COMMUTANT OF MULTIPLICATION OPERATORS IN WEIGHTED BERGMAN SPACES ON POLYDISK

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Abstract. We study a certain operator of multiplication by monomials in the weighted Bergman space both in the unit disk of the complex plane and in the polydisk of the n-dimensional complex plane. Characterization of the commutant of such operators is given.

Keywords: multiplication operator; commutant of an operator; weighted Bergman space

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1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane. We mean by polydisk the set

$$\mathbb{D}^n = \mathbb{D} \times \ldots \times \mathbb{D}$$

of the n-dimentional complex space. For $\alpha > -1$, we define the weighted Bergman space $A^2_\alpha(\mathbb{D})$ as the space of analytic functions f in $\mathbb{D}$ for which

$$\int_{\mathbb{D}} |f(z)|^2 \, \mathrm{d}A_\alpha(z) < \infty,$$

where

$$\mathrm{d}A_\alpha(z) = \pi^{-1}(\alpha+1)(1-|z|^2)^\alpha \, \mathrm{d}x \, \mathrm{d}y$$

is the normalized area measure in the complex plane. It is well-known that $A^2_\alpha(\mathbb{D})$ equipped with the inner product

$$\langle f, g\rangle = (\alpha+1) \int_{\mathbb{D}} f(z)\overline{g(z)}(1-|z|^2)^\alpha \, \mathrm{d}A(z)$$

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is a Hilbert space of analytic functions. It follows from the boundedness of point evaluation functional together with the Riesz' representation theorem that $A^2_\alpha(\mathbb{D})$ is a reproducing kernel Hilbert space of analytic functions, and that every function $f \in A^2_\alpha(\mathbb{D})$ can be written as

$$f(w) = \langle f, k_w \rangle = (\alpha+1) \int_{\mathbb{D}} f(z)\overline{k_w(z)}(1-|z|^2)^\alpha \,\mathrm{d}A(z), \quad w \in \mathbb{D},$$

where

$$k_w(z) = \frac{1}{(1-z\overline{w})^{\alpha+2}}$$

is the reproducing kernel for the Hilbert space $A^2_\alpha(\mathbb{D})$.

Now let $\mathrm{Hol}(\mathbb{D}^n)$ denote the space of holomorphic functions on the polydisk $\mathbb{D}^n$. The weighted Bergman space on the polydisk $\mathbb{D}^n$ is defined by

$$A^2_\alpha(\mathbb{D}^n) = \mathrm{Hol}(\mathbb{D}^n) \cap L^2(\mathbb{D}^n, \mathrm{d}V_\alpha),$$

where $\mathrm{d}V_\alpha = \mathrm{d}A_\alpha(z_1)\ldots\mathrm{d}A_\alpha(z_n)$. In other words, a function $f(z_1,\ldots,z_n) \in \mathrm{Hol}(\mathbb{D}^n)$ belongs to $A^2_\alpha(\mathbb{D}^n)$ if

$$\|f\|^2_{A^2_\alpha(\mathbb{D}^n)} = \int_{\mathbb{D}^n} |f(z_1,\ldots,z_n)|^2 \,\mathrm{d}A_\alpha(z_1)\ldots\mathrm{d}A_\alpha(z_n) < \infty,$$

where

$$\mathrm{d}A_\alpha(z_k) = \frac{\alpha+1}{\pi}(1-|z_k|^2)^\alpha \,\mathrm{d}x_k\,\mathrm{d}y_k.$$

It is well-known that $\{z^n/\gamma_n\}_{n=0}^\infty$ is an orthonormal basis for A^2_α, where

$$\gamma_n = \|z^n\|_\alpha = \sqrt{\frac{n!\,\Gamma(\alpha+2)}{\Gamma(\alpha+n+2)}}.$$

Then for $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ we have $\|f\|^2_\alpha = \sum\limits_{n=0}^{\infty} \gamma_n^2|a_n|^2$. Now, let $\beta = (\beta_1,\ldots,\beta_n)$ be a multi-index (each β_i is a nonnegative integer); in this case we write $\beta \geqslant 0$. For $z = (z_1,\ldots,z_n) \in \mathbb{D}^n$ we define $z^\beta = z_1^{\beta_1}\ldots z_n^{\beta_n}$ and $e_\beta = z^\beta/\gamma_{\beta_1}\ldots\gamma_{\beta_n}$. With this notation, $\{e_\beta\}_{\beta\geqslant 0}$ is an orthonormal basis for $A^2_\alpha(\mathbb{D}^n)$. The reproducing kernel associated to the points $(z_1,\ldots,z_n)$ and $w = (w_1,\ldots,w_n)$ of the polydisk is given by (see [13])

$$K_z(w) = \prod_{j=1}^{n} \frac{1}{(1-\overline{z_j}w_j)^{\alpha+2}} = k_{z_1}(w_1)\ldots k_{z_n}(w_n).$$

