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H-CONFORMAL ANTI-INVARIANT SUBMERSIONS FROM ALMOST QUATERNIONIC HERMITIAN MANIFOLDS

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Abstract. We introduce the notions of h-conformal anti-invariant submersions and h-conformal Lagrangian submersions from almost quaternionic Hermitian manifolds onto Riemannian manifolds as a generalization of Riemannian submersions, horizontally conformal submersions, anti-invariant submersions, h-anti-invariant submersions, h-Lagrangian submersion, conformal anti-invariant submersions. We investigate their properties: the integrability of distributions, the geometry of foliations, the conditions for such maps to be totally geodesic, etc. Finally, we give some examples of such maps.

Keywords: horizontally conformal submersion; quaternionic manifold; totally geodesic

MSC 2020: 53C15, 53C26, 53C43

1. Introduction

To study geometric structures and geometric properties on Riemannian manifolds with some additional structures, we usually use C^{∞}-maps. There are two ways: We take these ones as either base manifolds or target manifolds. As we know, isometric immersions are examples for studying target manifolds and Riemannian submersions are examples for investigating base manifolds. As a generalization of isometric immersions and Riemannian submersions, Riemannian maps were used to study both cases. The author introduced several types of new notions on this topic and by using them, the author obtained many interesting results on them. We recall some historical events on this topic, which are related with this paper.

In 1960s, Riemannian submersions between Riemannian manifolds were independently introduced by O'Neill in [20] and Gray in [11].

In 1976, Watson in [31] introduced the notion of almost Hermitian submersions between almost Hermitian manifolds. Given an almost Hermitian submer-

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sion $F\colon (M, g_M, J_M) \mapsto (N, g_N, J_N)$, we know that $J_M(\ker F_*) = \ker F_*$ and $J_M((\ker F_*)^\perp) = (\ker F_*)^\perp$, where $(\ker F_*)^\perp$ denotes the orthogonal complement of $\ker F_*$ in TM. Using this notion, he obtained some differential geometric properties among fibers, base manifolds, and total manifolds.

In 2010, by changing the invariance of $\ker F_*$ under almost complex structure J_M, Şahin in [26] defined an anti-invariant Riemannian submersion F from an almost Hermitian manifold (M, g_M, J_M) onto a Riemannian manifold (N, g_N). It satisfies $J_M(\ker F_*) \subset (\ker F_*)^\perp$. Using this notion, he also obtained lots of properties: the integrability of distributions, the geometry of foliations, the condition for such a map to be totally geodesic, some decomposition theorems, etc.

In 2017, as a generalization of an anti-invariant Riemannian submersion from an almost Hermitian manifold, the author in [23] introduced the notions of an h-anti-invariant submersion and an h-Lagrangian submersion from an almost quaternionic Hermitian manifold.

In 1970s, as a generalization of Riemannian submersions, a horizontally conformal submersion was introduced independently by Fuglede in [10] and Ishihara in [17].

In 1997, Gudmundsson and Wood in [13] studied conformal holomorphic submersions between almost Hermitian manifolds. They found the condition for a conformal holomorphic submersion to be a harmonic morphism.

In 2016, Akyol and Şahin in [1] defined a conformal anti-invariant submersion from an almost Hermitian manifold onto a Riemannian manifold. And they obtained some interesting propertis on it.

In 2016, Jin and Lee in [18] investigated a conformal anti-invariant submersion from a hyperkähler manifold.

Given a C^∞-submersion F from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N), with some additional structures, we get several types of submersions, see [1], [3], [7], [9], [11], [12], [14], [20], [21], [22], [23], [24], [26], [27], [28], [29], [31].

Riemannian submersions are related with physics and have their applications in the Yang-Mills theory (see [6], [32]), Kaluza-Klein theory (see [5], [15]), supergravity and superstring theories (see [16], [19]). We know that the quaternionic Kähler manifolds have applications in physics as the target spaces for nonlinear σ-models with supersymmetry, see [8].

The paper is organized as follows. In Section 2 we recall some notions, which are needed in the following sections. In Section 3 we introduce the notions of h-conformal anti-invariant submersions and h-conformal Lagrangian submersions and obtain some properties on them: the characterization of such maps, the harmonicity of such maps, the conditions for such maps to be totally geodesic, the integrability of distributions, the geometry of foliations, etc. In Section 4 we obtain some decomposition theorems.

In Section 5 we give some examples of h-conformal anti-invariant submersions and h-conformal Lagrangian submersions.

2. Preliminaries

In this section we recall some notions, which will be used in the following sections.

Let (M, g_M) and (N, g_N) be Riemannian manifolds, where g_M and g_N are Riemannian metrics on C^∞-manifolds M and N, respectively.

Let $F\colon (M, g_M) \mapsto (N, g_N)$ be a C^∞-map.

We call the map F a *C^∞-submersion* if F is surjective and the differential $(F_*)_p$ has maximal rank for any $p \in M$.

Then the map F is said to be a *Riemannian submersion* (see [9], [20]) if F is a C^∞-submersion and

$$(F_*)_p\colon ((\ker(F_*)_p)^\perp, (g_M)_p) \mapsto (T_{F(p)}N, (g_N)_{F(p)})$$

is a linear isometry for any $p \in M$, where $(\ker(F_*)_p)^\perp$ is the orthogonal complement of the space $\ker(F_*)_p$ in the tangent space T_pM to M at p.

The map F is called *horizontally weakly conformal* at $p \in M$ if it satisfies either (i) $(F_*)_p = 0$ or (ii) $(F_*)_p$ is surjective and there exists a number $\lambda(p) > 0$ such that

$$g_N((F_*)_pX, (F_*)_pY) = \lambda^2 g_M(X, Y) \quad \text{for } X, Y \in (\ker(F_*)_p)^\perp. \tag{2.1}$$

We call the point p a *critical point* if it satisfies the type (i) and call the point p a *regular point* if it satisfies the condition (ii). And the positive number $\lambda(p)$ is said to be *dilation* of F at p. The map F is called *horizontally weakly conformal* if it is horizontally weakly conformal at any point of M. If the map F is horizontally weakly conformal and it has no critical points, then we call the map F a *horizontally conformal submersion.*

Let $F\colon (M, g_M) \mapsto (N, g_N)$ be a horizontally conformal submersion.

Given any vector field $U \in \Gamma(TM)$, we have

$$U = \mathcal{V}U + \mathcal{H}U, \tag{2.2}$$

where $\mathcal{V}U \in \Gamma(\ker F_*)$ and $\mathcal{H}U \in \Gamma((\ker F_*)^\perp)$.

Define the (O'Neill) tensors $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \tag{2.3}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F \tag{2.4}$$

for $E, F \in \Gamma(TM)$, where ∇ is the Levi-Civita connection of g_M, see [9], [20]. Then it is well-known that

$$g_M(\mathcal{T}_U V, W) = -g_M(V, \mathcal{T}_U W), \tag{2.5}$$

$$g_M(\mathcal{A}_U V, W) = -g_M(V, \mathcal{A}_U W) \tag{2.6}$$

for $U, V, W \in \Gamma(TM)$.

Define $\widehat{\nabla}_X Y := \mathcal{V}\nabla_X Y$ for $X, Y \in \Gamma(\ker F_*)$.

Let $F\colon (M, g_M) \mapsto (N, g_N)$ be a C^∞-map.

Then the *second fundamental form* of F is given by

$$(\nabla F_*)(X, Y) := \nabla^F_X F_* Y - F_*(\nabla_X Y) \quad \text{for } X, Y \in \Gamma(TM),$$

where ∇^F is the pullback connection and we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N, see [3].

Remind that F is said to be *harmonic* if the tension field $\tau(F) = \mathrm{trace}(\nabla F_*) = 0$ and F is called a *totally geodesic* map if $(\nabla F_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$, see [3].

