial conditions,

$$P_0(x) = 1, \quad P_{-1}(x) = 0,$$

where $Q_1(x)$ and $Q_2(x)$ are arbitrary real polynomials. Give necessary and sufficient conditions on $(Q_1(x), Q_2(x))$ guaranteeing that all the zeros of $P_i(x)$ are real for all i.

To formulate our main result, we need to look at the curve defined by the first condition in (1.4). We shall view $\mathbb{C}P^1$ as $\mathbb{C} \cup \{\infty\}$, the extended complex plane, and $\mathbb{R}P^1$ as the union of the real line in $\mathbb{C}$ with $\{\infty\}$.

Let $f\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ be the rational function defined by $f = Q_1^2(x)/Q_2(x)$, where $Q_1(x)$ and $Q_2(x)$ are real polynomials. Denote by $\widetilde{\Gamma_Q} \subset \mathbb{C}P^1$ the curve given by $\mathrm{Im}(f) = 0$, that is

$$\widetilde{\Gamma_Q} = f^{-1}(\mathbb{R}P^1).$$

We note that for real polynomials $Q_1(x)$ and $Q_2(x)$, the curve $\widetilde{\Gamma_Q}$ contains Γ_Q since $[0, 4] \subset \mathbb{R}P^1$.

Lemma 1. *The curve $\widetilde{\Gamma_Q}$ has the following properties:*
(a) $\widetilde{\Gamma_Q} \supset \mathbb{R}P^1$,
(b) $\widetilde{\Gamma_Q}$ *is invariant under complex conjugation,*
(c) *except $\mathbb{R}P^1$, $\widetilde{\Gamma_Q}$ might contain ovals disjoint with $\mathbb{R}P^1$ (which come in complex-conjugate pairs) and ovals crossing $\mathbb{R}P^1$ which are mapped to themselves by complex conjugation,*
(d) *the intersection points of the second type of ovals with $\mathbb{R}P^1$ are exactly the real critical points of f.*

Figures 1 and 2 illustrate the properties of $\widetilde{\Gamma_Q}$ claimed in Lemma 1 (a)–(d). The main result of the present paper is as follows.

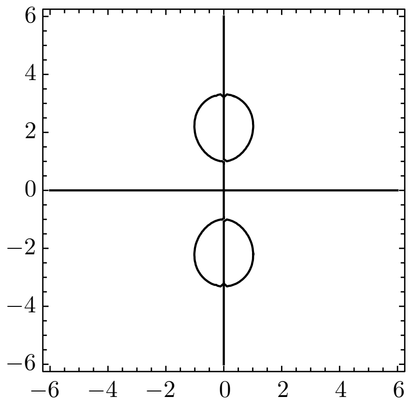

Figure 1. The curve $\widetilde{\Gamma_Q}$ for $f = Q_1^2(x)/Q_2(x)$, where $Q_1(x) = x^2 + 1$ and $Q_2(x) = x^2 + 6$.

Theorem 4. *Let $\{P_i(x)\}$ be a sequence of polynomials whose generating function is*

$$\sum_{i=0}^{\infty} P_i(x)t^i = \frac{1}{1 + Q_1(x)t + Q_2(x)t^2}, \tag{1.5}$$

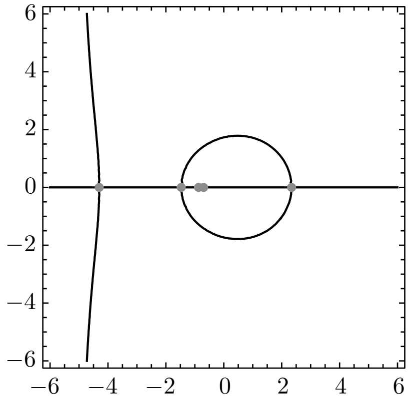

Figure 2. The curve $\widetilde{\Gamma_Q}$ for $f = Q_1^2(x)/Q_2(x)$ where $Q_1(x) = x^2 + 5x + 3$ and $Q_2(x) = 5x^2 - 3$. The grey points are the real critical points of f.

where $Q_1(x)$ and $Q_2(x)$ are arbitrary coprime real polynomials in x. Then, for all positive integers i, all the zeros of $P_i(x)$ are real if and only if the following conditions are satisfied:

