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q-ANALOGUES OF TWO SUPERCONGRUENCES OF Z.-W. SUN

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Abstract. We give several different q-analogues of the following two congruences of Z.-W. Sun:

$$\sum_{k=0}^{(p^r-1)/2} \frac{1}{8^k}\binom{2k}{k} \equiv \Big(\frac{2}{p^r}\Big) \pmod{p^2} \quad \text{and} \quad \sum_{k=0}^{(p^r-1)/2} \frac{1}{16^k}\binom{2k}{k} \equiv \Big(\frac{3}{p^r}\Big) \pmod{p^2},$$

where p is an odd prime, r is a positive integer, and $(\frac{m}{n})$ is the Jacobi symbol. The proofs of them require the use of some curious q-series identities, two of which are related to Franklin's involution on partitions into distinct parts. We also confirm a conjecture of the latter author and Zeng in 2012.

Keywords: congruences; q-binomial coefficient; cyclotomic polynomial; Franklin's involution

MSC 2020: 11B65, 05A10, 05A30, 11A07

1. Introduction

Among other things, Sun in [14], (1.7) and (1.8) proved the congruences

$$\sum_{k=0}^{(p^r-1)/2} \frac{1}{8^k}\binom{2k}{k} \equiv \Big(\frac{2}{p^r}\Big) \pmod{p^2}, \tag{1.1}$$

$$\sum_{k=0}^{(p^r-1)/2} \frac{1}{16^k}\binom{2k}{k} \equiv \Big(\frac{3}{p^r}\Big) \pmod{p^2}, \tag{1.2}$$

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where p is an odd prime, r is a positive integer, and $(\frac{m}{n})$ is the Jacobi symbol. Recently, the latter author and Liu in [6], Theorem 1.2 gave the following q-analogue of (1.1): for odd n,

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k q^{k^2}}{(q^4;q^4)_k} \equiv (-q)^{(1-n^2)/8} \pmod{\Phi_n(q)^2}. \tag{1.3}$$

Here and in what follows, $(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$ and $\Phi_n(q)$ is the nth cyclotomic polynomial in q.

The first aim of this paper is to give q-analogues of (1.1) and (1.2) as follows.

Theorem 1.1. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k q^{2k}}{(q^2;q^2)_k(-q;q^2)_k} \equiv \left(\frac{2}{n}\right) q^{2\lfloor (n+1)/4\rfloor^2} \pmod{\Phi_n(q)^2}, \tag{1.4}$$

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k q^{2k}}{(q^4;q^4)_k(-q;q^2)_k} \equiv \left(\frac{3}{n}\right) q^{(n^2-1)/12} \pmod{\Phi_n(q)^2}, \tag{1.5}$$

where $\lfloor x\rfloor$ denotes the largest integer not exceeding x.

It is easy to see that the congruences (1.4) and (1.5) reduce to (1.1) and (1.2), respectively, when $q \to 1$ and $n = p^r$.

Recall that the *q-binomial coefficients* $\begin{bmatrix} n \\ k \end{bmatrix}$ are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} & \text{if } 0 \leqslant k \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the *q-integer* is defined as $[n] = [n]_q = (1-q^n)/(1-q)$. The second aim of this paper is to give the following result, which in the case $n = p^r$ confirms a conjecture of the latter author and Zeng, see [8], Conjecture 5.13.

Theorem 1.2. *Let n be a positive integer. Then*

$$\sum_{k=0}^{n-1} q^{(n-k)^2} \begin{bmatrix} n+k \\ k \end{bmatrix}^2 \begin{bmatrix} n-1 \\ k \end{bmatrix}^2 \equiv q[n] \pmod{\Phi_n(q)^2}. \tag{1.6}$$

Note that, exactly similarly to the proof of Theorem 5.3 in [8], we can show that

$$\sum_{k=0}^{n-1} q^{(n-k)^2} \begin{bmatrix} n+k \\ k \end{bmatrix}^2 \begin{bmatrix} n-1 \\ k \end{bmatrix}^2 \equiv 0 \pmod{[n]}. \tag{1.7}$$

Therefore, combining the congruences (1.6) and (1.7), we see that the congruence (1.6) also holds modulo $[n]\Phi_n(q)$. We refer the reader to [7] and references therein for other congruences on sums involving q-binomial coefficients.

Suggested by the referee, we would like to make the following conjecture.