Given a bounded linear operator T on a Hilbert space $\mathcal{H}$, we mean by the commutant of T the set of all bounded linear operators on $\mathcal{H}$ which commute with T. If we denote the algebra of all bounded linear operators on $\mathcal{H}$ by $B(\mathcal{H})$, then the commutant of T which is denoted by $(T)'$ is by definition

$$(T)' = \{S \in B(\mathcal{H})\colon\ ST = TS\}.$$

The operator of multiplication by z^k, where k is a positive integer, is the operator $M_{z^k}\colon\ \mathcal{H} \to \mathcal{H}$ defined by $f \mapsto M_{z^k}(f) = z^k f$. In [12], Kehe Zhu, among other things, proved that a bounded linear operator T on the Bergman space $A^2(\mathbb{D})$ (this is the space $A^2_\alpha(\mathbb{D})$ for $\alpha = 0$) commutes with M_{z^2} if and only if there exist two bounded analytic functions F and G such that

$$Tf = Ff_{\mathrm{e}} + Gf_{\mathrm{o}}/z,$$

where $f = f_{\mathrm{e}} + f_{\mathrm{o}}$ is the even-odd decomposition of f; that is,

$$f_{\mathrm{e}}(z) = \frac{f(z) + f(-z)}{2}, \quad f_{\mathrm{o}}(z) = \frac{f(z) - f(-z)}{2}.$$

In a later paper, the current author proved that the same result is true for the weighted Bergman spaces $A^2_\alpha(\mathbb{D})$ too, see [3]. The question as to what happens if we instead consider the operator of multiplication by z^k for a positive integer $k \geqslant 3$ seems to be more interesting. Here we intend to consider this problem for both one-dimensional complex plane and n-dimensional complex plane. More precisely, and for the sake of simplicity, we shall characterize the commutant of the operator M_{z^3} on $A^2_\alpha(\mathbb{D})$, as well as the commutant of the operator of multiplication by z_1^3 on the polydisk $A^2_\alpha(\mathbb{D}^n)$ (just for simplicity, we take $n = 2$). This latter is the operator

$$f(z_1, z_2) \mapsto M_{z_1^3} f(z_1, z_2) = z_1^3 f(z_1, z_2).$$

It is proved that T commutes with $M_{z_1^3}$ if and only if there exist three bounded analytic functions h_1, h_2, h_3 on the polydisk $A^2_\alpha(\mathbb{D}^2)$ satisfying a certain equality in terms of even-odd-odd decomposition of f.

To tackle this problem, we first need to have an alternative decomposition theory of functions into k summands. This will be done in the next section. Second, we need to know that $\ker(M^*_{\lambda^3 - z^3})$ is spanned by three functions. This will be explained in Section 3.

The importance of this sort of problems is due to the fact that a knowledge of the commutant of a specific operator will result into a knowledge of the reducing subspaces of the given operator (a closed subspace M is said to be *reducing for* *the operator* T if it is invariant both for T and its adjoint T^*). This information in turn has applications in the decomposition theory of operators (for more information see [8]).

For detailed information on the theory of Bergman spaces we refer the reader to the books [9] and [10]. For a different approach of investigation, we refer the reader to [1], [2], [4], [5], [6], [7], [11] and the references therein.

2. A general even-odd decomposition

As indicated in the previous section, the first step to the main result of this paper is to find a more general decomposition of functions into n summands, where the first summand is even and the rest are odd functions (here we use the terms even and odd in a more general sense). Recall that $\{\gamma_k z^k\}_{k=0}^{\infty}$ is an orthonormal basis for $A^2_\alpha(\mathbb{D})$, where

$$\gamma_k = \Big(\frac{\Gamma(\alpha+k+2)}{k!\,\Gamma(\alpha+2)}\Big)^{1/2}.$$

Let $n \geqslant 2$ be fixed and define for $0 \leqslant j \leqslant n-1$,

$$M_j = \operatorname{span}\{z^{j+kn}\}_{k=0}^{\infty}.$$

It follows that these subspaces are orthogonal to each other, moreover,

$$A^2_\alpha(\mathbb{D}) = M_0 + M_1 + \ldots + M_{n-1},$$

or each $f \in A^2_\alpha(\mathbb{D})$ can be represented as

$$f = f_0 + f_1 + \ldots + f_{n-1}, \quad f_j \in M_j. \tag{2.1}$$

In the case when $n = 2$, we get $A^2_\alpha(\mathbb{D}) = M_0 + M_1$, where M_0 is the subspace of even functions and M_1 is the subspace of odd functions.