(a) *All the zeros of the polynomial $Q_1(x)$ must be real and simple.*
(b) *No ovals γ of $\widetilde{\Gamma_Q}$ disjoint with $\mathbb{R}P^1$ should exist.*
(c) *All the zeros of the discriminant $D(x)$ of the characteristic polynomial $1 + Q_1(x)t + Q_2(x)t^2$ must be real.*
(d) *No real critical values of f should belong to the interval $(0, 4)$.*
(e) *The polynomial $Q_2(x)$ must be non-negative at the zeros of $Q_1(x)$.*

Remark. The situation when $Q_1(x)$ and $Q_2(x)$ have a common real zero is not interesting to consider since from $P_i(x) + Q_1(x)P_{i-1}(x) + Q_2(x)P_{i-2}(x) = 0$, such a zero would necessarily be a zero of $P_i(x)$ for all i.

2. Proofs

Let us begin with the following definitions and remarks.

Definition 1. For a non-constant rational function $R(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are coprime polynomials, the degree of $R(x)$ is defined as the maximum of the degrees of $P(x)$ and $Q(x)$.

Equivalently, the degree of $R(x)$ is the number of distinct preimages of any generic point.

Definition 2. A point x_0 is called a critical point of $R(x)$ if $R(x)$ fails to be injective in a neighbourhood of x_0, that is, $R'(x_0) = 0$. A critical value of $R(x)$ is the image of a critical point. The order of a critical point x_0 of $R(x)$ is the order of the zero of $R'(x)$ at x_0.

Now, if d is the degree of $R(x)$ and if w is not a critical value then $R^{-1}(w) = \{x_1, x_2, \ldots, x_d\}$ with $x_i \neq x_j$ for all $i \neq j$. Since the points x_j are non-critical there is a neighbourhood of each of these points such that $R(x)$ is injective on that neighbourhood. The function $R(x)$ has $2d - 2$ critical points in $\mathbb{C}P^1$ counting multiplicities. This follows from the fact that in the complex plane, $\deg(R') = \deg(P'Q - Q'P) = \deg(P) + \deg(Q) - 1$ while the order of the critical point at infinity is $|\deg(P) - \deg(Q)| - 1$, see [4].

Definition 3. Given a pair (P, Q) of polynomials, define their Wronskian as

$$W(P, Q) = P'Q - Q'P.$$

An interesting thing about the Wronskian is that if P and Q are coprime, then the zeros of $W(P, Q)$ are exactly the critical points of the rational map $R = P/Q$. In [7] we find that if P and Q have all real, simple and interlacing zeros, then all zeros of $W(P, Q)$ are non-real. In addition, if we know that α is a zero of R of multiplicity not less than 2, then α is also a (multiple) zero of the Wronskian. More information about the Wronski map can be found in [5] and [7].

Proof of Theorem 4. (a) Substitution of the initial conditions $P_0(x) = 1$ and $P_{-1}(x) = 0$ in the recurrence relation

$$P_i(x) + Q_1(x)P_{i-1}(x) + Q_2(x)P_{i-2}(x) = 0$$

gives for $i = 1$ that

$$P_1(x) + Q_1(x)P_0(x) + Q_2(x)P_{-1}(x) = 0$$

or

$$P_1(x) = -Q_1(x).$$

Therefore $Q_1(x)$ must have all its zeros real since we require all the zeros of $P_i(x)$ to be real for all i and in particular for $i = 1$. These zeros must be simple (see part (e) for the justification).

(b) Suppose there exists an oval γ of $\widetilde{\Gamma_Q}$ which does not intersect $\mathbb{R}P^1$. From Lemma 1 (c), γ is the type one oval contained in $\widetilde{\Gamma_Q}$. We note that all the points on γ and its interior are of the form $z = x + iy$ where $x, y \in \mathbb{R}$, $y \neq 0$, and this is a connected component with γ as its boundary. This component is mapped by f onto the half plane with degree $\geqslant 1$ depending on the number of critical points of f it strictly contains. The boundary γ of the component is mapped onto $\mathbb{R}P^1$ (the boundary of the half plane). In particular, the image $f(\gamma)$ covers the interval $[0,4] \subset \mathbb{R}P^1$. Therefore $\Gamma_Q = f^{-1}([0,4])$ must contain an arc of the boundary γ. From [9], Theorem 2 all the zeros of $P_i(x)$ are contained in Γ_Q for all i and are dense in Γ_Q as $i \to \infty$. Now since we require all the zeros of $P_i(x)$ to be real, it must hold that $\Gamma_Q \subseteq \mathbb{R}P^1$. This is not possible as we already have that Γ_Q contains an arc of γ yet this arc is not contained in $\mathbb{R}P^1$, hence a contradiction.