Lemma 2.1 ([30]). *Let (M, g_M) and (N, g_N) be Riemannian manifolds and $F\colon (M, g_M) \mapsto (N, g_N)$ a C^∞-map. Then we have*

$$\nabla^F_X F_* Y - \nabla^F_Y F_* X - F_*([X, Y]) = 0 \tag{2.7}$$

for $X, Y \in \Gamma(TM)$.

Remark 2.2.
(1) By (2.7), we see that the second fundamental form ∇F_* is symmetric.
(2) By (2.7), we obtain

$$[V, X] \in \Gamma(\ker F_*) \tag{2.8}$$

for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

Proposition 2.3 ([12]). *Let $F\colon (M, g_M) \mapsto (N, g_N)$ be a horizontally conformal submersion with dilation λ. Then we obtain*

$$\mathcal{A}_X Y = \frac{1}{2}\Big\{\mathcal{V}[X, Y] - \lambda^2 g_M(X, Y)\nabla_{\mathcal{V}}\Big(\frac{1}{\lambda^2}\Big)\Big\} \tag{2.9}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$.

Here, $\nabla_{\mathcal{V}}$ denotes the gradient vector field in the distribution $\ker F_* \subset TM$ (i.e. $\nabla_{\mathcal{V}} f = \sum_{i=1}^{m} V_i(f)V_i$ for $f \in C^\infty(M)$ and a local orthonormal frame $\{V_1, \ldots, V_m\}$ of $\ker F_*$).

Lemma 2.4 ([3]). *Let $F\colon (M, g_M) \mapsto (N, g_N)$ be a horizontally conformal submersion with dilation λ. Then we have*

$$(\nabla F_*)(X, Y) = X(\ln\lambda)F_*Y + Y(\ln\lambda)F_*X - g_M(X, Y)F_*(\nabla \ln\lambda) \tag{2.10}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$.

We remind some notions. Let (M, g_M, J) be an almost Hermitian manifold, where J is an almost complex structure on M (i.e. $J^2 = -\mathrm{id}$, $g_M(JX, JY) = g_M(X, Y)$ for $X, Y \in \Gamma(TM)$).

We call a horizontally conformal submersion $F\colon (M, g_M, J) \mapsto (N, g_N)$ a *conformal anti-invariant submersion* (see [1]) if $J(\ker F_*) \subset (\ker F_*)^\perp$.

Let M be a $4m$-dimensional C^∞-manifold and let E be a rank 3 subbundle of $\mathrm{End}(TM)$ such that for any point $p \in M$ with a neighborhood U, there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -\mathrm{id}, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1}J_\alpha = J_{\alpha+2},$$

where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Then we call E an *almost quaternionic structure* on M and (M, E) an *almost quaternionic manifold*, see [2].

Moreover, let g be a Riemannian metric on M such that for any point $p \in M$ with a neighborhood U, there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -\mathrm{id}, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1}J_\alpha = J_{\alpha+2}, \tag{2.11}$$

$$g(J_\alpha X, J_\alpha Y) = g(X, Y) \tag{2.12}$$

for all vector fields $X, Y \in \Gamma(TM)$, where the indices are taken from $\{1, 2, 3\}$ modulo 3.

Then we call (M, E, g) an *almost quaternionic Hermitian manifold*, see [14].

Conveniently, the above basis $\{J_1, J_2, J_3\}$ satisfying (2.11) and (2.12) is said to be a *quaternionic Hermitian basis*.

Let (M, E, g) be an almost quaternionic Hermitian manifold.

We call (M, E, g) a *quaternionic Kähler manifold* if there exist locally defined 1-forms ω_1, ω_2, ω_3 such that for $\alpha \in \{1, 2, 3\}$

$$\nabla_X J_\alpha = \omega_{\alpha+2}(X)J_{\alpha+1} - \omega_{\alpha+1}(X)J_{\alpha+2}$$

for $X \in \Gamma(TM)$, where the indices are taken from $\{1, 2, 3\}$ modulo 3, see [14].

If there exists a global parallel quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of sections of E on M (i.e. $\nabla J_\alpha = 0$ for $\alpha \in \{1,2,3\}$, where ∇ is the Levi-Civita connection of the metric g), then (M, E, g) is said to be a *hyperkähler manifold*. Furthermore, we call (J_1, J_2, J_3, g) a *hyperkähler structure* on M and g a *hyperkähler metric*, see [4].

Let (M, E_M, g_M) and (N, E_N, g_N) be almost quaternionic Hermitian manifolds.

A map $F\colon M \mapsto N$ is called a *(E_M, E_N)-holomorphic map* if given a point $x \in M$ for any $J \in (E_M)_x$ there exists $J' \in (E_N)_{F(x)}$ such that

$$F_* \circ J = J' \circ F_*.$$

A Riemannian submersion $F\colon M \mapsto N$ which is a (E_M, E_N)-holomorphic map is called a *quaternionic submersion*, see [14].

Moreover, if (M, E_M, g_M) is a quaternionic Kähler manifold (or a hyperkähler manifold), then we say that F is a *quaternionic Kähler submersion* (or a *hyperkähler submersion*), see [14]. It is well known that any quaternionic Kähler submersion is a harmonic map, see [14].

Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold.

A Riemannian submersion $F\colon (M, E, g_M) \mapsto (N, g_N)$ is called an *h-anti-invariant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that $R(\ker F_*) \subset (\ker F_*)^\perp$ for $R \in \{I, J, K\}$, see [23].

We call such a basis $\{I, J, K\}$ an *h-anti-invariant basis*.

A Riemannian submersion $F\colon (M, E, g_M) \mapsto (N, g_N)$ is called an *h-Lagrangian submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that $I(\ker F_*) = (\ker F_*)^\perp$, $J(\ker F_*) = \ker F_*$, and $K(\ker F_*) = (\ker F_*)^\perp$, see [23].

We call such a basis $\{I, J, K\}$ an *h-Lagrangian basis*.

Throughout this paper, we will use the above notations.

3. Almost h-conformal anti-invariant submersions

In this section, we introduce the notions of h-conformal anti-invariant submersions and h-conformal Lagrangian submersions from almost quaternionic Hermitian manifolds onto Riemannian manifolds. We investigate their properties: the integrability of distributions, the geometry of foliations, the conditions for such maps to be totally geodesic, etc.

Definition 3.1. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Let $F\colon (M, E, g_M) \mapsto (N, g_N)$ be a horizontally conformal submersion. We call the map F an *h-conformal anti-invariant submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that $R(\ker F_*) \subset (\ker F_*)^\perp$ for $R \in \{I, J, K\}$.

We call such a basis $\{I, J, K\}$ an *h-conformal anti-invariant basis.*

Remark 3.2. Let F be an h-conformal anti-invariant submersion from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N). Then it is impossible to satisfy the condition $\dim(\ker F_*) = \dim((\ker F_*)^\perp)$.

If not, then we choose a local quaternionic Hermitian basis $\{I, J, K\}$ of E with $R(\ker F_*) \subset (\ker F_*)^\perp$ for $R \in \{I, J, K\}$. This means

$$R(\ker F_*) = (\ker F_*)^\perp \quad \text{for } R \in \{I, J, K\},$$

so

$$K(\ker F_*) = IJ(\ker F_*) = I((\ker F_*)^\perp) = (\ker F_*),$$

contradiction!

Due to Remark 3.2, we have:

Definition 3.3. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Let $F\colon (M, E, g_M) \mapsto (N, g_N)$ be a horizontally conformal submersion. We call the map F an *h-conformal Lagrangian submersion* if given a point $p \in M$ with a neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that $I(\ker F_*) = (\ker F_*)^\perp$, $J(\ker F_*) = \ker F_*$, and $K(\ker F_*) = (\ker F_*)^\perp$.

