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MAXIMAL NON λ-SUBRINGS

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Abstract. Let R be a commutative ring with unity. The notion of maximal non λ-subrings is introduced and studied. A ring R is called a maximal non λ-subring of a ring T if $R \subset T$ is not a λ-extension, and for any ring S such that $R \subset S \subseteq T$, $S \subseteq T$ is a λ-extension. We show that a maximal non λ-subring R of a field has at most two maximal ideals, and exactly two if R is integrally closed in the given field. A determination of when the classical $D + M$ construction is a maximal non λ-domain is given. A necessary condition is given for decomposable rings to have a field which is a maximal non λ-subring. If R is a maximal non λ-subring of a field K, where R is integrally closed in K, then K is the quotient field of R and R is a Prüfer domain. The equivalence of a maximal non λ-domain and a maximal non valuation subring of a field is established under some conditions. We also discuss the number of overrings, chains of overrings, and the Krull dimension of maximal non λ-subrings of a field.

Keywords: maximal non λ-subring; λ-extension; integrally closed extension; valuation domain

MSC 2010: 13B02, 13B22, 13A18

1. Introduction

All rings considered below are commutative with nonzero identity and all ring extensions are unital. By an overring of R, we mean a subring of the total quotient ring of R containing R. By a local ring, we mean a ring with unique maximal ideal. The symbol $\subseteq$ is used for inclusion, while $\subset$ is used for proper inclusion. Throughout this paper, $\mathrm{qf}(R)$ denotes the quotient field of an integral domain R and R' the integral closure of R in $\mathrm{qf}(R)$. Our work is motivated by the work of Gilbert on λ-extensions (see [13]). A ring extension $R \subseteq T$ is said to be a λ-extension (equivalently, T is a λ-extension of R or R is a λ-subring of T) if the set of all subrings

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of T containing R is linearly ordered by inclusion. Moreover, if $T = \mathrm{qf}(R)$, then R is said to be a λ-domain. It is obvious that if $R \subseteq T$ is a λ-extension and S is a ring such that $R \subseteq S \subseteq T$, then $S \subseteq T$ is a λ-extension. This leads us to think on subrings R of a given ring T such that $R \subset T$ is not a λ-extension and R is maximal with this property. Motivated by this idea, we introduce the notion of maximal non λ-subrings of a ring. A ring R is called a maximal non λ-subring of a ring T if $R \subset T$ is not a λ-extension, and for any ring S such that $R \subset S \subseteq T$, $S \subseteq T$ is a λ-extension. Further, if $T = \mathrm{qf}(R)$, then R is called a maximal non λ-domain. In this paper, we establish some characterizations of a maximal non λ-subring.

In Section 3, we discuss the properties of a maximal non λ-subring R of a ring T and necessary conditions for R to be a maximal non λ-subring of T. We prove that if R is a maximal non λ-subring of a field K, then R has at most two maximal ideals (see Proposition 3.1), and if R is a maximal non λ-subring of a ring T, then there are at most two maximal ideals of R containing the contraction of any maximal ideal in T (see Proposition 3.4). We characterize the maximal non λ-subrings of a field K. A determination of when the classical $D + M$ construction is a maximal non λ-domain is given in Theorem 3.3. It is also shown that if a field R is a maximal non λ-subring of $T = \prod\limits_{i \in \Delta} T_i$, where T_i's are rings for all $i \in \Delta$ and $|\Delta| \geqslant 2$, then $|\Delta| = 2$ (see Corollary 3.2), and when R is not a field then $|\Delta| = 2$ under some conditions (see Proposition 3.5).

In Section 4, we discuss maximal non λ-subrings R of T when R is integrally closed in T. We prove that if R is a maximal non λ-subring of T and is integrally closed in T, then T is an overring of R (see Theorem 4.1), and if T is a field, then $T = \mathrm{qf}(R)$ (see Corollary 4.1). For an integrally closed domain R, a necessary and sufficient condition is given for R to be a maximal non λ-domain (see Theorem 4.3). We show that if R is an integral domain and R' is not local, then R is a maximal non λ-domain if and only if R is a maximal non valuation subring of $\mathrm{qf}(R)$ (see Theorem 4.4). We discuss the spectra of maximal non λ-subrings of a field K. We show that either both $\mathrm{Spec}(R)$ and $\mathrm{Spec}(R')$ are chains of the same dimension or both $\mathrm{Spec}(R)$ and $\mathrm{Spec}(R')$ are (a, b)-Y graphs (see Theorem 4.7). Under some conditions, we also discuss the number of overrings, chains of overrings, and the Krull dimension of maximal non λ-subrings of a field.