(c) It is known, see [2], that the endpoints of Γ_Q are the points where $t_1 = t_2$ (see the notation in Theorem 2). In our case equation (1.3) has degree 2 in t. Therefore every x, for which the roots of (1.3) coincide, belongs to Γ_Q. These x are exactly the zeros of $D(x) = Q_1^2(x) - 4Q_2(x)$. Since we require that $\Gamma_Q \subset \mathbb{R}P^1$, all the zeros of $D(x)$ must be real.

(d) Suppose there exists a critical value $w \in (0,4)$. Then there must exist a real critical point x_c such that $f(x_c) = w$. Clearly, x_c is a point on Γ_Q. It is known, see [4], that a point $x_c \in \mathbb{C}P^1$ is a critical point of order k for a rational function $R(x)$ if and only if there are open sets U containing x_c and V containing $w = R(x_c)$ such that each $w_0 \in V$, $w \neq w_0$, has exactly $k+1$ distinct preimages in U.

In our scenario, let x_c be such a critical point of order k for $f(x)$ and V be the real interval $(w - \varepsilon, w + \varepsilon)$ for a sufficiently small $\varepsilon > 0$. Then $V \subset (0,4)$. Note that since x_c is a critical point of order k for $f(x)$ and any point $z \in V$, $z \neq w$, has exactly $k+1$ distinct preimages in U, we have locally at x_c that the preimage $U = f^{-1}(V)$ of V consists of $k+1$ distinct curves (arcs) with a common intersection only at x_c. One of these curves is a line segment on the real line while the remaining k curves are complex, i.e. apart from x_c, the points on these k curves are of the form $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$, $y \neq 0$.

Now since complex arcs are formed in the preimage of V, then some of the zeros of $P_i(x)$ are contained in the complex arcs since all the zeros of $P_i(x)$ are contained in $\Gamma_Q = f^{-1}([0,4] \supset V)$ and are dense there as $i \to \infty$. This contradicts our requirement that Γ_Q is contained in $\mathbb{R}P^1$. Therefore, in order to have all the real roots of $P_i(x)$ for all i, no real critical values of f can be in the real interval $(0,4)$. Otherwise the condition that $\Gamma_Q \subset \mathbb{R}P^1$ cannot hold.

(e) Let x_0 be a zero of $Q_1(x)$, i.e. $Q_1(x_0) = 0$. Note that $Q_2(x)$ and $Q_1(x)$ do not have a common zero since they are coprime. Therefore, at the point x_0 we have $Q_2(x_0) \neq 0$. It remains to show that $Q_2(x_0) > 0$. We note that all the zeros of $Q_1(x)$ are the critical points of f. In addition, they belong to Γ_Q because at x_0 we have that $f(x_0) = Q_1^2(x_0)/Q_2(x_0) = 0$; therefore both the real and the imaginary part of f vanish, hence satisfying (1.4) of Theorem 2. Suppose that x_0 is a simple critical point of f and let $Q_2 > 0$ at x_0. Then locally $f = Q_1^2/Q_2 \geqslant 0$ on an interval in $\mathbb{R}$ and if $Q_2 < 0$ at x_0, then locally $f = Q_1^2/Q_2 \geqslant 0$ on the complex arc, i.e., there exists an interval $I \subset [0,4]$ such that $f^{-1}(I)$ contains a complex arc. On the other hand, if x_0 is a critical point of order greater than 1, then locally $f = Q_1^2/Q_2 \geqslant 0$ on some complex arc irrespective of whether $Q_2 > 0$ or $Q_2 < 0$. However, it is known that the zeros of $P_i(x)$ are contained in $\Gamma_Q = f^{-1}([0,4])$ and are dense there as $i \to \infty$, and so for the reality of all the zeros of $P_i(x)$ we require that $\Gamma_Q \subset \mathbb{R}P^1$. This is not possible if $Q_2 < 0$ at simple zeros of $Q_1(x)$ or when zeros of Q_1 have multiplicity greater than 1 since in either case there will be some zeros of $P_i(x)$ on the complex arc which is a contradiction. Therefore, the polynomial $Q_2(x)$ must be non-negative at the zeros of $Q_1(x)$ as a necessary condition for the reality of all the zeros of $P_i(x)$ for all i. Furthermore all the zeros of $Q_1(x)$ must be simple otherwise, as explained above, some zeros of $P_i(x)$ would be on the complex arc. (This last part settles part (a) of the theorem where we require all the zeros of $P_i(x)$ to be simple.) □