Conjecture 1.3. *The congruence* (1.6) *still holds modulo* $[n]\Phi_n(q)^2$.

The paper is organized as follows. In the next section, we give a new proof of a curious q-series identity of Liu, see [12] and also provide two similar identities. We prove Theorems 1.1 and 1.2 in Sections 3 and 4, respectively. Finally in Section 5, motivated by the recent work of the latter author and Zudilin, see [9], we give parameter generalizations of (1.3)–(1.5) and show more complicated q-analogues of (1.1) and (1.2).

2. A curious q-series identity of J.-C. Liu

Liu in [12], (2.3) presented the following q-series identity:

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} 2n-k \\ k \end{bmatrix} (-q;q)_{n-k} = (-1)^{\binom{n}{2}} q^{\binom{n}{2}+\lfloor (n+1)/2 \rfloor^2}, \tag{2.1}$$

which will be used in our proof of (1.4). We give a new proof (2.1) here for two reasons. Firstly, Liu's proof of (1.4) is a little complicated and cannot be generalized to prove similar identities. Secondly, we want the paper to be more self-contained.

Proof of (2.1). By the q-binomial theorem (see, for example, [1], page 36, Theorem 3.3), we have

$$(x;q)_N = \sum_{k=0}^{N} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} N \\ k \end{bmatrix} x^k, \qquad \frac{1}{(x;q)_N} = \sum_{k=0}^{\infty} \begin{bmatrix} N+k-1 \\ k \end{bmatrix} x^k,$$

and so

$$\left(\sum_{k=0}^{n+a} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n+a \\ k \end{bmatrix} x^k\right) \left(\sum_{k=0}^{\infty} \begin{bmatrix} n+a+k \\ k \end{bmatrix} x^k\right) = \frac{(x;q)_{n+a}}{(x;q)_{n+a+1}} = \frac{1}{1-xq^{n+a}}. \tag{2.2}$$

Equating the coefficients of x^{n-a} on both sides of (2.2), we obtain

$$\sum_{k=0}^{n-a}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n+a\\k\end{bmatrix}\begin{bmatrix}2n-k\\n-a-k\end{bmatrix}=q^{n^2-a^2},$$

which can be written as

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}2n-k\\k\end{bmatrix}\begin{bmatrix}2n-2k\\n-k+a\end{bmatrix}=q^{n^2-a^2}. \tag{2.3}$$

With the help of (2.3), we are now able to prove (2.1). By Slater's Bailey pair C(1) in [13], we have

$$(-q;q)_n=\sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^k q^{3k^2+k}\begin{bmatrix}2n\\n+2k\end{bmatrix}.$$

It follows that

$$\begin{aligned}\sum_{k=0}^{n}&(-1)^k q^{\binom{k}{2}}\begin{bmatrix}2n-k\\k\end{bmatrix}(-q;q)_{n-k}\\&=\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}2n-k\\k\end{bmatrix}\sum_{j=-\lfloor (n-k)/2\rfloor}^{\lfloor (n-k)/2\rfloor}(-1)^j q^{3j^2+j}\begin{bmatrix}2n-2k\\n-k+2j\end{bmatrix}.\end{aligned} \tag{2.4}$$

Interchanging the summation order on the right-hand side of (2.4) and using (2.3), we get

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}2n-k\\k\end{bmatrix}(-q;q)_{n-k}=\sum_{j=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^j q^{n^2-j^2+j}. \tag{2.5}$$

Since the jth and $(1-j)$th terms on the right-hand side of (2.5) cancel each other for $j=1,\dots,\lfloor n/2\rfloor$, only the term corresponding to $j=-\lfloor n/2\rfloor$ on the right-hand side of (2.5) survives. This proves (2.1). $\square$

Similarly, we can show that

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}}\begin{bmatrix}2n-k\\k\end{bmatrix}(-q;q)_{n-k}=\sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor}(-1)^k q^{3k^2+k}. \tag{2.6}$$

There are many more identities similar to (2.1) and (2.6). For example, using the identities (see [10], Proposition 2)

$$(-q;q^2)_n = \sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor} (-1)^k q^{2k^2} \begin{bmatrix} 2n \\ n+2k \end{bmatrix},$$

$$(1+q^n)(-q^2;q^2)_{n-1} = \sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor} (-1)^k q^{2k^2+k} \begin{bmatrix} 2n \\ n+2k \end{bmatrix},$$

we can prove that

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} 2n-k \\ k \end{bmatrix} (-q;q^2)_{n-k} q^{\binom{k}{2}} = \sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor} (-1)^k q^{n^2-2k^2}, \tag{2.7}$$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} 2n-k \\ k \end{bmatrix} (1+q^{n-k})(-q^2;q^2)_{n-k-1} q^{\binom{k}{2}} = \sum_{k=-\lfloor n/2\rfloor}^{\lfloor n/2\rfloor} (-1)^k q^{n^2-2k^2+k},$$

where $(-q^2;q^2)_{-1} = 1/2$.