Now let $\omega = \exp(2\pi\mathrm{i}/n)$, then for $f \in M_j$ we have $f(\omega z) = \omega^j f(z)$. Put it another way, we let $n > 1$ be an integer and consider the additive cyclic group $\mathbb{Z}/n\mathbb{Z} \approx \mathbb{Z}_n$. Define φ in the following way: φ sends an element $k \in \mathbb{Z}_n$ to the operator R_k given by $R_k f(z) = f(\omega^k z)$, where $\omega = \exp(2\pi\mathrm{i}/n)$. The operator R_k acts on the weighted Bergman space $A^2_\alpha(\mathbb{D})$. Note that each R_k is unitary (a surjective isometry). We may just look at R_1 for the moment; this is just a rotation operator, indeed, $R_1 f(z) = (f \circ r)(z)$, where $r(z) = \omega z$ with $|\omega| = 1$. We observe that $R_k = R_1^k$, so if we understand R_1, we understand R_k as well. As such, we might be interested in the spectrum of R_1 and the corresponding eigenspaces. Now, since R_1 has the property $R_1^n = \mathrm{id}$, the only eigenvalues are the nth roots of unity. To see this let λ satisfy $R_1 f(z) = \lambda f(z)$. It follows that $f(\omega z) = \lambda f(z)$. Applying R_1 to both sides of this equality we get $f(\omega^2 z) = \lambda f(\omega z) = \lambda^2 f(z)$. In this way, we obtain

$$f(z) = f(\omega^n z) = \lambda^n f(z)$$

from which it follows that $\lambda^n = 1$, or $\lambda = \omega^j$, $j = 1, \ldots, n$. Now let $\lambda = \omega^j$ for some $1 \leqslant j \leqslant n$ be an eigenvalue of R_1. The corresponding eigenspace

$$M_j = \{f \in A^2_\alpha(\mathbb{D})\colon\ R_1 f(z) = \omega^j f(z)\}$$

consists of functions in the weighted Bergman space satisfying $f(\omega z) = \omega^j f(z)$. These eigenspaces are necessarily orthogonal, by unitarity, and span the whole space (a result of general spectral theory). In the case that $n = 2$, we get $\omega = -1$, $R_1 f(z) = f(-z)$ and $R_2 = \mathrm{id}$. Therefore, M_1, M_2 will become the space of odd and even functions, respectively. Indeed,

$$f_{\mathrm{e}}(z) = \frac{R_2 f(z) + R_1 f(z)}{2}$$

and

$$f_{\mathrm{o}}(z) = \frac{R_2 f(z) - R_1 f(z)}{2}.$$

In brief, the Bergman space can be written as the sum of its eigenspaces. In this way, we have proved the following theorem.

Theorem 2.1. *Let f be a function in the weighted Bergman space $A^2_\alpha(\mathbb{D})$. Then there are n functions $f_1, \ldots, f_n$ in $A^2_\alpha(\mathbb{D})$ such that*

$$f = f_1 + f_2 + \ldots + f_n,$$

where f_j satisfies $f(\omega z) = \omega^j f(z)$ and ω is an nth root of unity.

3. Multiplication operators by monomials

In this section we shall provide a characterization for the commutant of the operator of multiplication by z^k. We shall see that T commutes with M_{z^k} if and only if there exist k bounded analytic functions φ_j, $1 \leqslant j \leqslant k$, such that T can be written as

$$Tf = \varphi_1 f_1 + \varphi_2 f_2/z + \ldots + \varphi_k f_k/z^{k-1}, \quad f \in A^2_\alpha(\mathbb{D}),$$

where $f = f_1 + \ldots + f_k$ is the even-odd decomposition of f given by Theorem 2.1. We begin with the following proposition. For the sake of simplicity, we often assume that $k = 3$.

Proposition 3.1. *Let φ_1, φ_2 and φ_3 be bounded analytic functions in the unit disk and let $f = f_1 + f_2 + f_3$ be the decomposition of $f \in A^2_\alpha(\mathbb{D})$ into three functions as indicated above. Then $T\colon\ A^2_\alpha(\mathbb{D}) \to A^2_\alpha(\mathbb{D})$ defined by*

$$Tf = \varphi_1 f_1 + \varphi_2 f_2/z + \varphi_3 f_3/z^2, \quad f \in A^2_\alpha(\mathbb{D})$$

is a bounded linear operator.

Proof. Since φ_j's are bounded functions and $\|f_1\| \leqslant \|f\|$, it suffices to verify that there is a positive constant C such that

$$\max\left\{\left\|\frac{f_2}{z}\right\|, \left\|\frac{f_3}{z^2}\right\|\right\} \leqslant C\|f\|.$$

Assume that $f \in A^2_\alpha(\mathbb{D})$. Now we have

$$\|f\|^2_{A^2_\alpha(\mathbb{D})} = \sum_{n=0}^\infty \frac{n!\,\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)}|a_n|^2$$

and similarly,

$$\|zf\|^2_{A^2_\alpha(\mathbb{D})} = \sum_{n=0}^\infty \frac{(n+1)!\,\Gamma(\alpha+2)}{\Gamma(\alpha+n+3)}|a_n|^2.$$

Since $\alpha + 2 \geqslant 1$, we have $1 + n/(\alpha+2) \leqslant n+1$ or $\alpha + 2 + n \leqslant (\alpha+2)(n+1)$, from which it follows that

$$\frac{\Gamma(\alpha+n+3)}{\Gamma(\alpha+n+2)} \leqslant (\alpha+2)(n+1).$$

This is equivalent to

$$\frac{1}{(\alpha+2)\Gamma(\alpha+n+2)} \leqslant \frac{n+1}{\Gamma(\alpha+n+3)}.$$

