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INEQUALITIES FOR GENERAL WIDTH-INTEGRALS OF BLASCHKE-MINKOWSKI HOMOMORPHISMS

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Abstract. We establish some inequalities for general width-integrals of Blaschke-Minkowski homomorphisms. As applications, inequalities for width-integrals of projection bodies are derived.

Keywords: general width-integral; volume difference type inequality; Blaschke-Minkowski homomorphism; Brunn-Minkowski type inequality; projection body

MSC 2020: 52A20, 52A40

1. Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. The n-dimensional volume of the body $M \in \mathcal{K}^n$ is denoted by $V(M)$. For the standard unit ball U, write $V(U) = \omega_n$. The unit sphere, i.e. the boundary of U, is denoted by S^{n-1} and the surface area measure on S^{n-1} is denoted by $S(\cdot)$.

A convex body $M \in \mathcal{K}^n$ is uniquely determined by its support function $h(M,\cdot)\colon \mathbb{R}^n \to \mathbb{R}$, which is defined by

$$h(M,x) = \max\{x \cdot y\colon\ y \in M\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$.

For $M, N \in \mathcal{K}^n$ and $\lambda, \mu \geqslant 0$ (not both zero), the Minkowski linear combination $\lambda M + \mu N$ of M and N is defined by

$$\lambda M + \mu N = \{\lambda x + \mu y\colon\ x \in M,\ y \in N\},$$

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which is equivalent to

$$h(\lambda M + \mu N, \cdot) = \lambda h(M, \cdot) + \mu h(N, \cdot). \tag{1.1}$$

We refer to the extensive monographs (see [12], [24]) for more background on convex geometry.

Width-integrals were first proposed by Blaschke, see [3]. In 1975, Lutwak in [20] introduced width-integrals of index i and the mixed width-integral for convex bodies, see [21]. In 2010, Lv in [23] studied the width-integral difference. Later on, Zhao and Mihály in [36] established some Brunn-Minkowski inequalities for width-integrals of mixed projection bodies.

In 2016, Feng in [6] introduced the concept of general mixed width-integrals for convex bodies, and established the inequality of isoperimetric type, the Aleksandrov-Fenchel type inequality and the cyclic inequality. He also considered the general width-integrals of order i and showed its related properties and inequalities. Recently, Zhou in [37] researched the general L_p-mixed width-integrals of convex bodies, and gave its extremal values and extended Feng's results. See [4], [9], [18], [30] for more related results on the width-integrals of convex bodies.

For $M_1, \ldots, M_n \in \mathcal{K}^n$, $\tau \in (-1, 1)$, the general mixed width-integral $B^{(\tau)}(M_1, \ldots, M_n)$ of $M_1, \ldots, M_n$ is defined by

$$B^{(\tau)}(M_1, \ldots, M_n) = \frac{1}{n} \int_{S^{n-1}} b^{(\tau)}(M_1, u) \ldots b^{(\tau)}(M_n, u) \, \mathrm{d}S(u), \tag{1.2}$$

where

$$b^{(\tau)}(M, u) = f_1(\tau) h(M, u) + f_2(\tau) h(M, -u) \tag{1.3}$$

for all $u \in S^{n-1}$ and the functions $f_1(\tau)$ and $f_2(\tau)$ are given by

$$f_1(\tau) = \frac{(1+\tau)^2}{2(1+\tau^2)}, \quad f_2(\tau) = \frac{(1-\tau)^2}{2(1+\tau^2)}. \tag{1.4}$$

Clearly,

$$f_1(\tau) + f_2(\tau) = 1. \tag{1.5}$$

The case $\tau = 0$ in (1.2) is just Lutwak's mixed width-integral $B(M_1, \ldots, M_n)$. Convex bodies M and N are said to have similar general width if there exists a constant $\lambda > 0$ such that $b^{(\tau)}(M, u) = \lambda b^{(\tau)}(N, u)$ for all $u \in S^{n-1}$.