We call such a basis $\{I, J, K\}$ an *h-conformal Lagrangian basis.*

Remark 3.4.

(1) We easily check that $J(\ker F_*) = \ker F_*$ implies $J((\ker F_*)^\perp) = (\ker F_*)^\perp$.

(2) In a similar way to Remark 3.2, there does not exist a horizontally conformal submersion F from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N) such that

$$I(\ker F_*) = \ker F_*, \quad J(\ker F_*) = \ker F_*, \quad K(\ker F_*) = (\ker F_*)^\perp$$

for a local quaternionic Hermitian basis $\{I, J, K\}$ of E, i.e. $K(\ker F_*) = IJ(\ker F_*) = \ker F_*$, contradiction!

Let F be an h-conformal anti-invariant submersion (or an h-conformal Lagrangian submersion) from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N). Given a point $p \in M$ with a neighborhood U, we have an h-conformal anti-invariant basis (or an h-conformal Lagrangian basis) $\{I, J, K\}$ of sections of E on U.

Then given $X \in \Gamma((\ker F_*)^\perp)$ and $R \in \{I, J, K\}$, we write

$$RX = B_R X + C_R X, \tag{3.1}$$

where $B_R X \in \Gamma(\ker F_*)$ and $C_R X \in \Gamma((\ker F_*)^\perp)$.

If $F\colon (M, E, g_M) \mapsto (N, g_N)$ is an h-conformal anti-invariant submersion, then we have the orthogonal decomposition

$$(\ker F_*)^\perp = R(\ker F_*) \oplus \mu^R$$

for $R \in \{I, J, K\}$. Then it is easy to check that μ^R is R-invariant for $R \in \{I, J, K\}$.

Given $X \in \Gamma((\ker F_*)^\perp)$ and $R \in \{I, J, K\}$, we obtain

$$X = P_R X + Q_R X, \tag{3.2}$$

where $P_R X \in \Gamma(R(\ker F_*))$ and $Q_R X \in \Gamma(\mu^R)$.

Furthermore, given $R \in \{I, J, K\}$, we get

$$C_R X \in \Gamma(\mu^R) \quad \text{for } X \in \Gamma((\ker F_*)^\perp) \tag{3.3}$$

and

$$g_M(C_R X, RV) = 0 \tag{3.4}$$

for $X \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.

Then we easily obtain:

Lemma 3.5. *Let F be an h-conformal anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal anti-invariant basis. Then we have:*

(1) *for $V, W \in \Gamma(\ker F_*)$ and $R \in \{I, J, K\}$*

$$\mathcal{T}_V RW = B_R \mathcal{T}_V W, \quad \mathcal{H}\nabla_V RW = C_R \mathcal{T}_V W + R\widehat{\nabla}_V W,$$

(2) *for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $R \in \{I, J, K\}$*

$$\mathcal{A}_X C_R Y + \mathcal{V}\nabla_X B_R Y = B_R \mathcal{H}\nabla_X Y, \quad \mathcal{H}\nabla_X C_R Y + \mathcal{A}_X B_R Y = R\mathcal{A}_X Y + C_R \mathcal{H}\nabla_X Y,$$

(3) *for $V \in \Gamma(\ker F_*)$, $X \in \Gamma((\ker F_*)^\perp)$, and $R \in \{I, J, K\}$*

$$\mathcal{A}_X RV = B_R \mathcal{A}_X V, \quad \mathcal{H}\nabla_X RV = C_R \mathcal{A}_X V + R\mathcal{V}\nabla_X V.$$

Now, we will study the integrability of distributions and the geometry of foliations. From [18], we obtain:

Theorem 3.6. *Let F be an h-conformal anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal anti-invariant basis. Then the following conditions are equivalent:*

(a) *the distribution $(\ker F_*)^\perp$ is integrable,*

(b) *for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$*

$$\begin{aligned}\frac{1}{\lambda^2} g_N(\nabla^F_Y F_* C_I X - \nabla^F_X F_* C_I Y, F_* IV) \\ = g_M(\mathcal{A}_Y B_I X - \mathcal{A}_X B_I Y - C_I Y(\ln\lambda)X + C_I X(\ln\lambda)Y \\ + 2g_M(X, C_I Y)\nabla(\ln\lambda), IV),\end{aligned}$$

(c) *for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$*

$$\begin{aligned}\frac{1}{\lambda^2} g_N(\nabla^F_Y F_* C_J X - \nabla^F_X F_* C_J Y, F_* JV) \\ = g_M(\mathcal{A}_Y B_J X - \mathcal{A}_X B_J Y - C_J Y(\ln\lambda)X + C_J X(\ln\lambda)Y \\ + 2g_M(X, C_J Y)\nabla(\ln\lambda), JV),\end{aligned}$$

(d) *for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$*

$$\begin{aligned}\frac{1}{\lambda^2} g_N(\nabla^F_Y F_* C_K X - \nabla^F_X F_* C_K Y, F_* KV) \\ = g_M(\mathcal{A}_Y B_K X - \mathcal{A}_X B_K Y - C_K Y(\ln\lambda)X + C_K X(\ln\lambda)Y \\ + 2g_M(X, C_K Y)\nabla(\ln\lambda), KV).\end{aligned}$$

We deal with the condition for an h-conformal anti-invariant submersion to be horizontally homothetic.

Theorem 3.7. *Let F be an h-conformal anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal anti-invariant basis. Assume that the distribution $(\ker F_*)^\perp$ is integrable. Then the following conditions are equivalent:*

(a) *the map F is horizontally homothetic,*

(b) *for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$*

$$\lambda^2 g_M(\mathcal{A}_X B_I Y - \mathcal{A}_Y B_I X, IV) = g_N(\nabla^F_Y F_* C_I X - \nabla^F_X F_* C_I Y, F_* IV),$$

(c) *for* $X,Y\in\Gamma((\ker F_*)^\perp)$ *and* $V\in\Gamma(\ker F_*)$

$$\lambda^2 g_M(\mathcal{A}_X B_J Y-\mathcal{A}_Y B_J X, JV)=g_N(\nabla^F_Y F_* C_J X-\nabla^F_X F_* C_J Y, F_* JV),$$

(d) *for* $X,Y\in\Gamma((\ker F_*)^\perp)$ *and* $V\in\Gamma(\ker F_*)$

$$\lambda^2 g_M(\mathcal{A}_X B_K Y-\mathcal{A}_Y B_K X, KV)=g_N(\nabla^F_Y F_* C_K X-\nabla^F_X F_* C_K Y, F_* KV).$$

Proof. Given $X,Y\in\Gamma((\ker F_*)^\perp)$, $V\in\Gamma(\ker F_*)$ and $R\in\{I,J,K\}$ with some computation we have

$$\begin{aligned}0=g_M([X,Y],V)&=g_M(\mathcal{A}_X B_R Y-\mathcal{A}_Y B_R X+C_R X(\ln\lambda)Y\\&\quad-C_R Y(\ln\lambda)X+2g_M(X,C_R Y)\nabla(\ln\lambda), RV)\\&\quad-\frac{1}{\lambda^2}g_N(\nabla^F_Y F_* C_R X-\nabla^F_X F_* C_R Y, F_* RV).\end{aligned}\tag{3.5}$$

Using (3.5), we easily get (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c), (a) $\Rightarrow$ (d).