Multiplying both sides by $n!\,\Gamma(\alpha+2)|a_n|^2$ we obtain

$$\frac{n!\,\Gamma(\alpha+2)}{(\alpha+2)\Gamma(\alpha+n+2)}|a_n|^2 \leqslant \frac{(n+1)!\,\Gamma(\alpha+2)}{\Gamma(\alpha+n+3)}|a_n|^2,$$

from which it follows that

$$\|f\|^2_{A^2_\alpha(\mathbb{D})} \leqslant (\alpha+2)\|zf\|^2_{A^2_\alpha(\mathbb{D})}.$$

Assume now that $f(0)=0$ and put $g(z)=f(z)/z$. It follows from the above argument that

$$\|g\|_{A^2_\alpha(\mathbb{D})} \leqslant \sqrt{\alpha+2}\|zg\|_{A^2_\alpha(\mathbb{D})},$$

and in particular (since $f_2(0)=0$),

$$\left\|\frac{f_2}{z}\right\|_{A^2_\alpha(\mathbb{D})} \leqslant \sqrt{\alpha+2}\|f_2\|_{A^2_\alpha(\mathbb{D})}.$$

Again, if $f(0)=f'(0)=0$, we consider $g(z)=f(z)/z^2$ to obtain

$$\|g\|_{A^2_\alpha(\mathbb{D})} \leqslant \sqrt{\alpha+2}\|zg\|_{A^2_\alpha(\mathbb{D})} \leqslant (\alpha+2)\|z^2 g\|_{A^2_\alpha(\mathbb{D})}.$$

This latter inequality implies that

$$\left\|\frac{f_3}{z^2}\right\|_{A^2_\alpha(\mathbb{D})} \leqslant (\alpha+2)\|f_3\|_{A^2_\alpha(\mathbb{D})}.$$

Now, it is clear that

$$\max\left\{\left\|\frac{f_2}{z}\right\|, \left\|\frac{f_3}{z^2}\right\|\right\} \leqslant (\alpha+2)\|f\|,$$

from which the boundedness of T follows. □

Lemma 3.1. *Let f be a function in $A^2_\alpha(\mathbb{D})$ and let $w_1,\ldots,w_n$ be a finite sequence of points in the open unit disk. Then the following are equivalent:*

(a) *f vanishes at the points $w_1,\ldots,w_n$, counting multiplicities,*

(b) *$f=qg$, where g is in $A^2_\alpha(\mathbb{D})$ and q is the polynomial of degree n whose zeros are $w_1,\ldots,w_n$ and the highest order coefficient is 1.*

Proof. Since (b) implies (a) trivially, it remains to obtain (b) from (a). It is enough to do this for polynomials of degree one, by iteration and the well-known factorization of polynomials. So, if f is in the Bergman space and $f(w_0)=0$ for some w_0 in the open unit disk, we claim that $g(z)=f(z)/(z-w_0)$ is in the Bergman space. Clearly, g is holomorphic around w_0, and moreover, $|z-w_0| \geqslant \frac{1}{2}(1-|w_0|)$ for z in the annulus $\frac{1}{2}(1+|w_0|) < |z| < 1$. Then g is in $A^2_\alpha(\mathbb{D})$ since it is holomorphic in the unit disk and has the integrability property in the annulus (inside the disk $D(0,\frac{1}{2}(1+|w_0|))$ the function is also integrable trivially). □

An important observation in the proof of the main result is to see that for $\lambda \neq 0$, the subspace $\ker M^*_{\lambda^3-z^3}$ of the weighted Bergman space is spanned by three Bergman kernel functions associated to the roots of $z^3-\lambda^3=0$. For $k=2$, this was observed by Kehe Zhu (see [12]) who used a direct method to find them; see also [3].

Proposition 3.2. *Let $w_1,\ldots,w_n$ be simple zeros of a polynomial q and M_q be the operator of multiplication by q on $A^2_\alpha(\mathbb{D})$. Then $\ker M^*_q$ is spanned by the Bergman kernel functions $\{k_{w_1},\ldots,k_{w_n}\}$.*

P r o o f. Assume $f \in \ker M_q^*$. Then for any $h \in A_\alpha^2(\mathbb{D})$ we have $\langle M_q^* f, h\rangle = 0$, which is equivalent to saying that $\langle f, qh\rangle = 0$. This means that

$$f \in \ker M_q^* \iff f \in (qA_\alpha^2)^\perp.$$

According to part (b) of Lemma 3.1,

$$N := qA_\alpha^2 = \{g \in A_\alpha^2 \colon\ g(w_1) = \ldots = g(w_n) = 0\}.$$

This means that each function in N annihilates $k_{w_1}, \ldots, k_{w_n}$, or

$$N^\perp = \operatorname{span}\{k_{w_1}, \ldots, k_{w_n}\}.$$

$\square$

Theorem 3.1. *Let T be a bounded operator on $A_\alpha^2(\mathbb{D})$. Then T commutes with M_{z^k} if and only if there exist k bounded analytic functions φ_j, $1 \leqslant j \leqslant k$, such that T can be written as*

$$Tf = \varphi_1 f_1 + \varphi_2 f_2/z + \ldots + \varphi_k f_k/z^{k-1}, \quad f \in A_\alpha^2(\mathbb{D}),$$

where $f = f_1 + \ldots + f_k$ is the even-odd decomposition of f given by Theorem 2.1.