The set of all R-subalgebras of T (that is, of rings S such that $R \subseteq S \subseteq T$) is denoted by $[R, T]$. For any ring R, let $\mathrm{Spec}(R)$ (and $\mathrm{Max}(R)$) denote, respectively, the set of all prime (and maximal) ideals of R. As usual, $|X|$ denotes the cardinality of a set X and the dimension of a ring refers to the Krull dimension. If $R \subseteq T$ is a ring extension, then $(R : T) = \{r \in R\colon\ rT \subseteq R\}$ denotes the conductor of $R \subseteq T$.

2. Preliminaries

In this section we recall some results on λ-extensions from [13] which are used throughout the paper frequently.

(A) Let $R \subseteq K$ be a λ-extension, where K is a field. Then either (i) R is a field or (ii) R is not a field, $K = \mathrm{qf}(R)$, and R' is a valuation domain. See [13], Proposition 1.3.

(B) An integrally closed domain R is a λ-domain if and only if it is a valuation domain. See [13], Corollary 1.5.

(C) Let (V, M) be a valuation domain containing a field F such that $V = F + M$. Let D be a proper subring of F and set $R = D + M$. Then R is a λ-domain if and only if $D \subseteq F$ is a λ-extension. See [13], Proposition 1.6 (c).

(D) Let $R \subseteq T = \prod_{i\in\Delta} T_i$ be a λ-extension, where T_i's are rings for all $i \in \Delta$ and $|\Delta| \geqslant 2$. Let $\pi_i\colon T \to T_i$ be the canonical projection and let $I_i = \mathrm{Ker}(\pi_i) \cap R$ for all $i \in \Delta$. Assume that $I_i + I_j$ is a proper ideal of R for all pairs $i, j \in \Delta$. Then $|\Delta| = 2$. See [13], Proposition 2.8.

(E) Let K be a field and n a positive integer. Then the ring extension $K \subseteq K[X]/(X^n)$ is a λ-extension if and only if $n \leqslant 3$. See [13], Proposition 3.5.

(F) Let $R \subseteq T$ be a ring extension and J an ideal of T. Then $R/(J \cap R) \subseteq T/J$ is a λ-extension if and only if $R + J \subseteq T$ is a λ-extension. See [13], Proposition 3.9.

(G) Let R be a one-dimensional Prüfer domain with property (#). Then:
  (i) Each overring of R has property (#).
  (ii) Define the map $\Phi\colon$ {overrings of R} $\to$ {subsets of the set of valuation overrings of R} by $\Phi(T) =$ {valuation overrings of T} and the map $\Psi\colon$ {subsets of the set of valuation overrings of R} $\to$ {overrings of R} by $\Psi(\{V_\alpha\}) = \bigcap V_\alpha$. Then Φ and Ψ are inverse maps and both are inclusion-reversing.

See [13], Proposition 4.8 case (1), Proposition 4.9.

(H) Let R be a one-dimensional Prüfer domain with property (#). Then:
  (i) The overrings of R which are the minimal ring extension of R are precisely those overrings which are the intersection of all but one of the valuation overrings of R.
  (ii) Let T be a proper overring of R. Then T is a λ-extension of R if and only if T is a minimal ring extension of R.

See [13], Corollary 4.10.

(I) Let R be a principal ideal domain not equal to its quotient field. Then the minimal overrings of R are precisely the rings $R[1/p]$, where p is an irreducible element of R. See [13], Proposition 4.11.

3. Properties and characterizations

First, we define the maximal non λ-subring of a ring T formally.

Definition 3.1. A proper subring R of a ring T is said to be a maximal non λ-subring of T if $R \subset T$ is not a λ-extension and R is maximal with this property, that is, if $R \subset T$ is not a λ-extension and for any ring S such that $R \subset S \subseteq T$, $S \subseteq T$ is a λ-extension. Further, if $T = \mathrm{qf}(R)$, then R is called a maximal non λ-domain.

First, we discuss the cardinality of $\mathrm{Max}(R)$, where R is a maximal non λ-subring of a field K.

Proposition 3.1. *Let R be a maximal non λ-subring of a field K. Then R has at most two maximal ideals.*

Proof. Suppose M, N and P are distinct maximal ideals of R. Then we have $R \subseteq R_M \cap R_N \cap R_P \subset R_M \cap R_N$. Since R is a maximal non λ-subring of K, $R_M \cap R_N \subset K$ is a λ-extension. Therefore, $R_M \subseteq R_N$ or $R_N \subseteq R_M$, which is a contradiction. Thus, R has at most two maximal ideals. $\square$

In view of case (A), the following result is evident.