Let $M_1 = \ldots = M_{n-i} = M$ and $M_{n-i+1} = \ldots = M_n = U$ in (1.2), and allow i to be any real and notice that $b^{(\tau)}(U, \cdot) = 1$. Then the general width-integrals of index i, $B_i^{(\tau)}(M)$, of $M \in \mathcal{K}^n$ is given by

$$B_i^{(\tau)}(M) = \frac{1}{n}\int_{S^{n-1}} b^{(\tau)}(M,u)^{n-i}\,\mathrm{d}S(u). \tag{1.6}$$

Obviously, $B_i^{(\tau)}(U) = \omega_n$ and when $i = n$ in (1.6), we have

$$B_n^{(\tau)}(M) = \frac{1}{n}\int_{S^{n-1}} \mathrm{d}S(u) = \omega_n. \tag{1.7}$$

Among other results, Feng in [6] established the following:

Theorem 1.A. *If $\tau \in (-1,1)$ and $M \in \mathcal{K}^n$, then*

$$B_{2n}^{(\tau)}(M) \leqslant V(M^*)$$

with equality if and only if M is origin-symmetric.

Here M^* denotes polar body of M which is defined by (see [12], [24]): If M is a convex body that contains the origin in the interior, then

$$M^* = \{x \in \mathbb{R}^n \colon\ x \cdot y \leqslant 1,\ y \in M\}.$$

Notice that the case $\tau = 0$ of Theorem 1.A was given by Lutwak, see [20].

The projection bodies were introduced by Minkowski at the turn of the previous century. For every $M \in \mathcal{K}^n$, the projection body ΠM of M is an origin-symmetric convex body whose support function is defined by (see [12])

$$h(\Pi M, u) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v|\,\mathrm{d}S(M,v)$$

for all $u \in S^{n-1}$. Here $S(M,\cdot)$ denotes the surface area measure of M. The projection body is a very important concept in the Brunn-Minkowski theory, see references [2], [13], [19], [22], [27], [28], [29].

Based on the properties of projection bodies, Schuster in [25] introduced the notion of Blaschke-Minkowski homomorphisms as follows:

A map $\Phi\colon\ \mathcal{K}^n \to \mathcal{K}^n$ is called a Blaschke-Minkowski homomorphism if it satisfies the following conditions:

(a) Φ is continuous,

(b) for all $M, N \in \mathcal{K}^n$,

$$\Phi(M\#N) = \Phi M + \Phi N, \tag{1.8}$$

(c) for all $M \in \mathcal{K}^n$ and every $\vartheta \in SO(n)$, $\Phi(\vartheta M) = \vartheta \Phi M$.

Here, $SO(n)$ is the group of rotations in n dimensions. $\Phi M + \Phi N$ denotes the Minkowski sum of ΦM and ΦN, moreover, $M\#N$ denotes the Blaschke addition of convex bodies M and N, that is, the up to translation uniquely determined convex body with the surface area measure $S(M\#N, \cdot) = S(M, \cdot) + S(N, \cdot)$, see [12], [24].

A Blaschke-Minkowski homomorphism is a Minkowski valuation, i.e. a convex body valued valuation. Blaschke-Minkowski homomorphisms have attracted considerable interest, see for example [7], [8], [10], [15], [16], [26], [31], [32], [33], [34], [35].

In this paper, we continue the research on the general width-integrals of index i. First, we establish the following Brunn-Minkowski type inequalities.

Theorem 1.1. *Let $M, N \in \mathcal{K}^n$, $\tau \in (-1, 1)$, $i, j \in \mathbb{R}$ and $i \neq j$. If $i \leqslant n-1 \leqslant j \leqslant n$, then*

$$\left(\frac{B_i^{(\tau)}(\Phi(M\#N))}{B_j^{(\tau)}(\Phi(M\#N))}\right)^{1/(j-i)} \leqslant \left(\frac{B_i^{(\tau)}(\Phi M)}{B_j^{(\tau)}(\Phi M)}\right)^{1/(j-i)} + \left(\frac{B_i^{(\tau)}(\Phi N)}{B_j^{(\tau)}(\Phi N)}\right)^{1/(j-i)}, \tag{1.9}$$

if $n-1 \leqslant i \leqslant n \leqslant j$, then inequality (1.9) is reversed. Equality holds in (1.9) for $i \neq n-1$ or $j \neq n$ if and only if ΦM and ΦN have similar general width. For $i = n-1$ and $j = n$, (1.9) is an identity.

