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HAUSDORFF OPERATOR ON MORREY SPACES AND CAMPANATO SPACES

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Abstract. We study the high-dimensional Hausdorff operators on the Morrey space and on the Campanato space. We establish their sharp boundedness on these spaces. Particularly, our results solve an open question posted by E. Liflyand (2013).

Keywords: Hausdorff operator; Morrey space; Campanato space

MSC 2010: 42B35, 42B30, 46E30

1. Introduction

Let $\mathbb{R}^n$ be the n dimensional Euclidean space with $n \geqslant 2$. For a suitable function Φ, Lerner and Liflyand in [6] studied the Hausdorff operator $\mathcal{H}_{\Phi,A}$ defined initially on the Schwartz space in the form of

$$\mathcal{H}_{\Phi,A}(f)(x) = \int_{\mathbb{R}^n} \Phi(y) f(A(y)x)\,\mathrm{d}y,$$

where $A(y)$ is an $n \times n$ matrix which is invertible for almost all y lying in the support of Φ. A special case is $A(y) = \mathrm{diag}[1/|y|, 1/|y|, \ldots, 1/|y|]$ for which $\mathcal{H}_{\Phi,A}$ is reduced to the well-studied operator

$$\mathcal{H}_{\Phi}(f)(x) = \int_{\mathbb{R}^n} \Phi(y) f(x/|y|)\,\mathrm{d}y.$$

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The operator $\mathcal{H}_\Phi$ has received extensive attentions in recent years. For instance, the reader can see [1], [3], [7], [8], [10], [11], [12], [13], [14], [16] for studies of $\mathcal{H}_\Phi$ on various function spaces such as Lebesgue spaces L^p, the Hardy space H^1 and the BMO space. Two recent survey papers [2] and [9] might also provide a good source of information.

The situation clearly becomes much more involved if we study $\mathcal{H}_{\Phi,A}$ with a general non-singular matrix A. Hence, for a fixed normed function space X, finding reasonable conditions on Φ related to A to guarantee the boundedness of $\mathcal{H}_{\Phi,A}$ on X is an interesting research subject. Based on this motivation, the aim of this article is to obtain the boundedness of $\mathcal{H}_{\Phi,A}$ on the Morrey space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ and on the Campanato space $\mathcal{E}^{p,\lambda}(\mathbb{R}^n)$, extending the known results on spaces $L^p(\mathbb{R}^n)$ and on the $\mathrm{BMO}(\mathbb{R}^n)$ space, respectively.

For a matrix $A = (a_{ij})_{n\times n}$,

$$\|A\| = \Big(\sum_{i,j=1}^{n} |a_{ij}|^2\Big)^{1/2}$$

is a norm of A. If A is invertible then

$$\|A^{-1}\|^{-n} \leqslant |\det A| \leqslant \|A\|^n. \tag{1.1}$$

We recall the following result in [6].

Theorem A ([6]). *If A is invertible, then*

$$\|\mathcal{H}_{\Phi,A}(f)\|_{\mathrm{BMO}(\mathbb{R}^n)} \preceq \Big(\int_{\mathbb{R}^n} |\Phi(y)| \frac{\|A(y)\|^n}{|\det A(y)|}\,\mathrm{d}y\Big)\|f\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Thus, $\mathcal{H}_{\Phi,A}$ is bounded on the $\mathrm{BMO}(\mathbb{R}^n)$ space if Φ satisfies the size condition

$$\int_{\mathbb{R}^n} |\Phi(y)| \frac{\|A(y)\|^n}{|\det A(y)|}\,\mathrm{d}y < \infty,$$

where $\mathrm{BMO}(\mathbb{R}^n)$ denotes the space of functions bounded mean oscillation and it is also the dual space of the Hardy space $H^1(\mathbb{R}^n)$.

In [9], Liflyand posed the following question (see [9], 6.2 c, page 135):

Prove (or disprove) the sharpness of the condition in Theorem A for the boundedness of $\mathcal{H}_{\Phi,A}$ on the space $\mathrm{BMO}(\mathbb{R}^n)$.

We will disprove the sharpness of the size condition in Theorem A by obtaining a weaker sufficient condition on Φ. To state our main results, we first introduce the definitions of the Morrey space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ and the Campanato space $\mathcal{E}^{p,\lambda}(\mathbb{R}^n)$.