Conversely, from (3.5) we have

$$g_M(C_R X(\ln\lambda)Y-C_R Y(\ln\lambda)X+2g_M(X,C_R Y)\nabla(\ln\lambda), RV)=0.\tag{3.6}$$

Applying $Y=RV$ to (3.6) and using (3.4) we obtain

$$g_M(\nabla(\ln\lambda), C_R X)g_M(RV,RV)=0,$$

so

$$g_M(\nabla(\lambda),X)=0\quad\text{for } X\in\Gamma(\mu^R).\tag{3.7}$$

Applying $Y=C_R X$, $X\in\Gamma(\mu^R)$ to (3.6) we have

$$2g_M(X,C_R^2 X)g_M(\nabla(\ln\lambda),RV)=-2g_M(X,X)g_M(\nabla(\ln\lambda),RV)=0,$$

so

$$g_M(\nabla(\lambda),RV)=0\quad\text{for } V\in\Gamma(\mathcal{D}_2^R).\tag{3.8}$$

By (3.7) and (3.8), we get (b) $\Rightarrow$ (a), (c) $\Rightarrow$ (a), (d) $\Rightarrow$ (a). Therefore, the result follows. $\square$

Lemma 3.8. *Let F be an h-conformal Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal Lagrangian basis. Then the following conditions are equivalent:*

(a) *the distribution* $(\ker F_*)^\perp$ *is integrable,*
(b) $\mathcal{A}_X IY = \mathcal{A}_Y IX$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$,
(c) $\mathcal{A}_X KY = \mathcal{A}_Y KX$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$,
(d) $\mathcal{A}_X JY = \mathcal{A}_Y JX$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$.

Proof. Since $B_R = R$ and $C_R = 0$ on $(\ker F_*)^\perp$ for $R \in \{I, K\}$, from Theorem 3.6 we obtain (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c).

Given $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$, since $J(\ker F_*) = \ker F_*$, we get

$$g_M([X, Y], JV) = -g_M(\nabla_X JY - \nabla_Y JX, V) = g_M(\mathcal{A}_Y JX - \mathcal{A}_X JY, V),$$

which implies (a) $\Leftrightarrow$ (d). Therefore, the result follows. $\square$

From [18], we obtain:

Theorem 3.9. *Let F be an h-conformal anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal anti-invariant basis. Then the following conditions are equivalent:*

(a) *the distribution* $(\ker F_*)^\perp$ *defines a totally geodesic foliation on* M,
(b) *for* $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$

$$g_N(\nabla^F_X F_* IV, F_* C_I Y) = \lambda^2 g_M(\mathcal{A}_X B_I Y - C_I Y(\ln\lambda)X + g_M(X, C_I Y)\nabla(\ln\lambda), IV),$$

(c) *for* $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$

$$g_N(\nabla^F_X F_* JV, F_* C_J Y) = \lambda^2 g_M(\mathcal{A}_X B_J Y - C_J Y(\ln\lambda)X + g_M(X, C_J Y)\nabla(\ln\lambda), JV),$$

(d) *for* $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$

$$g_N(\nabla^F_X F_* KV, F_* C_K Y) = \lambda^2 g_M(\mathcal{A}_X B_K Y - C_K Y(\ln\lambda)X + g_M(X, C_K Y)\nabla(\ln\lambda), KV).$$

Theorem 3.10. *Let F be an h-conformal anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal anti-invariant basis. Assume that the distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation on M. Then the following conditions are equivalent:*

(a) *the map F is horizontally homothetic,*

(b) *for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$*

$$g_N(F_* C_I Y, \nabla^F_X F_* IV) = \lambda^2 g_M(\mathcal{A}_X B_I Y, IV),$$

(c) *for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$*

$$g_N(F_* C_J Y, \nabla^F_X F_* JV) = \lambda^2 g_M(\mathcal{A}_X B_J Y, JV),$$

(d) *for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$*

$$g_N(F_* C_K Y, \nabla^F_X F_* KV) = \lambda^2 g_M(\mathcal{A}_X B_K Y, KV).$$

Proof. Given $X, Y \in \Gamma((\ker F_*)^\perp)$, $V \in \Gamma(\ker F_*)$, and $R \in \{I, J, K\}$, by Theorem 3.9, we get

$$g_N(\nabla^F_X F_* RV, F_* C_R Y) = \lambda^2 g_M(\mathcal{A}_X B_R Y - C_R Y(\ln\lambda)X + g_M(X, C_R Y)\nabla(\ln\lambda), RV). \tag{3.9}$$

Hence, it means (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c), (a) $\Rightarrow$ (d).

Conversely, from (3.9) we obtain

$$0 = g_M(-C_R Y(\ln\lambda)X + g_M(X, C_R Y)\nabla(\ln\lambda), RV). \tag{3.10}$$

Applying $X = C_R Y$ to (3.10) and using (3.4), we have

$$0 = g_M(C_R Y, C_R Y) g_M(\nabla(\ln\lambda), RV),$$

so

$$g_M(\nabla(\lambda), RV) = 0 \quad \text{for } V \in \Gamma(\ker F_*). \tag{3.11}$$

Applying $X = RV$ to (3.10) and using (3.4), we get

$$0 = g_M(\nabla(\ln\lambda), C_R Y) g_M(RV, RV),$$

so

$$g_M(\nabla(\lambda), Y) = 0 \quad \text{for } Y \in \Gamma(\mu_R). \tag{3.12}$$

By (3.11) and (3.12), we have (b) $\Rightarrow$ (a), (c) $\Rightarrow$ (a), (d) $\Rightarrow$ (a).

Therefore, the result follows. □

Lemma 3.11. *Let F be an h-conformal Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal Lagrangian basis. Then the following conditions are equivalent:*
(a) *the distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation on M,*
(b) $\mathcal{A}_X IY = 0$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$,
(c) $\mathcal{A}_X KY = 0$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$,
(d) $\mathcal{A}_X JY = 0$ *for* $X, Y \in \Gamma((\ker F_*)^\perp)$.

Proof. Since $B_R = R$ and $C_R = 0$ on $(\ker F_*)^\perp$ for $R \in \{I, K\}$, from Theorem 3.9 we get (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c).

Given $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$, since $J(\ker F_*) = \ker F_*$ we have

$$g_M(\nabla_X Y, JV) = -g_M(\nabla_X JY, V) = -g_M(\mathcal{A}_X JY, V),$$

which implies (a) $\Leftrightarrow$ (d). Therefore, we get the result. $\square$

From [18], we obtain:

Theorem 3.12. *Let F be an h-conformal anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal anti-invariant basis. Then the following conditions are equivalent:*
(a) *the distribution $\ker F_*$ defines a totally geodesic foliation on M,*
(b) *for $X \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$*

$$-\frac{1}{\lambda^2} g_N(\nabla^F_{IW} F_* IV, F_* IC_I X) = g_M(\mathcal{T}_V IW, B_I X) + g_M(W, V) g_M(\nabla(\ln\lambda), IC_I X),$$

(c) *for $X \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$*

$$-\frac{1}{\lambda^2} g_N(\nabla^F_{JW} F_* JV, F_* JC_J X) = g_M(\mathcal{T}_V JW, B_J X) + g_M(W, V) g_M(\nabla(\ln\lambda), JC_J X),$$

(d) *for $X \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$*

$$-\frac{1}{\lambda^2} g_N(\nabla^F_{KW} F_* KV, F_* KC_K X) = g_M(\mathcal{T}_V KW, B_K X) + g_M(W, V) g_M(\nabla(\ln\lambda), KC_K X).$$

Lemma 3.13. *Let F be an h-conformal Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal Lagrangian basis. Then the following conditions are equivalent:*
(a) *the distribution $\ker F_*$ defines a totally geodesic foliation on M,*
(b) $\mathcal{T}_V IW = 0$ *for* $V, W \in \Gamma(\ker F_*)$,
(c) $\mathcal{T}_V KW = 0$ *for* $V, W \in \Gamma(\ker F_*)$,
(d) $\mathcal{T}_V JW = 0$ *for* $V, W \in \Gamma(\ker F_*)$.