Let $j = n$ in (1.9). Combining with (1.7), we have the following fact:

Corollary 1.1. *If $M, N \in \mathcal{K}^n$, i is any real and $\tau \in (-1, 1)$, then for $i < n-1$,*

$$B_i^{(\tau)}(\Phi(M\#N))^{1/(n-i)} \leqslant B_i^{(\tau)}(\Phi M)^{1/(n-i)} + B_i^{(\tau)}(\Phi N)^{1/(n-i)} \tag{1.10}$$

with equality if and only if ΦM and ΦN have similar general width; for $i > n-1$ and $i \neq n$, inequality (1.10) is reversed. For $i = n-1$, (1.10) is an identity.

Since the projection body is a special example of a Blaschke-Minkowski homomorphisms, by Corollary 1.1 we obtain the following result:

Corollary 1.2. *Let $M, N \in \mathcal{K}^n$, i be any real and $\tau \in (-1,1)$. Then for $i < n-1$,*

$$B_i^{(\tau)}(\Pi(M\#N))^{1/(n-i)} \leqslant B_i^{(\tau)}(\Pi M)^{1/(n-i)} + B_i^{(\tau)}(\Pi N)^{1/(n-i)};$$

this inequality is reversed for $i > n-1$ and $n \neq i$. Equality holds if and only if ΠM and ΠN have similar general width. For $i = n-1$, this inequality is an identity.

Since ΠM is origin-symmetric, we have that $B_{2n}^{(\tau)}(\Pi M) = V(\Pi^* M)$ by the equality conditions of Theorem 1.A. Thus, if $i = 2n$, then Corollary 1.2 yields:

Corollary 1.3. *If $M, N \in \mathcal{K}^n$, then*

$$V(\Pi^*(M\#N))^{-1/n} \geqslant V(\Pi^* M)^{-1/n} + V(\Pi^* N)^{-1/n}$$

with equality if and only if ΠM and ΠN are homothetic.

In addition, if $K = \Pi M$ and $L = \Pi N$, then $(K + L)^* = \Pi^*(M\#N)$ since $\Pi M + \Pi N = \Pi(M\#N)$. Hence, Corollary 1.3 can also be obtained directly by the following classical result of Firey in [11] (also see the case of $i = 0$ in [14], Theorem 1.1).

Corollary 1.4. *If K and L are convex bodies that contain the origin in their interior, then*

$$V((K+L)^*)^{-1/n} \geqslant V(K^*)^{-1/n} + V(L^*)^{-1/n}$$

with equality if and only if K and L are dilates.

Next, we establish another form of the Brunn-Minkowski type inequality for general width-integrals.

Theorem 1.2. *If $M \in \mathcal{K}^n$ and N is a ball in $\mathbb{R}^n$, $\tau \in (-1,1)$, then for all $i = 0, \ldots, n-1$,*

$$\frac{B_i^{(\tau)}(\Phi(M\#N))}{B_{i+1}^{(\tau)}(\Phi(M\#N))} \leqslant \frac{B_i^{(\tau)}(\Phi M)}{B_{i+1}^{(\tau)}(\Phi M)} + \frac{B_i^{(\tau)}(\Phi N)}{B_{i+1}^{(\tau)}(\Phi N)}. \tag{1.11}$$

Finally, as an application of Corollary 1.1 and its equality condition, we give an analogous version of the volume differences inequality, which is related to the Blaschke-Minkowski homomorphism for the general width-integral of index i.