Definition 1.1. Let $1 \leqslant p < \infty$, $-n/p \leqslant \lambda < \infty$. A function $f \in L^p_{\rm loc}(\mathbb{R}^n)$ is said to belong to the Morrey space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} = \sup_{r>0,x_0\in\mathbb{R}^n} \frac{1}{|Q(x_0,r)|^{\lambda/n}} \left(\frac{1}{|Q(x_0,r)|} \int_{Q(x_0,r)} |f(x)|^p \,\mathrm{d}x \right)^{1/p} < \infty,$$

where $Q(x_0,r)$ denotes the cube centered at x_0 with the side length r.

It is easy to see that $\mathcal{L}^{p,-n/p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $\mathcal{L}^{p,0}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. Also, we may easily check that $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ reduces to $\{0\}$ when $\lambda > 0$.

Definition 1.2. Let $1 \leqslant p < \infty$, $-n/p \leqslant \lambda < \infty$. A function $f \in L^p_{\rm loc}(\mathbb{R}^n)$ is said to be in the Campanato space $\mathcal{E}^{p,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)} = \sup_{r>0,x_0\in\mathbb{R}^n} \frac{1}{|Q(x_0,r)|^{\lambda/n}} \left(\frac{1}{|Q(x_0,r)|} \int_{Q(x_0,r)} |f(x) - f_Q|^p \,\mathrm{d}x \right)^{1/p} < \infty,$$

where $f_Q = \int_{Q(x_0,r)} f(x)\,\mathrm{d}x / |Q(x_0,r)|$.

When $\lambda = 0$, we have that

$$\|f\|_{\mathcal{E}^{p,0}(\mathbb{R}^n)} \simeq \|f\|_{\mathrm{BMO}(\mathbb{R}^n)}$$

so that $\mathcal{E}^{p,0}(\mathbb{R}^n)$ is the well known $\mathrm{BMO}(\mathbb{R}^n)$ space. When $0 < \lambda \leqslant 1$, $\mathcal{E}^{p,\lambda}(\mathbb{R}^n)$ is the Lipschitz space $\mathrm{Lip}_\lambda(\mathbb{R}^n)$ with

$$\|f\|_{\mathrm{Lip}_\lambda(\mathbb{R}^n)} = \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x-y|^\lambda} \simeq \|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)}.$$

For $1 < \lambda < \infty$, $\mathcal{E}^{p,\lambda}(\mathbb{R}^n)$ contains only constant functions. And if $-n/p \leqslant \lambda < 0$, $\mathcal{E}^{p,\lambda}(\mathbb{R}^n)/\mathcal{C}$ is equivalent to the Morrey space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$, where $\mathcal{C}$ is the space of the constant functions.

Here and throughout this paper, we use the notation $A \simeq B$ if there exists a positive constant C independent of all essential values and variables such that $C^{-1}B \leqslant A \leqslant CB$. The notation $A \preceq B$ denotes that there is a constant $C > 0$ independent of all essential values and variables such that $A \leqslant CB$.

Now we are in a position to state our results.

Theorem 1.1. *Let $1 \leqslant p < \infty$ and $-n/p \leqslant \lambda \leqslant 0$. Then we have*

$$\|\mathcal{H}_{\Phi,A} f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \preceq C_1 \|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)},$$

where

$$C_1 = \int_{\mathbb{R}^n} |\Phi(y)| \|A(y)\|^\lambda \left(\frac{\|A(y)\|^n}{|\det A(y)|} \right)^{\min\{1/p,-\lambda\}} \mathrm{d}y.$$

Theorem 1.2. *Let $1 \leqslant p < \infty$ and $-n/p \leqslant \lambda \leqslant 1$.*

(i) *If $-n/p \leqslant \lambda < 0$, we have*

$$\|\mathcal{H}_{\Phi,A} f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)} \preceq C_1 \|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)},$$

where C_1 is the same as in Theorem 1.1.

(ii) *If $\lambda = 0$, we have*

$$\|\mathcal{H}_{\Phi,A} f\|_{\mathrm{BMO}(\mathbb{R}^n)} \preceq C_2 \|f\|_{\mathrm{BMO}(\mathbb{R}^n)},$$

where

$$C_2 = \int_{\mathbb{R}^n} |\Phi(y)| \left(1 + \log \frac{\|A(y)\|^n}{|\det A(y)|}\right) \mathrm{d}y.$$

(iii) *If $0 < \lambda \leqslant 1$, we have*

$$\|\mathcal{H}_{\Phi,A} f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)} \preceq C_3 \|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)},$$

where

$$C_3 = \int_{\mathbb{R}^n} |\Phi(y)| \|A(y)\|^{\lambda} \, \mathrm{d}y.$$

Since $\mathcal{E}^{p,0} = \mathrm{BMO}$, Theorem 1.2 clearly is an extension of Theorem A, while the second part of Theorem 1.2 improves the size condition in Theorem A, so that it gives a negative answer to the question by Liflyand. Also, unlike the proof for the operator $\mathcal{H}_\Phi$, where the Minkowski inequality might be directly applied, for $\mathcal{H}_{\Phi,A}$ we must be concerned with the geometric shape of image $A(y)Q$ for a cube when y runs over the support of Φ. It raises main difficulties in the proof of theorems. On the other hand, as a consequence of Theorem 1.2, by the dual argument we re-prove the following result, which is the main theorem in [3].