P r o o f. Since $B_R = R$ and $C_R = 0$ on $(\ker F_*)^\perp$ for $R \in \{I, K\}$, from Theorem 3.12 we have (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c).

Given $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$, since $J(\ker F_*) = \ker F_*$, we get

$$g_M(\nabla_V W, JX) = -g_M(\nabla_V JW, X) = -g_M(\mathcal{T}_V JW, X),$$

which implies (a) $\Leftrightarrow$ (d). Therefore, we obtain the result. $\square$

Lemma 3.14 ([3]). *Let F be a horizontally conformal submersion from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N) with dilation λ.*

Then the tension field $\tau(F)$ of F is given by

$$\tau(F) = -mF_*H + (2 - n)F_*(\nabla(\ln\lambda)), \tag{3.13}$$

where H is the mean curvature vector field of the distribution $\ker F_*$, $m = \dim\ker F_*$, $n = \dim N$.

Using Lemma 3.14, we easily get:

Corollary 3.15. *Let F be an h-conformal anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal anti-invariant basis. Assume that F is harmonic with $\dim\ker F_* > 0$ and $\dim N > 2$. Then the following conditions are equivalent:*

(a) *all the fibers of F are minimal,*

(b) *the map F is horizontally homothetic.*

Corollary 3.16. *Let F be an h-conformal anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal anti-invariant basis. Assume that $\dim\ker F_* > 0$ and $\dim N = 2$. Then the following conditions are equivalent:*

(a) *all the fibers of F are minimal,*

(b) *the map F is harmonic.*

Lemma 3.17. *Let F be an h-conformal Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal Lagrangian basis. Then we have*

$$\tau(F) = (2 - 2m)F_*(\nabla(\ln\lambda)), \tag{3.14}$$

where $2m = \dim\ker F_*$.

P r o o f. Since $J(\ker F_*) = \ker F_*$, it means $J((\ker F_*)^\perp) = (\ker F_*)^\perp$. So, we can choose a local orthonormal frame $\{e_1, Je_1, \ldots, e_m, Je_m\}$ of $\ker F_*$.

Given $V, W \in \Gamma(\ker F_*)$, we get

$$\mathcal{T}_V JW = \mathcal{H}\nabla_V JW = \mathcal{H}J\nabla_V W = \mathcal{H}J(\mathcal{T}_V W + \widehat{\nabla}_V W) = J\mathcal{T}_V W,$$

so

$$\begin{aligned} 2mH &= \sum_{i=1}^{m}(\mathcal{T}_{e_i} e_i + \mathcal{T}_{Je_i} Je_i) = \sum_{i=1}^{m}(\mathcal{T}_{e_i} e_i + J\mathcal{T}_{Je_i} e_i) \\ &= \sum_{i=1}^{m}(\mathcal{T}_{e_i} e_i + J\mathcal{T}_{e_i} Je_i) = \sum_{i=1}^{m}(\mathcal{T}_{e_i} e_i - \mathcal{T}_{e_i} e_i) = 0. \end{aligned}$$

By Lemma 3.14, we obtain the result. □

From Lemma 3.17, we easily have:

Lemma 3.18. *Let F be an h-conformal Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal Lagrangian basis. Assume that* $\dim \ker F_* > 2$.

Then the map F is harmonic if and only if F is horizontally homothetic.

Definition 3.19. Let F be an h-conformal anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal anti-invariant basis. Then given $R \in \{I, J, K\}$, we call the map F $(R\ker F_*, \mu^R)$*-totally geodesic* if it satisfies $(\nabla F_*)(RV, X) = 0$ for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma(\mu^R)$.

Theorem 3.20. *Let F be an h-conformal anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal anti-invariant basis. Then the following conditions are equivalent:*

(a) *the map F is horizontally homothetic,*
(b) *the map F is $(I\ker F_*, \mu^I)$-totally geodesic,*
(c) *the map F is $(J\ker F_*, \mu^J)$-totally geodesic,*
(d) *the map F is $(K\ker F_*, \mu^K)$-totally geodesic.*

P r o o f. Given $V \in \Gamma(\ker F_*)$, $X \in \Gamma(\mu^R)$ and $R \in \{I, J, K\}$, by (2.10), we have

$$\begin{aligned} (\nabla F_*)(RV, X) &= RV(\ln\lambda)F_*X + X(\ln\lambda)F_*RV - g_M(RV, X)F_*(\nabla\ln\lambda) \\ &= RV(\ln\lambda)F_*X + X(\ln\lambda)F_*RV. \end{aligned}$$

Since $g_N(F_*X, F_*RV) = \lambda^2 g_M(X, RV) = 0$, $\{F_*X, F_*RV\}$ is linearly independent for nonzero X, V.

Hence, we get (a) $\Leftrightarrow$ (b), (a) $\Leftrightarrow$ (c), (a) $\Leftrightarrow$ (d). Therefore, the result follows. □

From [18], we obtain:

Theorem 3.21. *Let F be an h-conformal anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal anti-invariant basis. Then the following conditions are equivalent:*

(a) *the map F is a totally geodesic map,*

(b) (i) $\mathcal{T}_V IW = 0$ *and* $\mathcal{H}\nabla_V IW \in \Gamma(I \ker F_*)$,

(ii) *F is horizontally homothetic,*

(iii) $\widehat{\nabla}_V B_I X + \mathcal{T}_V C_I X = 0$, $\mathcal{T}_V B_I X + \mathcal{H}\nabla_V C_I X \in \Gamma(I \ker F_*)$ *for* $V, W \in \Gamma(\ker F_*)$ *and* $X \in \Gamma((\ker F_*)^\perp)$,

(c) (i) $\mathcal{T}_V JW = 0$ *and* $\mathcal{H}\nabla_V JW \in \Gamma(J \ker F_*)$,

(ii) *F is horizontally homothetic,*

(iii) $\widehat{\nabla}_V B_J X + \mathcal{T}_V C_J X = 0$, $\mathcal{T}_V B_J X + \mathcal{H}\nabla_V C_J X \in \Gamma(J \ker F_*)$ *for* $V, W \in \Gamma(\ker F_*)$ *and* $X \in \Gamma((\ker F_*)^\perp)$,

(d) (i) $\mathcal{T}_V KW = 0$ *and* $\mathcal{H}\nabla_V KW \in \Gamma(K \ker F_*)$,

(ii) *F is horizontally homothetic,*

(iii) $\widehat{\nabla}_V B_K X + \mathcal{T}_V C_K X = 0$, $\mathcal{T}_V B_K X + \mathcal{H}\nabla_V C_K X \in \Gamma(K \ker F_*)$ *for* $V, W \in \Gamma(\ker F_*)$ *and* $X \in \Gamma((\ker F_*)^\perp)$.

Remark 3.22. Using the proof of Theorem 3.21, we can show that F is horizontally homothetic if and only if $(\nabla F_*)(X, Y) = 0$ for $X, Y \in \Gamma((\ker F_*)^\perp)$.

Lemma 3.23. *Let F be an h-conformal Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal Lagrangian basis. Then the following conditions are equivalent:*

(a) *the map F is a totally geodesic map,*

(b) (i) $\mathcal{T}_V IW = 0$,

(ii) *F is horizontally homothetic,*

(iii) $\mathcal{A}_X IV = 0$ *for* $V, W \in \Gamma(\ker F_*)$ *and* $X \in \Gamma((\ker F_*)^\perp)$,

(c) (i) $\mathcal{T}_V KW = 0$,

(ii) *F is horizontally homothetic,*

(iii) $\mathcal{A}_X KV = 0$ *for* $V, W \in \Gamma(\ker F_*)$ *and* $X \in \Gamma((\ker F_*)^\perp)$,

(d) (i) $\mathcal{T}_V JW = 0$,

(ii) *F is horizontally homothetic,*

(iii) $\mathcal{A}_X JV = 0$ *for* $V, W \in \Gamma(\ker F_*)$ *and* $X \in \Gamma((\ker F_*)^\perp)$.