Remark. Each of the conditions of Theorem 4 (a)–(e) is only a necessary (and not a sufficient) condition for the reality of all the zeros of $P_i(x)$. To guarantee the reality of all the zeros of $P_i(x)$ for all i, all the five conditions must be satisfied simultaneously. Some of the examples illustrating this claim are given below.

Example 1. Consider the sequence of polynomials $\{P_i(x)\}$ generated by the rational function

$$\sum_{i=0}^{\infty} P_i(x)t^i = \frac{1}{1 + (-x^2 + 2x)t + (5x^2 - 1)t^2}.$$

The corresponding f is given by

$$f(x) = \frac{(-x^2 + 2x)^2}{5x^2 - 1}.$$

The zeros of $Q_1(x)$ are 0 and 2 which are real and simple (see Theorem 4 (a)). There are no ovals disjoint with $\mathbb{R}P^1$ (see Theorem 4 (b)). The discriminant $D(x) = x^4 - 4x^3 - 16x^2 + 4$ has only real zeros (see Theorem 4 (c)). These are -2.39337, -0.54374, 0.47570 and 6.46141 (rounded to 5 decimal places and indicated by black dots in Figure 3). However, as seen from Figure 3, not all the zeros of $P_{100}(x)$ are real. This shows that if the above three conditions are satisfied they are not sufficient to guarantee that all the zeros of $P_i(x)$ will be real for all i.

Example 2. Consider the sequence of polynomials $\{P_i(x)\}$ generated by the rational function

$$\sum_{i=0}^{\infty} P_i(x)t^i = \frac{1}{1 + (2x^2 - 8x + 6)t + (-5x^3 + 37x^2 - 43x - 21)t^2}.$$

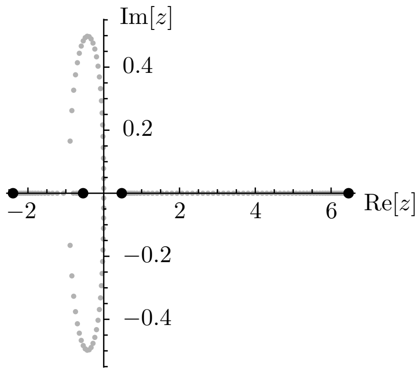

Figure 3. The zeros of $P_{100}(x)$ for the generating function $(1+(-x^2+2x)t+(5x^2-1)t^2)^{-1}$

The corresponding f is given by

$$f(x) = \frac{(2x^2-8x+6)^2}{-5x^3+37x^2-43x-21}.$$

The zeros of $Q_1(x)$ are 1 and 3 which are real and simple. Also there are no ovals disjoint with $\mathbb{R}P^1$. The discriminant $D(x) = 4(x^4-3x^3-15x^2+19x+30)$ has only real zeros, i.e. $x=-3$, $x=-1$, $x=2$ and $x=5$. However, f has a critical value of $3.50783 \in (0,4)$ corresponding to the critical point -1.66437 rounded to 5 decimal places, hence the condition of Theorem 4 (d) is violated. Consequently, some of the zeros of P_{100} are non-real (see Figure 4). The first three conditions of Theorem 4 are satisfied but not the fourth one. Therefore, having no critical value in $(0,4)$ is indeed a necessary condition for the reality of all the zeros of $P_i(x)$.