Corollary 1.1. *If*

$$C_4 = \int_{\mathbb{R}^n} |\Phi(y)| |\det A^{-1}(y)| \left(1 + \log \frac{\|A^{-1}(y)\|^n}{|\det A^{-1}(y)|}\right) \mathrm{d}y < \infty,$$

then $\mathcal{H}_{\Phi,A}$ is bounded on the Hardy space $H^1(\mathbb{R}^n)$ and

$$\|\mathcal{H}_{\Phi,A} f\|_{H^1(\mathbb{R}^n)} \preceq C_4 \|f\|_{H^1(\mathbb{R}^n)}.$$

We remark that the result in the above corollary was obtained in [3] by using the atomic decomposition of the Hardy space. Here, we will prove it by a different method. We do not know if the size condition on Φ in Theorem 1.1 is sharp or not. It looks not an easy problem for a general matrix function $A(y)$. However, if $\|A(y)\|^n$ and $|\det A(y)|$ are comparable, we obtain the following sharp result.

Theorem 1.3. *Let $1 \leqslant p < \infty$, $-n/p \leqslant \lambda < 0$ and Φ be a nonnegative function. Suppose that there is a constant C independent of y such that*

$$\|A(y)\|^n \leqslant C|\det A(y)|$$

for all $y \in \operatorname{supp}(\Phi)$. Then $\mathcal{H}_{\Phi,A}$ is bounded on $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ if and only if

$$\int_{\mathbb{R}^n} \Phi(y)\|A(y)\|^{\lambda} \, \mathrm{d}y < \infty.$$

Furthermore, if the matrix is diagonal, we have the following sharp results on $\mathrm{BMO}(\mathbb{R}^n)$.

Theorem 1.4. *Assume that Φ is a nonnegative function. Suppose that $A(y) = \operatorname{diag}[1/\lambda_1(y), \ldots, 1/\lambda_n(y)]$ and $\lambda_{i_0}(y) > 0$ (or $\lambda_{i_0}(y) < 0$) uniformly on $y \in \operatorname{supp}(\Phi)$ for some $i_0 \in \{1, 2, \ldots, n\}$. Denote*

$$M(y) = \max\{|\lambda_1(y)|, \ldots, |\lambda_n(y)|\}, \quad m(y) = \min\{|\lambda_1(y)|, \ldots, |\lambda_n(y)|\}.$$

If there is a constant $C \geqslant 1$ independent of y such that $M(y) \leqslant Cm(y)$ uniformly on $\operatorname{supp}(\Phi)$, then $\mathcal{H}_{\Phi,A}$ is bounded on $\mathrm{BMO}(\mathbb{R}^n)$ if and only if $\Phi \in L^1(\mathbb{R}^n)$.

2. Proof of the theorems

We first introduce some necessary lemmas.

Lemma 2.1 ([3])**.** *Any bounded convex domain $\mathfrak{D} \subset \mathbb{R}^n$ can be contained in a rectangle $\mathfrak{R}$ satisfying $|\mathfrak{R}| \leqslant n!\,|\mathfrak{D}|$.*

Lemma 2.2. *Let $1 < \eta < \infty$ and assume that the cube $\widetilde{Q} := Q(\widetilde{x}, \eta\varrho)$ contains the cube $Q := Q(x, \varrho)$ and has the same orientation as Q. Suppose that $f \in \mathcal{E}^{p,\lambda}(\mathbb{R}^n)$ with $1 \leqslant p < \infty$ and $-n/p \leqslant \lambda \leqslant 1$.*

(i) *If* $-n/p \leqslant \lambda < 0$, *then*

$$\frac{1}{|Q|^{\lambda/n}}\left(\frac{1}{|Q|}\int_Q |f(x)-f_{\widetilde{Q}}|^p\,\mathrm{d}x\right)^{1/p} \preceq \|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)}.$$

(ii) *If* $\lambda = 0$, *then*

$$\frac{1}{|Q|^{\lambda/n}}\left(\frac{1}{|Q|}\int_Q |f(x)-f_{\widetilde{Q}}|^p\,\mathrm{d}x\right)^{1/p} \preceq \|f\|_{\mathrm{BMO}(\mathbb{R}^n)}(1+\log\eta).$$