P r o o f. First assume that T is given by the above equality. According to Proposition 3.1, T is bounded. On the other hand,

$$TM_{z^k} f = T(M_{z^k})(f_1 + \ldots + f_k) = z^k\varphi_1 f_1 + z^k\varphi_2 f_2/z + \ldots + z^k\varphi_k f_k/z^{k-1} = M_{z^k} Tf,$$

that is, T commutes with M_{z^k}. For the converse, assume that $TM_{z^k} = M_{z^k}T$, so that $TM_{\lambda^k - z^k} = M_{\lambda^k - z^k}T$. This implies that T^* commutes with $M^*_{\lambda^k - z^k}$, from which it follows that $\ker M^*_{\lambda^k - z^k}$ is invariant under T^*.

From now on, for simplicity, let $k = 3$ and let λ be a nonzero complex number. Assume that ω_k, $k = 0, 1, 2$, are three roots of the equation $z^3 = 1$. It follows from Proposition 3.2 that

$$\ker M^*_{\lambda^3 - z^3} = \operatorname{span}\{k_\lambda(z), k_{\lambda\omega_1}(z), k_{\lambda\omega_2}(z)\}.$$

This means that for each $f \in \ker M^*_{\lambda^3 - z^3}$ there are functions $a(\lambda)$, $b(\lambda)$ and $c(\lambda)$ such that

$$f(z) = a(\lambda)k_\lambda(z) + b(\lambda)k_{\lambda\omega_1}(z) + c(\lambda)k_{\lambda\omega_2}(z).$$

Now, let $\lambda \neq 0$. We may write

$$T^* k_\lambda = \overline{a(\lambda)} k_\lambda + \overline{b(\lambda)} k_{\lambda\omega_1} + \overline{c(\lambda)} k_{\lambda\omega_2},$$

from which it follows that for each $f \in A^2_\alpha(\mathbb{D})$ we have

$$Tf(z) = \langle Tf, k_z \rangle = \langle f, T^* k_z \rangle = a(z)f(z) + b(z)f(\omega_1 z) + c(z)f(\omega_2 z).$$

Note that if $f \in M_1$, then

$$f(\omega_1 z) = f(\omega_2 z) = f(z).$$

It is easy to check that if $f \in M_2$, then

$$f(\omega_1 z) = \sum_{n=0}^{\infty} (\mathrm{e}^{2\pi\mathrm{i}/3})^{3n+1} z^{3n+1} = \mathrm{e}^{2\pi\mathrm{i}/3} f(z)$$

and

$$f(\omega_2 z) = \sum_{n=0}^{\infty} (\mathrm{e}^{4\pi\mathrm{i}/3})^{3n+1} z^{3n+1} = \mathrm{e}^{4\pi\mathrm{i}/3} f(z).$$

Moreover, for $f \in M_3$ we have

$$f(\omega_1 z) = \sum_{n=0}^{\infty} (\mathrm{e}^{2\pi\mathrm{i}/3})^{3n+2} z^{3n+2} = \mathrm{e}^{4\pi\mathrm{i}/3} f(z)$$

and

$$f(\omega_2 z) = \sum_{n=0}^{\infty} (\mathrm{e}^{4\pi\mathrm{i}/3})^{3n+2} z^{3n+2} = \mathrm{e}^{2\pi\mathrm{i}/3} f(z).$$

Now, we assume that $f = f_1 + f_2 + f_3$, where $f_j \in M_j$, and write

$$\begin{aligned} Tf = T(f_1) + T(f_2) + T(f_3) &= [a(z) + b(z) + c(z)]f_1(z) \\ &+ [a(z) + \omega_1 b(z) + \omega_2 c(z)]f_2(z) + [a(z) + \omega_2 b(z) + \omega_1 c(z)]f_3(z). \end{aligned}$$

We now define for $z \neq 0$,

$$\begin{aligned} F(z) &= a(z) + b(z) + c(z), \\ G(z) &= z[a(z) + \omega_1 b(z) + \omega_2 c(z)] \end{aligned}$$

and finally

$$H(z) = z^2[a(z) + \omega_2 b(z) + \omega_1 c(z)].$$

This implies that

$$Tf = F(z)f_1(z) + G(z)f_2(z)/z + H(z)f_3(z)/z^2.$$

Note that $F = T(1)$, $G = T(z)$ and $H = T(z^2)$, so that these functions are analytic in $\mathbb{D} \setminus \{0\}$. If we set $F(0) = T(1)(0)$, $G(0) = T(z)(0)$ and $H(0) = T(z^2)(0)$, they become analytic on the whole unit disk. The last thing to be proved is the fact that F, G, H are bounded. This will be proved in the following way. Consider the following closed subspaces in the weighted Bergman space:

$$\begin{aligned} E &= \{f \in A^2_\alpha(\mathbb{D})\colon\ f(\omega_1 z) = f(z)\},\\ O_1 &= \{f \in A^2_\alpha(\mathbb{D})\colon\ f(\omega_1 z) = \omega_1 f(z)\},\\ O_2 &= \{f \in A^2_\alpha(\mathbb{D})\colon\ f(\omega_1 z) = \omega_2 f(z)\}. \end{aligned}$$