Theorem 1.3. *Let $M,N,D,D'\in\mathcal{K}^n$, $\tau\in(-1,1)$, $i\in\mathbb{R}$ and $\Phi D\subset\Phi M$, $\Phi D'\subset\Phi N$, ΦM and ΦN have similar general width. If $i<n-1$, then*

$$\begin{aligned}(1.12)\quad &(B_i^{(\tau)}(\Phi(M\#N))-B_i^{(\tau)}(\Phi(D\#D')))^{1/(n-i)}\\ &\qquad\geqslant (B_i^{(\tau)}(\Phi M)-B_i^{(\tau)}(\Phi D))^{1/(n-i)}+(B_i^{(\tau)}(\Phi N)-B_i^{(\tau)}(\Phi D'))^{1/(n-i)},\end{aligned}$$

if $i>n-1$ and $i\neq n$, inequality (1.12) *is reversed. Equality holds in* (1.12) *if and only if ΦD and $\Phi D'$ have similar general width and $(B_i^{(\tau)}(\Phi M),B_i^{(\tau)}(\Phi D))=\lambda(B_i^{(\tau)}(\Phi N),B_i^{(\tau)}(\Phi D'))$, where λ is a constant. For $i=n-1$,* (1.12) *is an identity.*

2. Proofs of results

In this part, we will give the proofs of Theorems 1.1–1.3. First, in order to prove Theorem 1.1, the following lemmas are required.

Lemma 2.1 (The Beckenbach-Dresher inequality [1], [5]). *Let functions $f,g\geqslant 0$, $\mathbb{E}$ be a bounded measurable subset in $\mathbb{R}^n$ and φ be a distribution function. If $p\geqslant 1\geqslant r\geqslant 0$, then*

$$(2.1)\qquad \left(\frac{\int_{\mathbb{E}}(f+g)^p\,\mathrm{d}\varphi}{\int_{\mathbb{E}}(f+g)^r\,\mathrm{d}\varphi}\right)^{1/(p-r)}\leqslant\left(\frac{\int_{\mathbb{E}}f^p\,\mathrm{d}\varphi}{\int_{\mathbb{E}}f^r\,\mathrm{d}\varphi}\right)^{1/(p-r)}+\left(\frac{\int_{\mathbb{E}}g^p\,\mathrm{d}\varphi}{\int_{\mathbb{E}}g^r\,\mathrm{d}\varphi}\right)^{1/(p-r)}$$

with equality if and only if the functions f and g are positively proportional. For $p=1$ and $r=0$, (2.1) *is an identity.*

Lemma 2.2 (The inverse Beckenbach-Dresher inequality [17]). *Let functions $f,g\geqslant 0$, $\mathbb{E}$ be a bounded measurable subset in $\mathbb{R}^n$ and φ be a distribution function. If $1\geqslant p\geqslant 0\geqslant r$, then*

$$(2.2)\qquad \left(\frac{\int_{\mathbb{E}}(f+g)^p\,\mathrm{d}\varphi}{\int_{\mathbb{E}}(f+g)^r\,\mathrm{d}\varphi}\right)^{1/(p-r)}\geqslant\left(\frac{\int_{\mathbb{E}}f^p\,\mathrm{d}\varphi}{\int_{\mathbb{E}}f^r\,\mathrm{d}\varphi}\right)^{1/(p-r)}+\left(\frac{\int_{\mathbb{E}}g^p\,\mathrm{d}\varphi}{\int_{\mathbb{E}}g^r\,\mathrm{d}\varphi}\right)^{1/(p-r)}$$

with equality if and only if the functions f and g are positively proportional. For $p=1$ and $r=0$, (2.2) *is an identity.*