(iii) *If* $0 < \lambda \leqslant 1$, *then*

$$\frac{1}{|Q|^{\lambda/n}}\left(\frac{1}{|Q|}\int_Q |f(x)-f_{\widetilde{Q}}|^p\,\mathrm{d}x\right)^{1/p} \preceq \|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)}\eta^{\lambda}.$$

Proof. Let ηQ be the cube with the same center as Q and having the side length $\eta\varrho$.

$$\tag{2.1} \begin{aligned}\frac{1}{|Q|^{\lambda/n}}&\left(\frac{1}{|Q|}\int_Q |f(x)-f_{\widetilde{Q}}|^p\,\mathrm{d}x\right)^{1/p}\\ &\preceq \frac{1}{|Q|^{\lambda/n}}\left(\frac{1}{|Q|}\int_Q |f(x)-f_{\eta Q}|^p\,\mathrm{d}x\right)^{1/p} + \frac{|f_{\eta Q}-f_{\widetilde{Q}}|}{|Q|^{\lambda/n}}.\end{aligned}$$

It is not difficult to see that

$$\begin{aligned}|f_{\eta Q}-f_{\widetilde{Q}}| &\preceq |f_{\eta Q}-f_{2\widetilde{Q}}| + |f_{\widetilde{Q}}-f_{2\widetilde{Q}}| \preceq \frac{1}{|2\widetilde{Q}|}\int_{2\widetilde{Q}} |f(x)-f_{2\widetilde{Q}}|\,\mathrm{d}x\\ &\preceq \frac{1}{|\widetilde{Q}|^{1/p}}\left(\int_{2\widetilde{Q}} |f(x)-f_{2\widetilde{Q}}|^p\,\mathrm{d}x\right)^{1/p} \preceq |\widetilde{Q}|^{\lambda/n}\|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)},\end{aligned}$$

which means that

$$\tag{2.2} \frac{|f_{\eta Q}-f_{\widetilde{Q}}|}{|Q|^{\lambda/n}} \preceq \eta^{\lambda}\|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)}.$$

On the other hand, for the given $\eta \in (1,\infty)$, there is a non-negative integer j_0 satisfying $2^{j_0} \leqslant \eta < 2^{j_0+1}$. Therefore

$$\tag{2.3} \begin{aligned}\frac{1}{|Q|^{\lambda/n}}\left(\frac{1}{|Q|}\int_Q |f(x)-f_{\eta Q}|^p\,\mathrm{d}x\right)^{1/p} &\preceq \frac{1}{|Q|^{1/p+\lambda/n}}\Bigg\{\left(\int_Q |f(x)-f_Q|^p\,\mathrm{d}x\right)^{1/p}\\ &\quad + \sum_{j=0}^{j_0}\|f_{2^jQ}-f_{2^{j+1}Q}\|_{L^p(Q)} + \|f_{2^{j_0+1}Q}-f_{\eta Q}\|_{L^p(Q)}\Bigg\}\\ &\preceq \|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)} + \frac{1}{|Q|^{\lambda/n}}\sum_{j=0}^{j_0}|f_{2^jQ}-f_{2^{j+1}Q}| + \frac{1}{|Q|^{\lambda/n}}|f_{2^{j_0+1}Q}-f_{\eta Q}|.\end{aligned}$$

The Jensen inequality yields that

$$\frac{1}{|Q|^{\lambda/n}}|f_{2^jQ}-f_{2^{j+1}Q}| \preceq \frac{1}{|Q|^{\lambda/n}}\bigg(\frac{1}{|2^jQ|}\int_{2^jQ}|f(x)-f_{2^{j+1}Q}|^p\bigg)^{1/p} \tag{2.4}$$
$$\preceq 2^{j\lambda}\|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)},$$

and

$$\frac{1}{|Q|^{\lambda/n}}|f_{2^{j_0+1}Q}-f_{\eta Q}| \preceq \frac{1}{|Q|^{\lambda/n}}\bigg(\frac{1}{|\eta Q|}\int_{\eta Q}|f(x)-f_{2^{j+1}Q}|^p\bigg)^{1/p} \tag{2.5}$$
$$\preceq 2^{(j_0+1)\lambda}\|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)}.$$

It follows from (2.1)–(2.5) that

$$\frac{1}{|Q|^{\lambda/n}}\bigg(\frac{1}{|Q|}\int_Q|f(x)-f_{\widetilde{Q}}|^p\,\mathrm{d}x\bigg)^{1/p} \preceq \|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)}\bigg(1+\eta^\lambda+\sum_{j=0}^{j_0+1}2^{j\lambda}\bigg). \tag{2.6}$$