Indeed, F, G and H are multipliers from subspaces E_1, O_1 and O_2, respectively, into the weighted Bergman space A^2_α. For $z \in \mathbb{D}$ and $h \in E$ we have

$$|F(z)\varphi_z(h)| = |F(z)h(z)| = |\varphi_z(Fh)| \leqslant \|\varphi_z\|\|M_F\|\|h\|,$$

where φ_z and M_F are point evaluation functional and multiplication operator by F, respectively. Note that these are bounded operators. This implies that

$$|F(z)|\|\varphi_z\| \leqslant \|\varphi_z\|\|M_F\|,$$

from which it follows that

$$\sup_{z\in\mathbb{D}} |F(z)| \leqslant \|M_F\|.$$

Similarly, one proves that G and H belong to $H^\infty(\mathbb{D})$. □

4. Multiplication operator on polydisk

For the sake of simplicity, we shall assume that $n = 2$, that is, we study the polydisk $A^2_\alpha(\mathbb{D}^2)$. More precisely, we want to address the commutant of the operator

$$M_{z_1^3}\colon\ A^2_\alpha(\mathbb{D}^2) \to A^2_\alpha(\mathbb{D}^2)$$

defined by

$$f(z_1, z_2) \mapsto z_1^3 f(z_1, z_2).$$

Keeping z_2 fixed and invoking the arguments of the previous section for the one variable function $g(z_1) = f(z_1, z_2)$, we will write

$$f(z_1, z_2) = f_0(z_1, z_2) + f_1(z_1, z_2) + f_2(z_1, z_2), \tag{4.1}$$

where f_0 is even (with respect to z_1) and f_1, f_2 are odd functions (with respect to z_1). Using the terminology of the previous section (with a little change in notation), $f_0 \in E$, $f_1 \in O_1$ and $f_2 \in O_2$. It is shown that a bounded operator T on $A^2_\alpha(\mathbb{D}^2)$ commutes with $M_{z_1^3}$ if and only if there are three bounded analytic functions φ_0, φ_1, φ_2 such that

$$Tf(z_1, z_2) = \varphi_0(z_1, z_2) f_0(z_1, z_2) + \varphi_1(z_1, z_2) f_1(z_1, z_2)/z_1 + \varphi_2(z_1, z_2) f_2(z_1, z_2)/z_1^2.$$

We begin with the following proposition.

Proposition 4.1. *Let φ_0, φ_1 and φ_2 be bounded analytic functions in the polydisk $\mathbb{D}^2$. Then the linear operator $T\colon A^2_\alpha(\mathbb{D}^2) \to A^2_\alpha(\mathbb{D}^2)$ defined by*

$$Tf = \varphi_0 f_0 + \varphi_1 f_1/z_1 + \varphi_2 f_2/z_1^2 \quad (f = f_0 + f_1 + f_2)$$

is bounded.

Proof. Since φ_j's are bounded functions and $\|f_0\| \leqslant \|f\|$, it suffices to verify that there is a positive constant C such that

$$\max\left\{\left\|\frac{f_1}{z_1}\right\|, \left\|\frac{f_2}{z_1^2}\right\|\right\} \leqslant C\|f\|.$$

Assume that $f \in A^2_\alpha(\mathbb{D}^2)$. For

$$f(z_1, z_2) = \sum_{(n,m)\in\mathbb{N}^*\times\mathbb{N}^*} a_{n,m} z_1^n z_2^m,$$

where $\mathbb{N}^*$ is the set of nonnegative integers, we have

$$\|f\|^2_{A^2_\alpha(\mathbb{D}^2)} = \sum_{m,n=0}^{\infty} \frac{n!\,\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)} \frac{m!\,\Gamma(\alpha+2)}{\Gamma(m+\alpha+2)} |a_{m,n}|^2.$$

Similarly, we compute

$$\|z_1 f\|^2_{A^2_\alpha(\mathbb{D}^2)} = \sum_{m,n=0}^{\infty} \frac{n!\,\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)} \frac{(m+1)!\,\Gamma(\alpha+2)}{\Gamma(m+\alpha+3)} |a_{m,n}|^2.$$