P r o o f of Theorem 1.1. Since $i\leqslant n-1\leqslant j\leqslant n$ and $i\neq j$, let $p=n-i$, $r=n-j$, then $0\leqslant r\leqslant 1\leqslant p$ and $p\neq r$, combining with (1.1), (1.3), (1.5), (1.6) and (1.8), we get for $M, N \in \mathcal{K}^n$ and $\tau \in (-1, 1)$,

$$\begin{aligned}
(2.3)\qquad B_{n-p}^{(\tau)}(\Phi(M\#N)) &= \frac{1}{n}\int_{S^{n-1}} b^{(\tau)}(\Phi(M\#N), u)^p \,\mathrm{d}S(u) \\
&= \frac{1}{n}\int_{S^{n-1}} (f_1(\tau)h(\Phi(M\#N), u) + f_2(\tau)h(\Phi(M\#N), -u))^p \,\mathrm{d}S(u) \\
&= \frac{1}{n}\int_{S^{n-1}} (f_1(\tau)h(\Phi M + \Phi N, u) + f_2(\tau)h(\Phi M + \Phi N, -u))^p \,\mathrm{d}S(u) \\
&= \frac{1}{n}\int_{S^{n-1}} (b^{(\tau)}(\Phi M, u) + b^{(\tau)}(\Phi N, u))^p \,\mathrm{d}S(u).
\end{aligned}$$

Similarly,

$$(2.4)\qquad B_{n-r}^{(\tau)}(\Phi(M\#N)) = \frac{1}{n}\int_{S^{n-1}} (b^{(\tau)}(\Phi M, u) + b^{(\tau)}(\Phi N, u))^r \,\mathrm{d}S(u).$$

From (2.1), (2.3) and (2.4) we have

$$\begin{aligned}
(2.5)\qquad &\left(\frac{B_{n-p}^{(\tau)}(\Phi(M\#N))}{B_{n-r}^{(\tau)}(\Phi(M\#N))}\right)^{1/(p-r)} \\
&= \left(\frac{\int_{S^{n-1}} (b^{(\tau)}(\Phi M, u) + b^{(\tau)}(\Phi N, u))^p \,\mathrm{d}S(u)}{\int_{S^{n-1}} (b^{(\tau)}(\Phi M, u) + b^{(\tau)}(\Phi N, u))^r \,\mathrm{d}S(u)}\right)^{1/(p-r)} \\
&\leqslant \left(\frac{\int_{S^{n-1}} b^{(\tau)}(\Phi M, u)^p \,\mathrm{d}S(u)}{\int_{S^{n-1}} b^{(\tau)}(\Phi M, u)^r \,\mathrm{d}S(u)}\right)^{1/(p-r)} + \left(\frac{\int_{S^{n-1}} b^{(\tau)}(\Phi N, u)^p \,\mathrm{d}S(u)}{\int_{S^{n-1}} b^{(\tau)}(\Phi N, u)^r \,\mathrm{d}S(u)}\right)^{1/(p-r)} \\
&= \left(\frac{B_{n-p}^{(\tau)}(\Phi M)}{B_{n-r}^{(\tau)}(\Phi M)}\right)^{1/(p-r)} + \left(\frac{B_{n-p}^{(\tau)}(\Phi N)}{B_{n-r}^{(\tau)}(\Phi N)}\right)^{1/(p-r)}.
\end{aligned}$$

Let $i = n - p$ and $j = n - r$ in (2.5), then inequality (1.9) is given.

Similarly to the above method for $n - 1 \leqslant i \leqslant n \leqslant j$ the inverse of (1.9) follows from inequalities (2.2), (2.3) and (2.4).

The equality condition of inequality (2.1) implies that equality holds in (1.9) for $i \neq n - 1$ or $j \neq n$ and any $u \in S^{n-1}$ if and only if $b^{(\tau)}(\Phi M, u)$ and $b^{(\tau)}(\Phi N, u)$ are positively proportional, i.e. ΦM and ΦN have similar general width.