Proof. We know $B_R = R$ and $C_R = 0$ on $(\ker F_*)^\perp$ for $R \in \{I, K\}$ and we get

$$\widehat{\nabla}_V RX = \mathcal{V}R\nabla_V X = \mathcal{V}R\nabla_X V = \mathcal{V}\nabla_X RV = \mathcal{A}_X RV$$

for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$.

By Theorem 3.21, we have (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c).

Given $V, W \in \Gamma(\ker F_*)$, since $J(\ker F_*) = \ker F_*$, we get

$$(\nabla F_*)(V,W) = F_*(J\nabla_V JW) = F_*(J(\mathcal{T}_V JW + \widehat{\nabla}_V JW)) = F_* J\mathcal{T}_V JW,$$

so

$$(\nabla F_*)(V,W) = 0 \Leftrightarrow \mathcal{T}_V JW = 0.$$

We claim that F is horizontally homothetic if and only if $(\nabla F_*)(X,Y) = 0$ for $X, Y \in \Gamma((\ker F_*)^\perp)$.

By (2.10), we have

$$(\nabla F_*)(X,Y) = X(\ln\lambda)F_*Y + Y(\ln\lambda)F_*X - g_M(X,Y)F_*(\nabla\ln\lambda) \tag{3.15}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$, so the part from left to right is obtained.

Conversely, from (3.15) we obtain

$$0 = X(\ln\lambda)F_*Y + Y(\ln\lambda)F_*X - g_M(X,Y)F_*(\nabla\ln\lambda). \tag{3.16}$$

Applying $X = Y$ to (3.16), we have

$$0 = 2X(\ln\lambda)F_*X - g_M(X,X)F_*(\nabla\ln\lambda). \tag{3.17}$$

Taking the inner product with F_*X at (3.17), we get

$$0 = \lambda^2 g_M(X,X) g_M(X, \nabla\ln\lambda),$$

which implies our result.

Given $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$, we obtain

$$(\nabla F_*)(X,V) = F_*(J\nabla_X JV) = F_*(J(\mathcal{A}_X JV + \mathcal{V}\nabla_X JV)) = F_* J\mathcal{A}_X JV,$$

so

$$(\nabla F_*)(X,V) = 0 \Leftrightarrow \mathcal{A}_X JV = 0.$$

Hence, we have (a) $\Leftrightarrow$ (d). Therefore, result follows. □

4. Decomposition theorems

We will consider some decomposition theorems and we need to remind some notions.

Let (M, g) be a Riemannian manifold and L a foliation of M. Let ξ be the tangent bundle of L considered as a subbundle of the tangent bundle TM of M.

We call L a *totally umbilic foliation* (see [25]) of M if

$$h(X,Y) = g(X,Y)H \quad \text{for } X, Y \in \Gamma(\xi), \tag{4.1}$$

where h is the second fundamental form of L in M and H is the mean curvature vector field of L in M.

The foliation L is said to be a *spheric foliation* (see [25]) if it is a totally umbilic foliation and

$$\nabla_X H \in \Gamma(\xi) \quad \text{for } X \in \Gamma(\xi), \tag{4.2}$$

where ∇ is the Levi-Civita connection of g.

We call L a *totally geodesic foliation* (see [25]) of M if

$$\nabla_X Y \in \Gamma(\xi) \quad \text{for } X, Y \in \Gamma(\xi). \tag{4.3}$$

Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, $f_i\colon M_1 \times M_2 \mapsto \mathbb{R}$ a positive C^∞-function, and $\pi_i\colon M_1 \times M_2 \mapsto M_i$ the canonical projection for $i = 1, 2$.

We call $M_1 \times_{(f_1,f_2)} M_2$ a *double-twisted product manifold* (see [25]) of (M_1, g_1) and (M_2, g_2) if it is the product manifold $M = M_1 \times M_2$ with a Riemannian metric g such that

$$g(X,Y) = f_1^2 \cdot g_1(\pi_{1*}X, \pi_{1*}Y) + f_2^2 \cdot g_2(\pi_{2*}X, \pi_{2*}Y) \quad \text{for } X, Y \in \Gamma(TM). \tag{4.4}$$

We call $M_1 \times_{(f_1,f_2)} M_2$ *nontrivial* if all the functions f_1 and f_2 are nonconstant.

A Riemannian manifold $M_1 \times_f M_2$ is said to be a *twisted product manifold* (see [25]) of (M_1, g_1) and (M_2, g_2) if $M_1 \times_f M_2 = M_1 \times_{(1,f)} M_2$.

We call $M_1 \times_f M_2$ *nontrivial* if f is nonconstant.

A twisted product manifold $M_1 \times_f M_2$ is said to be a *warped product manifold* (see [25]) of (M_1, g_1) and (M_2, g_2) if f depends only on the points of M_1 (i.e. $f \in C^\infty(M_1, \mathbb{R})$).

Let M_1 and M_2 be connected C^∞-manifolds and M the product manifold $M_1 \times M_2$. Let $\pi_i\colon M \mapsto M_i$ be the canonical projection for $i = 1, 2$. Let $\xi_i := \ker \pi_{3-i_*}$ and $P_i\colon TM \mapsto \xi_i$ the vector bundle projection such that $TM = \xi_1 \oplus \xi_2$. And let L_i be the canonical foliation of M by the integral manifolds of ξ_i for $i = 1, 2$.

Proposition 4.1 ([25]). *Let g be a Riemannian metric on the product manifold $M_1 \times M_2$ and assume that the canonical foliations L_1 and L_2 intersect perpendicularly everywhere. Then g is the metric of*

(a) *a double-twisted product manifold* $M_1 \times_{(f_1,f_2)} M_2$ *if and only if* L_1 *and* L_2 *are totally umbilic foliations,*

(b) *a twisted product manifold* $M_1 \times_f M_2$ *if and only if* L_1 *is a totally geodesic foliation and* L_2 *is a totally umbilic foliation,*

(c) *a warped product manifold* $M_1 \times_f M_2$ *if and only if* L_1 *is a totally geodesic foliation and* L_2 *is a spheric foliation,*

(d) *a (usual) Riemannian product manifold* $M_1 \times M_2$ *if and only if* L_1 *and* L_2 *are totally geodesic foliations.*

Let F be a horizontally conformal submersion from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N) such that the distributions $\ker F_*$ and $(\ker F_*)^\perp$ are integrable. Then we denote by $M_{\ker F_*}$ and $M_{(\ker F_*)^\perp}$ the integral manifolds of $\ker F_*$ and $(\ker F_*)^\perp$, respectively. We also denote by H and $H^\perp$ the mean curvature vector fields of $\ker F_*$ and $(\ker F_*)^\perp$, respectively, i.e. $H = m^{-1} \sum_{i=1}^{m} \mathcal{T}_{e_i} e_i$ and $H^\perp = n^{-1} \sum_{i=1}^{n} \mathcal{A}_{v_i} v_i$ for a local orthonormal frame $\{e_1, \ldots, e_m\}$ of $\ker F_*$ and a local orthonormal frame $\{v_1, \ldots, v_n\}$ of $(\ker F_*)^\perp$.