If $-n/p\leqslant\lambda<0$, then

$$\sum_{j=0}^{j_0+1}2^{j\lambda}\leqslant\frac{1}{1-2^{n\lambda}}. \tag{2.7}$$

If $\lambda=0$, then

$$\sum_{j=0}^{j_0+1}2^{j\lambda}=j_0+1\preceq 1+\log\eta. \tag{2.8}$$

If $0<\lambda\leqslant 1$, then

$$\sum_{j=0}^{j_0+1}2^{j\lambda}=\frac{2^{\lambda(j_0+2)}-1}{2^\lambda-1}\preceq\eta^\lambda. \tag{2.9}$$

Noting that $\mathcal{E}^{p,0}(\mathbb{R}^n)=\mathrm{BMO}(\mathbb{R}^n)$, we complete the proof by (2.6)–(2.9). □

2.1. Proof of Theorem 1.1. By the definition and the Minkowski inequality,
(2.10)

$$\begin{aligned}\|\mathcal{H}_{\Phi,A}f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} &= \sup_{r>0,x_0\in\mathbb{R}^n}\frac{1}{|Q(x_0,r)|^{\lambda/n+1/p}}\bigg\|\int_{\mathbb{R}^n}|\Phi(y)|f(A(y)\cdot)\,\mathrm{d}y\bigg\|_{L^p(Q(x_0,r))}\\ &\leqslant \sup_{r>0,x_0\in\mathbb{R}^n}\frac{1}{|Q(x_0,r)|^{\lambda/n+1/p}}\int_{\mathbb{R}^n}|\Phi(y)|\|f(A(y)\cdot)\|_{L^p(Q(x_0,r))}\,\mathrm{d}y.\end{aligned}$$

A change of variables gives that

$$\|f(A(y)\cdot)\|_{L^p(Q(x_0,r))} = |\det A^{-1}(y)|^{1/p}\Big(\int_{A(y)Q(x_0,r)} |f(x)|^p\,\mathrm{d}x\Big)^{1/p}. \tag{2.11}$$

Next we will estimate (2.11) in two directions.

First, since

$$\operatorname{diam}(A(y)Q(x_0,r)) = \sup_{x,z\in Q(x_0,r)} |A(y)(x-z)| \leqslant \sqrt{n}\|A(y)\|r := \widetilde{r}, \tag{2.12}$$

there is some $\widetilde{x}_0 \in \mathbb{R}^n$ such that

$$A(y)Q(x_0,\varrho) \subset Q(\widetilde{x}_0,\widetilde{r}),$$

which tells us that

$$\begin{aligned}\Big(\int_{A(y)Q(x_0,r)} |f(x)|^p\,\mathrm{d}x\Big)^{1/p} &\leqslant \Big(\int_{Q(\widetilde{x}_0,\widetilde{r})} |f(x)|^p\,\mathrm{d}x\Big)^{1/p} \\ &\leqslant |Q(\widetilde{x}_0,\widetilde{r})|^{\lambda/n+1/p}\|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \simeq (\|A(y)\|r)^{\lambda+n/p}\|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)}.\end{aligned} \tag{2.13}$$

Thus, we infer from (2.10), (2.11) and (2.13) that

$$\|\mathcal{H}_{\Phi,A}f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \preceq \|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\Phi(y)|\Big(\frac{\|A(y)\|^n}{|\det A(y)|}\Big)^{1/p}\|A(y)\|^{\lambda}\,\mathrm{d}y. \tag{2.14}$$

Secondly, for any given $A(y)Q(x_0,r)$, obviously, it is a convex domain. By Lemma 2.1, it is contained in a rectangle Ω satisfying $|\Omega| \leqslant n!\,|\det A(y)||Q(x_0,r)|$. Without loss of generality, we may assume that the side lengths of the rectangle Ω are $l_1 \leqslant l_2 \leqslant \ldots \leqslant l_n$ and denote $l = l_1$ and $l_n = L = \widetilde{r}$, where $\widetilde{r}$ is as in (2.12). According to the definition of l and L, we have that

$$L^{n-1}l \geqslant |\Omega| \geqslant |A(y)Q(x_0,r)| \simeq |\det A(y)|r^n,$$

which yields that

$$\frac{L}{l} = \frac{L^n}{L^{n-1}l} \leqslant \frac{\widetilde{r}^n}{|A(y)Q(x_0,r)|} \preceq \frac{\|A(y)\|^n}{|\det A(y)|}. \tag{2.15}$$