For $i = n - 1$ and $j = n$, by (1.7) we get that (1.9) is an identity. $\square$

Because the projection body is a special example of the Blaschke-Minkowski homomorphism, by Theorem 1.1 for $\tau = 0$ we can obtain the following result:

Corollary 2.1. *If $M, N \in \mathcal{K}^n$ and $i, j \in \mathbb{R}$, then for $i \leqslant n-1 \leqslant j \leqslant n$ and $i \neq j$,*

$$\left(\frac{B_i(\Pi(M\#N))}{B_j(\Pi(M\#N))}\right)^{1/(j-i)} \leqslant \left(\frac{B_i(\Pi M)}{B_j(\Pi M)}\right)^{1/(j-i)} + \left(\frac{B_i(\Pi N)}{B_j(\Pi N)}\right)^{1/(j-i)};$$

this inequality is reversed for $n-1 \leqslant i \leqslant n \leqslant j$ and $i \neq j$. Equality holds for $i \neq n-1$ and $j \neq n$ if and only if ΠM and ΠN have similar width.

Subsequently, we will give the proof of Theorem 1.2. The following lemmas are necessary.

Lemma 2.3 ([6]). *If $M \in \mathcal{K}^n$, $\tau \in (-1,1)$, $j = 0, 1, \ldots, n$ and $\mu > 0$, then*

$$B_j^{(\tau)}(M_\mu) = \sum_{i=0}^{n-j} \binom{n-j}{i} B_{j+i}^{(\tau)}(M)\mu^i,$$

where $M_\mu = M + \mu U$.

Lemma 2.4 ([6]). *If $M \in \mathcal{K}^n$, $\tau \in (-1,1)$ and reals i, j, k satisfy $i < j < k$, then*

$$B_j^{(\tau)}(M)^{k-i} \leqslant B_i^{(\tau)}(M)^{k-j} B_k^{(\tau)}(M)^{j-i}$$

with equality if and only if M is a ball in $\mathbb{R}^n$.

P r o o f of Theorem 1.2. For $M \in \mathcal{K}^n$, $\tau \in (-1,1)$ and $i = 0, 1, \ldots, n-2$, let

$$g_i(\mu) = B_i^{(\tau)}(\Phi M + \mu U), \quad \mu > 0. \tag{2.6}$$

From Lemma 2.3 it follows that

$$\begin{aligned} g_i(\mu + \varepsilon) &= B_i^{(\tau)}(\Phi M + \mu U + \varepsilon U) \\ &= B_i^{(\tau)}(\Phi M + \mu U) + \varepsilon(n-i)B_{i+1}^{(\tau)}(\Phi M + \mu U) + o(\varepsilon^2) \\ &= g_i(\mu) + \varepsilon(n-i)g_{i+1}(\mu) + o(\varepsilon^2). \end{aligned}$$

Thus,

$$g_i'(\mu) = (n-i)g_{i+1}(\mu).$$

By Lemma 2.4, for $i = 0, 1, \ldots, n-2$ we have

$$B_{i+1}^{(\tau)}(\Phi M + \mu U)^2 \leqslant B_i^{(\tau)}(\Phi M + \mu U) B_{i+2}^{(\tau)}(\Phi M + \mu U),$$

i.e.

$$g_{i+1}(\mu)^2 \leqslant g_i(\mu) g_{i+2}(\mu).$$

Now, we define

$$G_i(\mu) = \frac{g_i(\mu)}{g_{i+1}(\mu)}, \quad i = 0, 1, \ldots, n-2. \tag{2.7}$$

This implies

$$\begin{aligned} G_i'(\mu) &= \frac{g_i'(\mu) g_{i+1}(\mu) - g_i(\mu) g_{i+1}'(\mu)}{g_{i+1}(\mu)^2} \\ &= \frac{g_{i+1}(\mu)^2 + (n-i-1)(g_{i+1}(\mu)^2 - g_i(\mu) g_{i+2}(\mu))}{g_{i+1}(\mu)^2} \leqslant 1. \end{aligned}$$

Thus, for $\lambda > 0$ we obtain

$$\int_0^\lambda G_i'(\mu)\,\mathrm{d}\mu \leqslant \int_0^\lambda 1\,\mathrm{d}\mu,$$

i.e.