Proposition 3.2. *Let R be a maximal non λ-subring of a field K and $R \neq R'$, where R' is the integral closure of R in $\mathrm{qf}(R)$. Then R' is a valuation domain with quotient field K.*

Recall from [20] that an integral domain R is called an *i-domain* if for each overring T of R, the canonical contraction map $\mathrm{Spec}(T) \to \mathrm{Spec}(R)$ is injective. The next corollary is a direct consequence of Proposition 3.2 and [20], Corollary 2.15.

Corollary 3.1. *Let R be a maximal non λ-subring of a field K and $R \neq R'$. Then R is a local i-domain.*

A proper ideal I of R (defined in [5]) is said to be a 2-absorbing ideal of R if whenever $xyz \in I$ for $x, y, z \in R$, then either $xy \in I$, or $yz \in I$, or $xz \in I$. We will show that if R is a maximal non λ-subring of a ring T, then $\mathrm{Rad}_R((R : T))$ is a 2-absorbing ideal of R. First, we prove the following lemma.

Lemma 3.1. *Let R be a maximal non λ-subring of T and let $x, y, z \in R$ be such that $xyz \in (R : T)$. Then either $x^2y^2 \in (R : T)$, or $x^2z^2 \in (R : T)$, or $y^2z^2 \in (R : T)$.*

P r o o f. Assume $xyz \in (R:T)$. If $xy \in (R:T)$, then there is nothing to prove. Now, suppose that $xy \notin (R:T)$. Then $R \subset R + xyT$. Since R is a maximal non λ-subring of T, $R + xyT \subseteq T$ is a λ-extension. Thus, either $R + xT \subseteq R + yT$ or $R + yT \subseteq R + xT$. Let $R + xT \subseteq R + yT$. Then $xzR + x^2zT \subseteq xzR + xyzT \subseteq R$ and hence $x^2zT \subseteq R$. Therefore, $x^2z^2 \in (R:T)$. □

Theorem 3.1. *Let R be a maximal non λ-subring of T. Then $\mathrm{Rad}_R((R:T))$ is a 2-absorbing ideal of R.*

P r o o f. Let $x, y, z \in R$ be such that $xyz \in \mathrm{Rad}_R((R:T))$. Then $x^ny^nz^n \in (R:T)$ for some $n \in \mathbb{N}$. Now by Lemma 3.1, $x^{2n}y^{2n} \in (R:T)$ or $x^{2n}z^{2n} \in (R:T)$ or $y^{2n}z^{2n} \in (R:T)$. Therefore, $xy \in \mathrm{Rad}_R((R:T))$ or $xz \in \mathrm{Rad}_R((R:T))$ or $yz \in \mathrm{Rad}_R((R:T))$. Thus, $\mathrm{Rad}_R((R:T))$ is a 2-absorbing ideal of R. □

The next proposition discusses maximal non λ-subrings of quotient rings. The proof is routine and hence omitted.

Proposition 3.3. *Let $R \subset T$ be a ring extension and J an ideal of T. Set $I = J \cap R$. Then R/I is a maximal non λ-subring of T/J if and only if $R + J$ is a maximal non λ-subring of T.*

In Proposition 3.4, we show that the contraction of any maximal ideal in T is contained in at most two maximal ideals of R, if R is a maximal non λ-subring of T. First, we need the following lemma which is manifestly a consequence of (F).

Lemma 3.2. *Let R be a maximal non λ-subring of T and J an ideal of T. Set $I = J \cap R$. Then either*

(i) *J is an ideal of R, or*

(ii) *R/I is a λ-subring of T/J.*

Proposition 3.4. *Let R be a maximal non λ-subring of T and J a maximal ideal of T. Set $I = J \cap R$. Then there are at most two maximal ideals of R containing I.*

P r o o f. If $J \subset R$, then R/I is a maximal non λ-subring of T/J, by Proposition 3.3. Since T/J is a field, R/I has at most two maximal ideals, by Proposition 3.1. Therefore, there are at most two maximal ideals of R containing I. If $J \not\subset R$, then R/I is a λ-subring of T/J by Lemma 3.2. Since T/J is a field, $(R/I)'$ is a valuation domain, by (A). Therefore, R/I is local, hence the result holds. □

Gilbert in [13] proved that a field K is a λ-subring of $K[X]/(X^n)$ if and only if $n \leqslant 3$. Now, we show that K is a maximal non λ-subring of $K[X]/(X^n)$ if and only if $n = 4$.