3. Proof of Theorem 1.1

It is easy to see that

$$(1-q^{n-2j+1})(1-q^{n+2j-1}) + (1-q^{2j-1})^2 q^{n-2j+1} = (1-q^n)^2,$$

$1 - q^n \equiv 0 \pmod{\Phi_n(q)}$, and so

$$(1-q^{n-2j+1})(1-q^{n+2j-1}) \equiv -(1-q^{2j-1})^2 q^{n-2j+1} \pmod{\Phi_n(q)^2}.$$

Therefore,

$$\begin{aligned} (q^{1-n};q^2)_k (q^{n+1};q^2)_k &= (-1)^k \prod_{j=1}^{k} (1-q^{n-2j+1})(1-q^{n+2j-1}) \\ &\equiv q^{k^2-nk} \prod_{j=1}^{k} (1-q^{2j-1})^2 q^{n-2j+1} = (q;q^2)_k^2 \pmod{\Phi_n(q)^2}. \end{aligned} \tag{3.1}$$

It follows that

$$\begin{aligned}\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k q^{2k}}{(q^4;q^4)_k(-q;q^2)_k} &\equiv \sum_{k=0}^{(n-1)/2}\frac{(q^{1-n};q^2)_k(q^{n+1};q^2)_k}{(q^2;q^4)_k(q^4;q^4)_k}q^{2k}\\ &= \sum_{k=0}^{(n-1)/2}(-1)^k q^{k(k-n+2)}\begin{bmatrix}(n-1)/2+k\\(n-1)/2-k\end{bmatrix}_{q^2}\\ &= \sum_{k=0}^{(n-1)/2}(-1)^{(n-1)/2-k}q^{n-(n^2+3)/4+2\binom{k}{2}}\begin{bmatrix}n-1-k\\k\end{bmatrix}_{q^2}\\ &= (-1)^{(n-1)/2}\left(\frac{n}{3}\right)q^{(n^2-1)/12} \pmod{\Phi_n(q)^2}.\end{aligned}$$

The last equality holds because of the identity (see [2], [3], [4])

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n-k\\k\end{bmatrix} = \begin{cases}(-1)^{\lfloor n/3\rfloor}q^{n(n-1)/6}, & \text{if } n\not\equiv 2 \pmod 3,\\ 0, & \text{if } n\equiv 2 \pmod 3.\end{cases} \tag{3.2}$$

The proof of (1.5) then follows from the quadratic reciprocity law

$$\left(\frac{3}{n}\right) = (-1)^{(n-1)/2}\left(\frac{n}{3}\right).$$

Similarly, we have

$$\begin{aligned}&\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k q^{2k}}{(q^2;q^2)_k(-q;q^2)_k}\\ &\quad\equiv \sum_{k=0}^{(n-1)/2}(-1)^{(n-1)/2-k}q^{n-(n^2+3)/4+2\binom{k}{2}}\begin{bmatrix}n-1-k\\k\end{bmatrix}_{q^2}(-q^2;q^2)_{(n-1)/2-k}\\ &\quad= \left(\frac{2}{n}\right)q^{2\lfloor (n+1)/4\rfloor^2} \pmod{\Phi_n(q)^2}.\end{aligned}$$

The last equality follows from (2.1) by replacing n with $(n-1)/2$ and q with $q\to q^2$.