$$G_i(\lambda) \leqslant G_i(0) + \lambda. \tag{2.8}$$

From (2.6), (2.7) and (2.8), for $i = 0, 1, \ldots, n-2$, we have

$$\frac{B_i^{(\tau)}(\Phi M + \lambda U)}{B_{i+1}^{(\tau)}(\Phi M + \lambda U)} \leqslant \frac{B_i^{(\tau)}(\Phi M)}{B_{i+1}^{(\tau)}(\Phi M)} + \lambda. \tag{2.9}$$

But for the standard unit ball U we know $\Phi U = \lambda U$ for some $\lambda > 0$ (see [25], page 224). Hence, if N is a ball in $\mathbb{R}^n$, then ΦN is also a ball. From this, let $\Phi N = \lambda U$, and by (1.3) and (1.6) we obtain

$$\frac{B_i^{(\tau)}(\Phi N)}{B_{i+1}^{(\tau)}(\Phi N)} = \frac{B_i^{(\tau)}(\lambda U)}{B_{i+1}^{(\tau)}(\lambda U)} = \frac{n^{-1}\int_{S^{n-1}} b^{(\tau)}(\lambda U, u)^{n-i}\,\mathrm{d}S(u)}{n^{-1}\int_{S^{n-1}} b^{(\tau)}(\lambda U, u)^{n-i-1}\,\mathrm{d}S(u)} = \frac{\lambda^{n-i}}{\lambda^{n-i-1}} = \lambda.$$

This, together with (1.8) and (2.9), yields

$$\frac{B_i^{(\tau)}(\Phi(M\#N))}{B_{i+1}^{(\tau)}(\Phi(M\#N))} \leqslant \frac{B_i^{(\tau)}(\Phi M)}{B_{i+1}^{(\tau)}(\Phi M)} + \frac{B_i^{(\tau)}(\Phi N)}{B_{i+1}^{(\tau)}(\Phi N)}.$$

This is just inequality (1.11). □

Finally, by the following Bellman's inequality, we give the proof of Theorem 1.3.

Lemma 2.5 (Bellman's inequality [1]). *Let* $\mathbf{a} = \{a_1, \ldots, a_n\}$ *and* $\mathbf{b} = \{b_1, \ldots, b_n\}$ *be two series of positive real numbers. If* $a_1^p - \sum\limits_{i=2}^{n} a_i^p > 0$, $b_1^p - \sum\limits_{i=2}^{n} b_i^p > 0$, *then for* $p > 1$,

$$\left(a_1^p - \sum_{i=2}^{n} a_i^p\right)^{1/p} + \left(b_1^p - \sum_{i=2}^{n} b_i^p\right)^{1/p} \leqslant \left((a_1+b_1)^p - \sum_{i=2}^{n}(a_i+b_i)^p\right)^{1/p}.$$

This inequality is reversed for $p < 0$ *or* $0 < p < 1$ *with equality if and only if* $\mathbf{a} = c\mathbf{b}$, *where* c *is a constant.*

P r o o f of Theorem 1.3. For $M, N, D, D' \in \mathcal{K}^n$ and $\tau \in (-1,1)$, if $i < n-1$, then by (1.10),

$$B_i^{(\tau)}(\Phi(D\#D'))^{1/(n-i)} \leqslant B_i^{(\tau)}(\Phi D)^{1/(n-i)} + B_i^{(\tau)}(\Phi D')^{1/(n-i)} \tag{2.10}$$

with equality if and only if ΦD and $\Phi D'$ have similar general width. Since ΦM and ΦN have similar general width, according to the equality condition of inequality (1.10), we have

$$B_i^{(\tau)}(\Phi(M\#N))^{1/(n-i)} = B_i^{(\tau)}(\Phi M)^{1/(n-i)} + B_i^{(\tau)}(\Phi N)^{1/(n-i)}. \tag{2.11}$$