Setting

$$\gamma_i = \begin{cases} l_i/l & \text{if } l_i/l = [l_i/l], \\ [l_i/l]+1 & \text{if } l_i/l > [l_i/l], \end{cases}$$

where $i = 2, 3, \ldots, n$ and $[\cdot]$ denotes the integer function. Now we divide the rectangle Ω into $\Gamma = \gamma_2\gamma_3 \ldots \gamma_n$ cubes with the same side length. Precisely, there is a collection of mutually disjoint cubes $Q_1, Q_2, \ldots, Q_\Gamma$ in the interior, which have the same side length l and satisfy $\Omega \subset \bigcup\limits_{i=1}^{\Gamma} Q_i$. Therefore

$$\begin{aligned}(2.16)\quad \biggl(\int_{A(y)Q(x_0,r)} |f(x)|^p \,\mathrm{d}x\biggr)^{1/p} &\preceq \biggl(\sum_{i=1}^{\Gamma} \int_{Q_i} |f(x)|^p \,\mathrm{d}x\biggr)^{1/p} \\ &\simeq \biggl(\sum_{i=1}^{\Gamma} \frac{l^{n+\lambda p}}{|Q_i|^{1+\lambda p/n}} \int_{Q_i} |f(x)|^p \,\mathrm{d}x\biggr)^{1/p} \preceq \|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} (\Gamma l^n)^{1/p} l^\lambda.\end{aligned}$$

It follows from the definition of Γ that

$$(2.17)\qquad \Gamma l^n \simeq |\Omega| \simeq |\det A(y)| r^n,$$

and from (2.15) that

$$(2.18)\qquad l^\lambda \preceq \biggl(\frac{|\det A(y)| L}{\|A(y)\|^n}\biggr)^\lambda \simeq \biggl(\frac{|\det A(y)| r}{\|A(y)\|^{n-1}}\biggr)^\lambda.$$

Then, we infer from (2.10), (2.11), (2.16), (2.17) and (2.18) that

$$(2.19)\qquad \|\mathcal{H}_{\Phi,A} f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \preceq \|f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\Phi(y)| \biggl(\frac{\|A(y)\|^n}{|\det A(y)|}\biggr)^{-\lambda} \|A(y)\|^\lambda \,\mathrm{d}y.$$

By combining (2.14) and (2.19), we finish the proof of the theorem. $\square$

2.2. Proof of Theorem 1.2. Part (i) immediately follows from Theorem 1.1 and the observation below Definition 1.2. It remains to prove (ii) and (iii). Because of the Minkowski inequality,

$$\begin{aligned}(2.20)\qquad \|\mathcal{H}_{\Phi,A} f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)} &\leqslant \int_{\mathbb{R}^n} |\Phi(y)| \, \|f(A(y)\cdot)\|_{\mathcal{E}^{p,\lambda}((\mathbb{R}^n))} \,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} |\Phi(y)| \sup_{r>0, x_0\in\mathbb{R}^n} \frac{1}{|Q(x_0,r)|^{1/p+\lambda/n}} \\ &\quad \times \biggl(\int_{Q(x_0,r)} |f(A(y)x) - f(A(y)\cdot)_Q|^p\biggr)^{1/p} \mathrm{d}y,\end{aligned}$$

where $f(A(y)\cdot)_Q = \fint_{Q(x_0,r)} f(A(y)z)\,\mathrm{d}z/|Q(x_0,r)|$. A change of variables yields that

$$
\begin{aligned}
(2.21)\quad &\Big(\int_{Q(x_0,r)} |f(A(y)x) - f(A(y)\cdot)_Q|^p\,\mathrm{d}x\Big)^{1/p} \\
&\leqslant \Big(\int_{Q(x_0,r)} \Big(\frac{1}{|Q(x_0,r)|}\int_{Q(x_0,r)} |f(A(y)x) - f(A(y)z)|\,\mathrm{d}z\Big)^p\,\mathrm{d}x\Big)^{1/p} \\
&= \frac{|\det A^{-1}(y)|^{1+1/p}}{|Q(x_0,r)|}\Big(\int_{A(y)Q(x_0,r)}\Big(\int_{A(y)Q(x_0,r)} |f(v) - f(u)|\,\mathrm{d}u\Big)^p\,\mathrm{d}v\Big)^{1/p}.
\end{aligned}
$$

Next we will estimate the term on the right-hand side of (2.21).