4. Proof of Theorem 1.2

We can easily prove the congruence

$$(q^{1-n};q)_k(q^{n+1};q)_k = (q;q)_k^2 \pmod{\Phi_n(q)^2}$$

similar to (3.1). It follows that

$$\begin{bmatrix} n+k \\ k \end{bmatrix}^2 \begin{bmatrix} n-1 \\ k \end{bmatrix}^2 = \frac{(q^{n+1};q)_k^2 (q^{1-n};q)_k^2}{(q;q)_k^4} q^{2nk-k^2-k} \equiv q^{2nk-k^2-k} \pmod{\Phi_n(q)^2}$$

and so

$$\sum_{k=0}^{n-1} q^{(n-k)^2} \begin{bmatrix} n+k \\ k \end{bmatrix}^2 \begin{bmatrix} n-1 \\ k \end{bmatrix}^2 \equiv \sum_{k=0}^{n-1} q^{n^2-k} = q^{n^2-n+1}[n] \equiv q[n] \pmod{\Phi_n(q)^2}$$

as desired.

5. Concluding remarks

Very recently, the latter author and Zudilin in [9] developed a *creative microscoping* method to prove q-supercongruences by adding a parameter a (see also [5]). Along the same lines, we can generalize (1.3)–(1.5) as follows: for any positive odd integer n modulo $(1-aq^n)(a-q^n)$:

$$\sum_{k=0}^{(n-1)/2} \frac{(aq;q^2)_k (q/a;q^2)_k q^{k^2}}{(q;q^2)_k (q^4;q^4)_k} \equiv (-q)^{(1-n^2)/8}, \tag{5.1}$$

$$\sum_{k=0}^{(n-1)/2} \frac{(aq;q^2)_k (q/a;q^2)_k q^{2k}}{(q^2;q^2)_k (q^2;q^4)_k} \equiv \left(\frac{2}{n}\right) q^{2\lfloor (n+1)/4 \rfloor^2}, \tag{5.2}$$

$$\sum_{k=0}^{(n-1)/2} \frac{(aq;q^2)_k (q/a;q^2)_k q^{2k}}{(q^4;q^4)_k (q^2;q^4)_k} \equiv \left(\frac{3}{n}\right) q^{(n^2-1)/12}. \tag{5.3}$$

It is easy to see that, letting $a \to 1$ in (5.1)–(5.3), we recover (1.3)–(1.5), respectively.

Moreover, there are other different q-analogues of (1.1) and (1.2). For example, applying the identities (2.6)–(2.8) (replacing n with $(n-1)/2$ and q with q^2), we have the following more complicated q-analogues of (1.1) modulo $\Phi_n(q)^2$:

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k}{(q^2;q^2)_k (-q;q^2)_k} \equiv (-1)^{(n-1)/2} q^{(1-n^2)/4} \sum_{k=-\lfloor (n-1)/4 \rfloor}^{\lfloor (n-1)/4 \rfloor} (-1)^k q^{6k^2+2k},$$

$$\sum_{k=0}^{(n-1)/2} \frac{(-q^2;q^4)_k (q;q^2)_k q^{2k}}{(q^4;q^4)_k (-q;q^2)_k} \equiv (-1)^{(n-1)/2} q^{(n^2-1)/4} \sum_{k=-\lfloor (n-1)/4 \rfloor}^{\lfloor (n-1)/4 \rfloor} (-1)^k q^{-4k^2},$$

$$\sum_{k=0}^{(n-1)/2} \frac{(1+q^{2k})(-q^4;q^4)_{k-1} (q;q^2)_k q^{2k}}{(q^4;q^4)_k (-q;q^2)_k}$$
$$\equiv (-1)^{(n-1)/2} q^{(n^2-1)/4} \sum_{k=-\lfloor (n-1)/4 \rfloor}^{\lfloor (n-1)/4 \rfloor} (-1)^k q^{2k-4k^2}.$$

Similarly, applying the invariant of (3.2) (see, for example, [11], (1.5))

$$\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k q^{\binom{k+1}{2}}\begin{bmatrix} n-k\\ k\end{bmatrix} = \sum_{k=-\lfloor (n+1)/3\rfloor}^{\lfloor n/3\rfloor}(-1)^k q^{k(3k+1)/2},$$

which follows readily from Franklin's involution on partitions into distinct parts (see the proof of [1], Theorem 1.6), we have the following q-analogue of (1.2): modulo $\Phi_n(q)^2$,

$$\sum_{k=0}^{(n-1)/2}\frac{(q;q^2)_k}{(q^4;q^4)_k(-q;q^2)_k} \equiv (-1)^{(n-1)/2}q^{(1-n^2)/4}\sum_{k=-\lfloor n/3\rfloor}^{\lfloor (n-1)/3\rfloor}(-1)^k q^{3k^2+k}.$$

There are also parameter generalizations of the above four congruences, which are omitted here.

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