Since $\alpha+2 \geqslant 1$, we have $1+m/(\alpha+2) \leqslant m+1$ or $\alpha+2+m \leqslant (\alpha+2)(m+1)$, from which it follows that

$$\frac{\Gamma(\alpha+m+3)}{\Gamma(\alpha+m+2)} \leqslant (\alpha+2)(m+1).$$

This is equivalent to

$$\frac{1}{(\alpha+2)\Gamma(\alpha+m+2)} \leqslant \frac{m+1}{\Gamma(\alpha+m+3)}.$$

Multiplying both sides by $m!\,\Gamma(\alpha+2)|a_{m,n}|^2$ we obtain

$$\frac{m!\,\Gamma(\alpha+2)}{(\alpha+2)\Gamma(\alpha+m+2)}|a_{m,n}|^2 \leqslant \frac{(m+1)!\,\Gamma(\alpha+2)}{\Gamma(\alpha+m+3)}|a_{m,n}|^2.$$

This implies that

$$\|f\|^2_{A^2_\alpha(\mathbb{D}^2)} \leqslant (\alpha+2)\|z_1 f\|^2_{A^2_\alpha(\mathbb{D}^2)}.$$

Assume now that $f(0,z_2)=0$ and put

$$g(z_1,z_2)=\frac{f(z_1,z_2)}{z_1}.$$

It follows from the above argument that

$$\|g\|_{A^2_\alpha(\mathbb{D}^2)} \leqslant \sqrt{\alpha+2}\|z_1 g\|_{A^2_\alpha(\mathbb{D}^2)},$$

and in particular,

$$\left\|\frac{f_1}{z_1}\right\|_{A^2_\alpha(\mathbb{D}^2)} \leqslant \sqrt{\alpha+2}\|f_1\|_{A^2_\alpha(\mathbb{D}^2)}.$$

In the same way, by considering $g(z_1,z_2)=f(z_1,z_2)/z_1^2$ we obtain

$$\|g\|_{A^2_\alpha(\mathbb{D})} \leqslant \sqrt{\alpha+2}\|z_1 g\|_{A^2_\alpha(\mathbb{D})} \leqslant (\alpha+2)\|z_1^2 g\|_{A^2_\alpha(\mathbb{D})}.$$

This implies that

$$\left\|\frac{f_2}{z_1^2}\right\|_{A^2_\alpha(\mathbb{D})} \leqslant (\alpha+2)\|f_2\|_{A^2_\alpha(\mathbb{D})},$$

and finally

$$\max\left\{\left\|\frac{f_1}{z_1}\right\|, \left\|\frac{f_2}{z_1^2}\right\|\right\} \leqslant (\alpha+2)\|f\|,$$

from which the boundedness of T follows. □

Now, we state the mail result of this section.

Theorem 4.1. *Let T be a bounded operator on $A^2_\alpha(\mathbb{D}^2)$. Then T commutes with $M_{z_1^3}$ if and only if there exist three bounded analytic functions φ_j, $0 \leqslant j \leqslant 2$, such that*

$$Tf = \varphi_0 f_0 + \varphi_1 f_1/z_1 + \varphi_2 f_2/z_1^2,$$

where $f = f_0 + f_1 + f_2$ is the even-odd decomposition of f.

Proof. If f can be written in the form represented above, then T belongs to the commutant of $M_{z_1^3}$; this follows from Proposition 3.1. To prove the other direction, assume that $f(z_1, z_2)$ is given. Keeping z_2 fixed, we use the arguments made in Section 2 to write

$$f(z_1, z_2) = f_0(z_1, z_2) + f_1(z_1, z_2) + f_2(z_1, z_2),$$

where $f_1(z_1, z_2)/z_1$ and $f_2(z_1, z_2)/z_1^2$ are analytic functions in $\mathbb{D}^2$. Let λ be a nonzero complex number. Assume that ω_k for $k = 0, 1, 2$, are roots of $z^3 = 1$. As in one variable case, it follows from Proposition 3.2 that

$$\ker M^*_{\lambda^3 - z_1^3} = \operatorname{span}\{k_\lambda(z_1), k_{\lambda\omega_1}(z_1), k_{\lambda\omega_2}(z_1)\}.$$

It follows that every function $f(z_1, z_2)$ in $\ker M^*_{\lambda^3 - z_1^3}$ can be written as

$$f(z_1, z_2) = a(\lambda, z_2)k_\lambda(z_1) + b(\lambda, z_2)k_{\lambda\omega_1}(z_1) + c(\lambda, z_2)k_{\lambda\omega_2}(z_1).$$

Now, let $\lambda \neq 0$. We may write

$$T^* k_\lambda(\cdot) = \overline{a(\lambda, z_2)}k_\lambda(\cdot) + \overline{b(\lambda, z_2)}k_{\lambda\omega_1}(\cdot) + \overline{c(\lambda, z_2)}k_{\lambda\omega_2}(\cdot),$$

from which it follows that for fixed z_2 and $z_1 \neq 0$ and $f(z_1, z_2) \in A^2_\alpha(\mathbb{D}^2)$ we have

$$\begin{aligned} Tf(z_1, z_2) &= \langle Tf(\cdot, z_2), k_{z_1}\rangle = \langle f(\cdot, z_2), T^* k_{z_1}\rangle \\ &= \langle f(\cdot, z_2), \overline{a(z_1, z_2)}k_{z_1} + \overline{b(z_1, z_2)}k_{\omega_1 z_1} + \overline{c(z_1, z_2)}k_{\omega_2 z_1}\rangle \\ &= a(z_1, z_2)f(z_1, z_2) + b(z_1, z_2)f(\omega_1 z_1, z_2) + c(z_1, z_2)f(\omega_2 z_1, z_2). \end{aligned}$$