Since $\Phi D \subset \Phi M$, $\Phi D' \subset \Phi N$ and by formulæ (1.6) and (1.8), we conclude

$$B_i^{(\tau)}(\Phi M) > B_i^{(\tau)}(\Phi D), \quad B_i^{(\tau)}(\Phi N) > B_i^{(\tau)}(\Phi D'),$$
$$B_i^{(\tau)}(\Phi(M\#N)) = B_i^{(\tau)}(\Phi M + \Phi N) > B_i^{(\tau)}(\Phi D + \Phi D') = B_i^{(\tau)}(\Phi(D\#D')).$$

From these, since $n - i > 1$ and by (2.10), (2.11) and Lemma 2.5, we obtain

$$\begin{aligned}
&(B_i^{(\tau)}(\Phi(M\#N)) - B_i^{(\tau)}(\Phi(D\#D')))^{1/(n-i)} \\
&\quad \geqslant [(B_i^{(\tau)}(\Phi M)^{1/(n-i)} + B_i^{(\tau)}(\Phi N)^{1/(n-i)})^{n-i} \\
&\qquad - (B_i^{(\tau)}(\Phi D)^{1/(n-i)} + B_i^{(\tau)}(\Phi D')^{1/(n-i)})^{n-i}]^{1/(n-i)} \\
&\quad \geqslant (B_i^{(\tau)}(\Phi M) - B_i^{(\tau)}(\Phi D))^{1/(n-i)} + (B_i^{(\tau)}(\Phi N) - B_i^{(\tau)}(\Phi D'))^{1/(n-i)}.
\end{aligned}$$

This yields inequality (1.12).

Along the same line for $i > n-1$ and $i \neq n$ the reversed inequality of (1.12) can be deduced directly from (1.10) and the reversed case of Lemma 2.5.

By the equality conditions of inequalities (1.10) and Lemma 2.5, we see that equality holds in (1.12) if and only if ΦD and $\Phi D'$ have similar general width and there exists constant λ such that $(B_i^{(\tau)}(\Phi M), B_i^{(\tau)}(\Phi D)) = \lambda(B_i^{(\tau)}(\Phi N), B_i^{(\tau)}(\Phi D'))$. For $i = n-1$, (1.12) is an identity. □

Combined with the projection bodies, let $\tau = 0$ in Theorem 1.3 and we obtain the following result.

Corollary 2.2. *Let $M, N, D, D' \in \mathcal{K}^n$, $i \in \mathbb{R}$ and $D \subset M$, $D' \subset N$, ΠM and ΠN have similar width. If $i < n-1$, then*

$$(B_i(\Pi(M\#N)) - B_i(\Pi(D\#D')))^{1/(n-i)} \geqslant (B_i(\Pi M) - B_i(\Pi D))^{1/(n-i)} + (B_i(\Pi N) - B_i(\Pi D'))^{1/(n-i)};$$

this inequality is reversed for $i > n-1$ and $i \neq n$ and equality holds if and only if ΠD and $\Pi D'$ have similar width and $(B_i(\Pi M), B_i(\Pi D)) = \lambda(B_i(\Pi N), B_i(\Pi D'))$, where λ is a constant. For $i = n-1$, this inequality is an identity.

Since the projection body is a Blaschke-Minkowski homomorphism and since ΠM is origin-symmetric, we obtain the following result of Lv's (see [23]) as a special case of Theorem 1.3, i.e. for $i = 2n$ and since $B_{2n}^{(\tau)}(\Pi M) = V(\Pi^* M)$ by Theorem 1.A.

Corollary 2.3. *Let $M, N, D, D' \in \mathcal{K}^n$. If $D \subset M$, $D' \subset N$ and M is a homothetic copy of N, then*

$$(V(\Pi^*(M\#N)) - V(\Pi^*(D\#D')))^{-1/n} \leqslant (V(\Pi^* M) - V(\Pi^* D))^{-1/n} + (V(\Pi^* N) - V(\Pi^* D'))^{-1/n}$$

with equality if and only if D and D' are homothetic and $(V(\Pi^* M), V(\Pi^* D)) = \lambda(V(\Pi^* N), V(\Pi^* D'))$, where λ is a constant.

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