First, the Minkowski inequality shows that

$$
\begin{aligned}
&\Big(\int_{A(y)Q(x_0,r)}\Big(\int_{A(y)Q(x_0,r)} |f(v) - f(u)|\,\mathrm{d}u\Big)^p\,\mathrm{d}v\Big)^{1/p} \\
&\quad\preceq |A(y)Q(x_0,r)|\Big(\int_{A(y)Q(x_0,r)} |f(u) - f_{A(y)Q(x_0,r)}|^p\,\mathrm{d}u\Big)^{1/p} \\
&\quad= |A(y)Q(x_0,r)| \\
&\qquad\times\Big(\int_{A(y)Q(x_0,r)} |f(u) - f_{Q(\widetilde{x}_0,\widetilde{r})} + f_{Q(\widetilde{x}_0,\widetilde{r})} - f_{A(y)Q(x_0,r)}|^p\,\mathrm{d}u\Big)^{1/p} \\
(2.22)\qquad &\quad\preceq |A(y)Q(x_0,r)|\Big(\int_{A(y)Q(x_0,r)} |f(u) - f_{Q(\widetilde{x}_0,\widetilde{r})}|^p\,\mathrm{d}u\Big)^{1/p} \\
&\quad\preceq |A(y)Q(x_0,r)|\Big(\int_{Q(\widetilde{x}_0,\widetilde{r})} |f(u) - f_{Q(\widetilde{x}_0,\widetilde{r})}|^p\,\mathrm{d}u\Big)^{1/p} \\
&\quad\preceq \|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)}|A(y)Q(x_0,r)||Q(\widetilde{x}_0,\widetilde{r})|^{\lambda/n+1/p} \\
(2.23)\qquad &\quad\simeq \|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)}|\det A(y)||Q(x_0,r)|(\|A(y)\|r)^{\lambda+n/p},
\end{aligned}
$$

where $Q(\widetilde{x}_0,\widetilde{r})$ is as in the proof of Theorem 1.1. Hence, we infer from (2.20), (2.21), and (2.23) that

$$
(2.24)\qquad \|\mathcal{H}_{\Phi,A}f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)} \preceq \|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)}\int_{\mathbb{R}^n} |\Phi(y)|\Big(\frac{\|A(y)\|^n}{|\det A(y)|}\Big)^{1/p}\|A(y)\|^\lambda\,\mathrm{d}y.
$$

Secondly, an estimation similar to that in (2.22) and the argument in the proof of

Theorem 1.1 show that

$$\left(\int_{A(y)Q(x_0,r)}\left(\int_{A(y)Q(x_0,r)}|f(v)-f(u)|\,\mathrm{d}u\right)^p\mathrm{d}v\right)^{1/p} \tag{2.25}$$
$$\preceq |A(y)Q(x_0,r)|\left(\int_{A(y)Q(x_0,r)}|f(u)-f_{2Q(\widetilde{x}_0,\widetilde{r})}|^p\,\mathrm{d}u\right)^{1/p}$$
$$\preceq |A(y)Q(x_0,r)|\left(\sum_{i=1}^{\Gamma}\int_{Q_i}|f(u)-f_{2Q(\widetilde{x}_0,\widetilde{r})}|^p\,\mathrm{d}u\right)^{1/p},$$

where $\{Q_i\}_{i=1}^{\Gamma}$ is the family of cubes as in the proof of Theorem 1.1. On the other hand, Lemma 2.2 and (2.15) tell us that, if $\lambda=0$, then

$$\int_{Q_i}|f(u)-f_{2Q(\widetilde{x}_0,\widetilde{r})}|^p\,\mathrm{d}u \preceq \|f\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\left(1+\log\frac{L}{l}\right)^p l^n \tag{2.26}$$
$$\preceq \|f\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\left(1+\log\frac{\|A(y)\|^n}{|\det A(y)|}\right)^p l^n$$

and, if $0<\lambda\leqslant 1$, then

$$\int_{Q_i}|f(u)-f_{2Q(\widetilde{x}_0,\widetilde{r})}|^p\,\mathrm{d}u \preceq \|f\|^p_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)}\left(\frac{L}{l}\right)^{\lambda p} l^{n+\lambda p} = \|f\|^p_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)}L^{\lambda p}l^n. \tag{2.27}$$

Thus (2.17), (2.25), and (2.26) yield that, if $\lambda=0$, then

$$\left(\int_{A(y)Q(x_0,r)}\left(\int_{A(y)Q(x_0,r)}|f(v)-f(u)|\,\mathrm{d}u\right)^p\mathrm{d}v\right)^{1/p} \tag{2.28}$$
$$\preceq \|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)}\left(1+\log\frac{\|A(y)\|^n}{|\det A(y)|}\right)|\det A(y)|r^n(\Gamma l^n)^{1/p}$$
$$\preceq \|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)}\left(1+\log\frac{\|A(y)\|^n}{|\det A(y)|}\right)(|\det A(y)|r^n)^{1+1/p}.$$