Using Proposition 4.1, Theorems 3.9 and 3.12 we have:

Theorem 4.2. *Let* F *be an h-conformal anti-invariant submersion from a hyperkähler manifold* (M, I, J, K, g_M) *onto a Riemannian manifold* (N, g_N) *such that* (I, J, K) *is an h-conformal anti-invariant basis. Then the following conditions are equivalent:*

(a) (M, g_M) *is locally a Riemannian product manifold of the form* $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$,

(b) *for* $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V, W \in \Gamma(\ker F_*)$

$$\begin{aligned} g_N(\nabla^F_X F_* IV, F_* C_I Y) &= \lambda^2 g_M(\mathcal{A}_X B_I Y - C_I Y(\ln\lambda) X \\ &\quad + g_M(X, C_I Y)\nabla(\ln\lambda), IV), \\ -\frac{1}{\lambda^2} g_N(\nabla^F_{IW} F_* IV, F_* I C_I X) &= g_M(\mathcal{T}_V IW, B_I X) \\ &\quad + g_M(W, V) g_M(\nabla(\ln\lambda), I C_I X), \end{aligned}$$

(c) *for* $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V, W \in \Gamma(\ker F_*)$

$$\begin{aligned} g_N(\nabla^F_X F_* JV, F_* C_J Y) &= \lambda^2 g_M(\mathcal{A}_X B_J Y - C_J Y(\ln\lambda) X \\ &\quad + g_M(X, C_J Y)\nabla(\ln\lambda), JV), \\ -\frac{1}{\lambda^2} g_N(\nabla^F_{JW} F_* JV, F_* J C_J X) &= g_M(\mathcal{T}_V JW, B_J X) \\ &\quad + g_M(W, V) g_M(\nabla(\ln\lambda), J C_J X), \end{aligned}$$

(d) *for* $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V, W \in \Gamma(\ker F_*)$.

$$
\begin{aligned}
g_N(\nabla^F_X F_* KV, F_* C_K Y) &= \lambda^2 g_M(\mathcal{A}_X B_K Y - C_K Y(\ln\lambda)X \\
&\quad + g_M(X, C_K Y)\nabla(\ln\lambda), KV), \\
-\frac{1}{\lambda^2} g_N(\nabla^F_{KW} F_* KV, F_* K C_K X) &= g_M(\mathcal{T}_V KW, B_K X) \\
&\quad + g_M(W, V) g_M(\nabla(\ln\lambda), K C_K X).
\end{aligned}
$$

Using Proposition 4.1, Lemmas 3.11 and 3.13 we get:

Lemma 4.3. *Let F be an h-conformal Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal Lagrangian basis. Then the following conditions are equivalent:*

(a) *(M, g_M) is locally a Riemannian product manifold of the form $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$,*

(b) *$\mathcal{A}_X IY = 0$ and $\mathcal{T}_V IW = 0$ for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$,*

(c) *$\mathcal{A}_X KY = 0$ and $\mathcal{T}_V KW = 0$ for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$,*

(d) *$\mathcal{A}_X JY = 0$ and $\mathcal{T}_V JW = 0$ for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$.*

Remark 4.4. The necessary and sufficient conditions for the manifold (M, g_M) to be locally a Riemannian product manifold of the form $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$ in an h-anti-invariant submersion are quite different from the necessary and sufficient conditions for the manifold (M, g_M) to be locally a Riemannian product manifold of the form $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$ in an h-conformal anti-invariant submersion.

On the other hand, the conditions for the manifold (M, g_M) to be locally a Riemannian product manifold of the form $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$ in an h-Lagrangian submersion are the same as the necessary and sufficient conditions for the manifold (M, g_M) to be locally a Riemannian product manifold of the form $M_{(\ker F_*)^\perp} \times M_{\ker F_*}$ in an h-conformal Lagrangian submersion, [23].

We deal with the geometry of distributions $\ker F_*$ and $(\ker F_*)^\perp$.

Theorem 4.5. *Let F be a horizontally conformal submersion from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N) with dilation λ. Assume that the distribution $(\ker F_*)^\perp$ defines a totally umbilic foliation on M. Then we have*

$$
H^\perp = -\frac{\lambda^2}{2}\nabla_{\mathcal{V}}\Big(\frac{1}{\lambda^2}\Big),
$$

where $\nabla_{\mathcal{V}}$ denotes the gradient vector in the distribution $\ker F_*$.

P r o o f. Given $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$ we obtain

$$g_M(\nabla_X Y, V) = g_M(\mathcal{A}_X Y, V) = g_M(X, Y) g_M(H^\perp, V) \tag{4.5}$$

and

$$g_M(\nabla_X Y, V) = -g_M(Y, \nabla_X V) = -g_M(Y, \mathcal{A}_X V). \tag{4.6}$$

Comparing (4.5) and (4.6), we have $\mathcal{A}_X V = -g_M(H^\perp, V)X$, so

$$g_M(\mathcal{A}_X V, X) = -g_M(H^\perp, V) g_M(X, X). \tag{4.7}$$

On the other hand, by using (2.9), we get

$$\begin{aligned} g_M(\mathcal{A}_X V, X) &= g_M(\nabla_X V, X) = -g_M(V, \nabla_X X) \\ &= -g_M(V, \mathcal{A}_X X) = g_M\Big(V, \frac{\lambda^2}{2} g_M(X, X) \nabla_{\mathcal{V}}\Big(\frac{1}{\lambda^2}\Big)\Big) \\ &= \frac{\lambda^2}{2} g_M(X, X) g_M\Big(V, \nabla_{\mathcal{V}}\Big(\frac{1}{\lambda^2}\Big)\Big). \end{aligned} \tag{4.8}$$

Comparing (4.7) and (4.8), we obtain the result. □

Remark 4.6. In Theorem 4.5, if F is a Riemannian submersion, then we get $H^\perp = 0$, so the distribution $(\ker F_*)^\perp$ also defines a totally geodesic foliation on M, see [23].

Theorem 4.7. *Let F be an h-conformal anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal anti-invariant basis. Then the following conditions are equivalent:*

(a) *the distribution $\ker F_*$ defines a totally umbilic foliation on M,*

(b) *for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$*

$$\mathcal{T}_V B_I X + \mathcal{H}\nabla_V C_I X = -g_M(H, X) IV,$$

(c) *for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$*

$$\mathcal{T}_V B_J X + \mathcal{H}\nabla_V C_J X = -g_M(H, X) JV,$$

(d) *for $V \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$*

$$\mathcal{T}_V B_K X + \mathcal{H}\nabla_V C_K X = -g_M(H, X) KV.$$

P r o o f. Given $V, W \in \Gamma(\ker F_*)$, $X \in \Gamma((\ker F_*)^\perp)$ and $R \in \{I, J, K\}$, we have

$$\begin{aligned} g_M(\mathcal{T}_V W, X) = g_M(\nabla_V RW, RX) &= -g_M(RW, \nabla_V B_R X + \nabla_V C_R X) \\ &= -\, g_M(RW, \mathcal{T}_V B_R X + \mathcal{H}\nabla_V C_R X), \end{aligned}$$

so we easily obtain

$$\mathcal{T}_V W = g_M(V, W)H \Leftrightarrow \mathcal{T}_V B_R X + \mathcal{H}\nabla_V C_R X = -g_M(H, X)RV.$$

Hence, we get (a) $\Leftrightarrow$ (b), (a) $\Leftrightarrow$ (c), (a) $\Leftrightarrow$ (d). Therefore, the result follows. □

Lemma 4.8. *Let F be an h-conformal Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal Lagrangian basis. Then the following conditions are equivalent:*
(a) *the distribution $\ker F_*$ defines a totally umbilic foliation on M,*
(b) $\mathcal{T}_V IX = -g_M(H, X)IV$ *for* $X \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$,
(c) $\mathcal{T}_V KX = -g_M(H, X)KV$ *for* $X \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$,
(d) $\mathcal{T}_V JX = -g_M(H, X)JV$ *for* $X \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

P r o o f. Since $B_R = R$ and $C_R = 0$ on $(\ker F_*)^\perp$ for $R \in \{I, K\}$, from Theorem 4.7 we have (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c).