This implies that

$$Tf(z_1, z_2) = \begin{cases} [a(z_1, z_2) + b(z_1, z_2) + c(z_1, z_2)]f(z_1, z_2) & \text{if } f \in E, \\ [a(z_1, z_2) + \omega_1 b(z_1, z_2) + \omega_2 c(z_1, z_2)]f(z_1, z_2) & \text{if } f \in O_1, \\ [a(z_1, z_2) + \omega_2 b(z_1, z_2) + \omega_1 c(z_1, z_2)]f(z_1, z_2) & \text{if } f \in O_2. \end{cases}$$

Therefore, by setting (for $z_1 \neq 0$)

$$\varphi_0(z_1, z_2) = a(z_1, z_2) + b(z_1, z_2) + c(z_1, z_2)$$

and

$$\varphi_1(z_1,z_2)=z_1[a(z_1,z_2)+\omega_1 b(z_1,z_2)+\omega_2 c(z_1,z_2)]$$

and finally,

$$\varphi_2(z_1,z_2)=z_1^2[a(z_1,z_2)+\omega_2 b(z_1,z_2)+\omega_1 c(z_1,z_2)],$$

we have

$$\begin{aligned}Tf(z_1,z_2)&=T(f_0(z_1,z_2)+f_1(z_1,z_2)+f_2(z_1,z_2))\\&=\varphi_0(z_1,z_2)f_0(z_1,z_2)+\varphi_1(z_1,z_2)f_1(z_1,z_2)/z_1+\varphi_2(z_1,z_2)f_2(z_1,z_2)/z_1^2.\end{aligned}$$

Note that $\varphi_0(z_1,z_2)=T(1)$ and $\varphi_1(z_1,z_2)=T(g)$, where $g(z_1,z_2)=z_1$, $\varphi_2(z_1,z_2)=T(h)$, where $h(z_1,z_2)=z_1^2$. Therefore φ_0, φ_1 and φ_2 are analytic in $\mathbb{D}^2\setminus\{(0,z_2)\colon\ z_2\in\mathbb{D}\}$. Now, we define $\varphi_0(0,z_2)=T(1)(0,z_2)$ and $\varphi_1(0,z_2)=T(g)(0,z_2)$ and finally $\varphi_2(0,z_2)=T(h)(0,z_2)$. Then φ_0, φ_1 and φ_2 will be analytic on the whole domain $\mathbb{D}^2$.

The last step is to prove that φ_0, φ_1 and φ_2 are bounded. To see this, we consider the closed subspaces

$$\begin{aligned}E&=\{f\in A^2_\alpha(\mathbb{D}^2)\colon\ f(\omega_1 z_1,z_2)=f(z_1,z_2)\},\\O_1&=\{f\in A^2_\alpha(\mathbb{D}^2)\colon\ f(\omega_1 z_1,z_2)=\omega_1 f(z_1,z_2)\},\\O_2&=\{f\in A^2_\alpha(\mathbb{D}^2)\colon\ f(\omega_1 z_1,z_2)=\omega_2 f(z_1,z_2)\}.\end{aligned}$$

Note that φ_0, φ_1 and φ_2 are multipliers from E, O_1 and O_2 into the Bergman space $A^2_\alpha(\mathbb{D}^2)$, respectively. Let $(z_1,z_2)\in\mathbb{D}^2$ and $f\in E$, then we have

$$|\varphi_0(z_1,z_2)e_{(z_1,z_2)}(f)|=|\varphi_0(z_1,z_2)f(z_1,z_2)|=|e_{(z_1,z_2)}(\varphi_0 f)|\leqslant\|e_{(z_1,z_2)}\|\|M_{\varphi_0}\|\|f\|,$$

where $e_{(z_1,z_2)}$ is the evaluation functional at (z_1,z_2) and M_{φ_0} is the multiplication operator by φ_0. Since both evaluation functional and multiplication operator are bounded, by taking supremum over all functions f with $\|f\|\leqslant 1$ we conclude that

$$|\varphi_0(z_1,z_2)|\|e_{(z_1,z_2)}\|\leqslant\|e_{(z_1,z_2)}\|\|M_{\varphi_0}\|,$$

from which it follows that

$$\sup_{(z_1,z_2)\in\mathbb{D}^2}|\varphi_0(z_1,z_2)|\leqslant\|M_{\varphi_0}\|.$$

Similarly, one proves that φ_1, φ_2 are bounded functions on $\mathbb{D}^2$. □

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