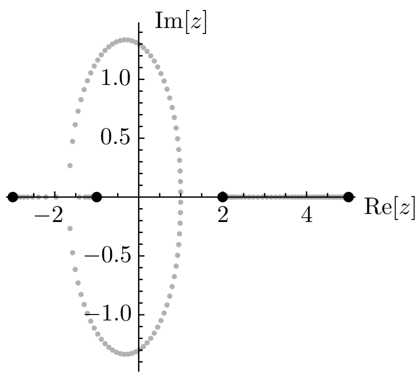

Figure 4. The zeros of $P_{100}(x)$ for the generating function $(1+(2x^2-8x+6)t+(-5x^3+37x^2-43x-21)t^2)^{-1}$

Example 3. Consider the sequence of polynomials $\{P_i(x)\}$ generated by the rational function

$$\sum_{i=0}^{\infty} P_i(x)t^i = \frac{1}{1+(2x^2-8x+6)t+(x^4-8x^3+21x^2-14x-16)t^2}.$$

The corresponding f is given by

$$f(x) = \frac{(2x^2-8x+6)^2}{x^4-8x^3+21x^2-14x-16}.$$

The zeros of $Q_1(x)$ are 1 and 3 which are real and simple. Also there are no ovals disjoint with $\mathbb{R}P^1$. The discriminant $D(x) = 4x^2 - 40x + 100$ has only real zeros, i.e., $x = 5$. In addition f has no critical value in the real interval $(0, 4)$. Thus the conditions of Theorem 4 (a) to (d) are satisfied. Note that on the zeros of Q_1 we have $Q_2(1) = -16 \not> 0$ and $Q_2(3) = -4 \not> 0$. Thus condition (e) of Theorem 4 is violated. Consequently some of the zeros of P_{100} are non-real as seen in Figure 5.

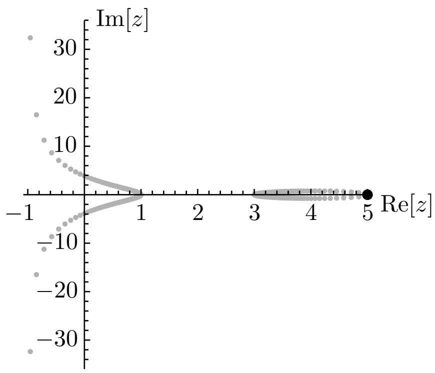

Figure 5. The zeros of $P_{100}(x)$ for the generating function $(1+(2x^2-8x+6)t+(x^4-8x^3+21x^2-14x-16)t^2)^{-1}$

Example 4. Consider the sequence of polynomials $\{P_i(x)\}$ generated by the rational function

$$\sum_{i=0}^{\infty} P_i(x)t^i = \frac{1}{1+(x^2-2x-5)t+x^2t^2}$$

The corresponding f is given by

$$f(x) = \frac{(x^2-2x-5)^2}{x^2}.$$

In this case, all the five conditions of Theorem 4 are satisfied and as seen in Figure 6, all the zeros of $P_{200}(x)$ are real. We used $i = 200$, but an arbitrary value of $i \in \mathbb{N}^+$ works. The black dots are the zeros of the discriminant and these are the endpoints of the intervals where all the zeros of $P_i(x)$ for all i are located, that is, all the zeros of $P_i(x)$ for all i are supported on the real axis and the support is a union of two disjoint real intervals given by $[-\sqrt{5}, -1] \cup [\sqrt{5}, 5] \subset \mathbb{R}P^1$. The zeros of $P_i(x)$ are dense on this support as $i \to \infty$.

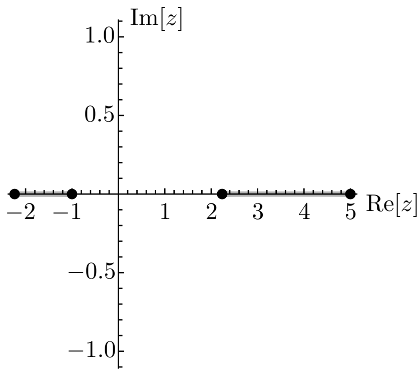

Figure 6. The zeros of $P_{200}(x)$ when the generating function is $(1+(x^2-2x-5)t+x^2t^2)^{-1}$.

3. Final remarks

Problem. Describe similar conditions guaranteeing reality of roots for all polynomials in the context of Theorem 3.

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