Given $V, W \in \Gamma(\ker F_*)$ and $X \in \Gamma((\ker F_*)^\perp)$, since $J(\ker F_*) = \ker F_*$, we obtain

$$g_M(\mathcal{T}_V W, X) = g_M(\nabla_V JW, JX) = -g_M(JW, \nabla_V JX) = -g_M(JW, \mathcal{T}_V JX),$$

so we get

$$\mathcal{T}_V W = g_M(V, W)H \Leftrightarrow \mathcal{T}_V JX = -g_M(H, X)JV,$$

which implies (a) $\Leftrightarrow$ (d). Therefore, we have the result. □

Using Proposition 4.1, Theorems 3.9 and 4.7 we obtain:

Theorem 4.9. *Let F be an h-conformal anti-invariant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal anti-invariant basis. Then the following conditions are equivalent:*
(a) (M, g_M) *is locally a twisted product manifold of the form* $M_{(\ker F_*)^\perp} \times_f M_{\ker F_*}$,
(b) *for* $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$

$$\begin{aligned} g_N(\nabla^F_X F_* IV, F_* C_I Y) &= \lambda^2 g_M(\mathcal{A}_X B_I Y - C_I Y(\ln\lambda)X \\ &\quad + g_M(X, C_I Y)\nabla(\ln\lambda), IV), \\ \mathcal{T}_V B_I X + \mathcal{H}\nabla_V C_I X &= -\, g_M(H, X)IV, \end{aligned}$$

(c) *for* $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$

$$\begin{aligned} g_N(\nabla^F_X F_* JV, F_* C_J Y) &= \lambda^2 g_M(\mathcal{A}_X B_J Y - C_J Y(\ln\lambda)X \\ &\quad + g_M(X, C_J Y)\nabla(\ln\lambda), JV), \\ \mathcal{T}_V B_J X + \mathcal{H}\nabla_V C_J X &= -g_M(H, X)JV, \end{aligned}$$

(d) *for* $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$

$$\begin{aligned} g_N(\nabla^F_X F_* KV, F_* C_K Y) &= \lambda^2 g_M(\mathcal{A}_X B_K Y - C_K Y(\ln\lambda)X \\ &\quad + g_M(X, C_K Y)\nabla(\ln\lambda), KV), \\ \mathcal{T}_V B_K X + \mathcal{H}\nabla_V C_K X &= -g_M(H, X)KV. \end{aligned}$$

Using Proposition 4.1, Lemmas 3.11 and 4.8, we get:

Lemma 4.10. *Let F be an h-conformal Lagrangian submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an h-conformal Lagrangian basis. Then the following conditions are equivalent:*

(a) *(M, g_M) is locally a twisted product manifold of the form $M_{(\ker F_*)^\perp} \times_f M_{\ker F_*}$,*

(b) *$\mathcal{A}_X IY = 0$ and $\mathcal{T}_V IX = -g_M(H, X)IV$ for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$,*

(c) *$\mathcal{A}_X KY = 0$ and $\mathcal{T}_V KX = -g_M(H, X)KV$ for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$,*

(d) *$\mathcal{A}_X JY = 0$ and $\mathcal{T}_V JX = -g_M(H, X)JV$ for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$.*

5. Examples

Note that given an Euclidean space $\mathbb{R}^{4m}$ with coordinates $(x_1, x_2, \ldots, x_{4m})$, we can canonically choose complex structures I, J, K on $\mathbb{R}^{4m}$ as follows:

$$\begin{aligned} I\Big(\frac{\partial}{\partial x_{4k+1}}\Big) &= \frac{\partial}{\partial x_{4k+2}}, & I\Big(\frac{\partial}{\partial x_{4k+2}}\Big) &= -\frac{\partial}{\partial x_{4k+1}}, \\ I\Big(\frac{\partial}{\partial x_{4k+3}}\Big) &= \frac{\partial}{\partial x_{4k+4}}, & I\Big(\frac{\partial}{\partial x_{4k+4}}\Big) &= -\frac{\partial}{\partial x_{4k+3}}, \\ J\Big(\frac{\partial}{\partial x_{4k+1}}\Big) &= \frac{\partial}{\partial x_{4k+3}}, & J\Big(\frac{\partial}{\partial x_{4k+2}}\Big) &= -\frac{\partial}{\partial x_{4k+4}}, \\ J\Big(\frac{\partial}{\partial x_{4k+3}}\Big) &= -\frac{\partial}{\partial x_{4k+1}}, & J\Big(\frac{\partial}{\partial x_{4k+4}}\Big) &= \frac{\partial}{\partial x_{4k+2}}, \end{aligned}$$

$$K\Big(\frac{\partial}{\partial x_{4k+1}}\Big) = \frac{\partial}{\partial x_{4k+4}}, \qquad K\Big(\frac{\partial}{\partial x_{4k+2}}\Big) = \frac{\partial}{\partial x_{4k+3}},$$
$$K\Big(\frac{\partial}{\partial x_{4k+3}}\Big) = -\frac{\partial}{\partial x_{4k+2}}, \quad K\Big(\frac{\partial}{\partial x_{4k+4}}\Big) = -\frac{\partial}{\partial x_{4k+1}}$$

for $k \in \{0, 1, \dots, m-1\}$.

Then we easily check that $(I, J, K, \langle\, ,\,\rangle)$ is a hyperkähler structure on $\mathbb{R}^{4m}$, where $\langle\, ,\,\rangle$ denotes the Euclidean metric on $\mathbb{R}^{4m}$. Throughout this section, we will use these notations.

Example 5.1. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Let $F\colon (M, E, g_M) \mapsto (N, g_N)$ be an h-anti-invariant submersion, see [23]. Then the map F is an h-conformal anti-invariant submersion with dilation $\lambda = 1$.

Example 5.2. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. Let $F\colon (M, E, g_M) \mapsto (N, g_N)$ be an h-Lagrangian submersion, see [23]. Then the map F is an h-conformal Lagrangian submersion with dilation $\lambda = 1$.

Example 5.3. Let (M, E, g_M) be a $4n$-dimensional almost quaternionic Hermitian manifold and (N, g_N) a $(4n-1)$-dimensional Riemannian manifold. Let $F\colon (M, E, g_M) \mapsto (N, g_N)$ be a horizontally conformal submersion with dilation λ. Then the map F is an h-conformal anti-invariant submersion with dilation λ.

Example 5.4. Let $F\colon \mathbb{R}^4 \mapsto \mathbb{R}^3$ be a horizontally conformal submersion with dilation λ. Then the map F is an h-conformal anti-invariant submersion with dilation λ.

Example 5.5. Define a map $F\colon \mathbb{R}^4 \mapsto \mathbb{R}^2$ by

$$F(x_1, \dots, x_4) = e^{34}(x_2, x_1).$$

Then the map F is an h-conformal Lagrangian submersion such that $I(\ker F_*) = \ker F_*$, $J(\ker F_*) = (\ker F_*)^{\perp}$, $K(\ker F_*) = (\ker F_*)^{\perp}$, and dilation $\lambda = e^{34}$.

Here, (K, I, J) is an h-conformal Lagrangian basis.

Example 5.6. Define a map $F\colon \mathbb{R}^8 \mapsto \mathbb{R}^6$ by

$$F(x_1, \dots, x_8) = \pi^{68}(x_2, \dots, x_7).$$

Then the map F is an h-conformal anti-invariant submersion with dilation $\lambda = \pi^{68}$.

Example 5.7. Define a map $F\colon \mathbb{R}^{12} \mapsto \mathbb{R}^9$ by

$$F(x_1,\dots,x_{12}) = \pi(x_5,x_7,x_4,x_8,x_{10},x_{11},x_1,x_2,x_{12}).$$

Then the map F is an h-conformal anti-invariant submersion with dilation $\lambda = \pi$.

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