And (2.17), (2.25), and (2.27) tell us that, if $0<\lambda\leqslant 1$, then

$$\left(\int_{A(y)Q(x_0,r)}\left(\int_{A(y)Q(x_0,r)}|f(v)-f(u)|\,\mathrm{d}u\right)^p\mathrm{d}v\right)^{1/p} \tag{2.29}$$
$$\preceq \|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)}|\det A(y)|r^n(\Gamma l^n)^{1/p}L^\lambda$$
$$\preceq \|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)}(|\det A(y)|r^n)^{1+1/p}(\|A(y)\|r)^\lambda.$$

Therefore, we infer from (2.20), (2.21), and (2.28) that, if $\lambda=0$, then

$$\|\mathcal{H}_{\Phi,A}f\|_{\mathrm{BMO}(\mathbb{R}^n)} \preceq \|f\|_{\mathrm{BMO}(\mathbb{R}^n)}\int_{\mathbb{R}^n}|\Phi(y)|\left(1+\log\frac{\|A(y)\|^n}{|\det A(y)|}\right)\mathrm{d}y \tag{2.30}$$

and from (2.20), (2.21), and (2.29) that, if $0 < \lambda \leqslant 1$, then

$$\|\mathcal{H}_{\Phi,A} f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)} \preceq \|f\|_{\mathcal{E}^{p,\lambda}(\mathbb{R}^n)} \int_{\mathbb{R}^n} |\Phi(y)| \|A(y)\|^{\lambda} \, \mathrm{d}y. \tag{2.31}$$

Thus we complete the proof of Theorem 1.2 by (2.24), (2.30), and (2.31). □

2.3. Proof of Corollary 1.1. Using the second part of Theorem 1.2 and the celebrated H^1-BMO inequality by Fefferman and Stein (see [4]), we finish the proof by the same argument as in the proof of Theorem 2.2 in [6]. □

2.4. Proof of Theorem 1.3. The sufficiency part is easily obtained by Theorem 1.1. It remains to prove the necessity part.

Since the space $\mathcal{L}^{p,-n/p}(\mathbb{R}^n)$ reduces to the Lebesgue space $L^p(\mathbb{R}^n)$ and the corresponding results were obtained in [15], we will just consider the case of $-n/p < \lambda < 0$. Let $f_0(x) = |x|^{\lambda}$. It follows from [5] that $f_0 \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ and $\|f_0\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n)} > 0$. Therefore

$$\begin{aligned} \mathcal{H}_{\Phi,A} f_0(x) &= \int_{\mathbb{R}^n} \Phi(y) |A(y)x|^{\lambda} \, \mathrm{d}y \\ &\geqslant |x|^{\lambda} \int_{\mathbb{R}^n} \Phi(y) \|A(y)\|^{\lambda} \, \mathrm{d}y = f_0(x) \int_{\mathbb{R}^n} \Phi(y) \|A(y)\|^{\lambda} \, \mathrm{d}y, \end{aligned}$$

which completes the proof of the theorem. □

2.5. Proof of Theorem 1.4. The sufficiency part is obvious in view of Theorem 1.2 (ii). It remains to prove the necessity part.

Without loss of generality, we assume that $\lambda_1(y) > 0$ uniformly on $\mathrm{supp}(\Phi)$. Let $f_0(x) = 1$ for $x \in \mathbb{R}^n_l$, $f_0(x) = -1$ for $x \in \mathbb{R}^n_r$, where $\mathbb{R}^n_l$ and $\mathbb{R}^n_r$ denote the left and right halves of $\mathbb{R}^n$, separated by the hyperplane $x_1 = 0$ (x_1 is the first coordinate of $x \in \mathbb{R}^n$). It follows from [17] that $f_0 \in \mathrm{BMO}(\mathbb{R}^n)$ and $\|f_0\|_{\mathrm{BMO}(\mathbb{R}^n)} > 0$. A simple calculation leads to

$$\mathcal{H}_{\Phi,A} f_0(x) = \begin{cases} \displaystyle\int_{\mathbb{R}^n} \Phi(y) \, \mathrm{d}y, & x \in \mathbb{R}^n_l, \\ \displaystyle -\int_{\mathbb{R}^n} \Phi(y) \, \mathrm{d}y, & x \in \mathbb{R}^n_r. \end{cases}$$

That is

$$\mathcal{H}_{\Phi,A} f_0(x) = f_0(x) \int_{\mathbb{R}^n} \Phi(y) \, \mathrm{d}y,$$

which finishes the proof of the theorem. □

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