Theorem 3.2. *Let K be a field and n a positive integer. Then K is a maximal non λ-subring of $K[X]/(X^n)$ if and only if $n = 4$.*

P r o o f. Let K be a maximal non λ-subring of $K[X]/(X^n)$. Then by (E), we have $n \geqslant 4$ as $K \subset K[X]/(X^n)$ is not a λ-extension. Note that $K[X]/(X^n) \cong K[u]$, where $u = X+(X^n)$ and $u^n = 0$. Thus, $\{1, u, u^2, \ldots, u^{n-1}\}$ is a basis of the K-vector space $K[u]$. Let $n > 6$. Since $K \subset K[u^6] \subset K[u]$, $K[u^6] \subset K[u]$ is a λ-extension. Therefore, either $K[u^2] \subseteq K[u^3]$ or $K[u^3] \subseteq K[u^2]$, which is a contradiction. Thus, $4 \leqslant n \leqslant 6$. Now, consider the following cases:

Case (i): $n = 4$. Then we have $K[X]/(X^4) \cong K[u]$, where $u = X + (X^4)$ and $u^4 = 0$. Let $x \in K[u] \setminus K$. Then $x = a_0 + a_1u + a_2u^2 + a_3u^3$ for some $a_0, a_1, a_2, a_3 \in K$. Now, $K[x] = K[a_1u + a_2u^2 + a_3u^3]$. Note that if $a_1 = 0$, then the dimension of K-vector space $K[x]$ is two and if $a_1 \neq 0$, then $K[x] = K[u]$. In any case, it follows that $K[x] \subseteq K[u]$ is a λ-extension. Thus, K is a maximal non λ-subring of $K[X]/(X^4)$.

Case (ii): $n = 5$. Then we have $K[X]/(X^5) \cong K[u]$, where $u = X + (X^5)$ and $u^5 = 0$. Now, $K[u^4] \subset K[u^2]$ and $K[u^4] \subset K[u^2 + u^3]$. Since $K[u^2]$ and $K[u^2 + u^3]$ are not comparable, $K[u^4] \subset K[u]$ is not a λ-extension. Thus, K is not a maximal non λ-subring of $K[X]/(X^5)$.

Case (iii): $n = 6$. Then we have $K[X]/(X^6) \cong K[u]$, where $u = X + (X^6)$ and $u^6 = 0$. Now, $K[u^4] \subset K[u^2]$ and $K[u^4] \subset K[u^2 + u^5]$. Since $K[u^2]$ and $K[u^2 + u^5]$ are not comparable, $K[u^4] \subset K[u]$ is not a λ-extension. Thus, K is not a maximal non λ-subring of $K[X]/(X^6)$. □

For a valuation domain (V, M) containing a field F such that $V = F + M$, we characterize the classical $D + M$ construction to be a maximal non λ-domain.

Theorem 3.3. *Let (V, M) be a valuation domain containing a field F such that $V = F + M$. Let D be a proper subring of F and set $R = D + M$. Then R is a maximal non λ-domain if and only if D is a maximal non λ-subring of F.*

P r o o f. If R is a maximal non λ-domain, then $D \subset F$ is not a λ-extension, by (C). Let $D \subset B \subseteq F$ and let $x \in B \setminus D$. We assert that $D + M \subset B + M$. Suppose instead that $D + M = B + M$. Then $x = y + z$ for some $y \in D$ and $z \in M$. Therefore, $x - y \in M$. Since $B \cap M = \{0\}$, $x = y \in D$, which is a contradiction. Hence, $D + M \subset B + M$. Now, by [6], Theorem 3.1, $B + M$ is an overring of R. Therefore, $B + M \subseteq \mathrm{qf}(R)$ is a λ-extension. Thus, $B \subseteq F$ is a λ-extension, by (C). Hence, D is a maximal non λ-subring of F.

Conversely, if D is a maximal non λ-subring of F, then R is not a λ-domain, by (C). Let $R \subset S \subseteq \mathrm{qf}(R)$. Then by [6], Theorem 3.1, either S is an overring of V or $S = B + M$, where $D \subset B \subseteq F$. If S is an overring of V, then $S \subseteq \mathrm{qf}(R)$ is a λ-extension, by (B). Let $S = B + M$, where $D \subset B \subseteq F$. Since D is a maximal non λ-subring of F, $B \subseteq F$ is a λ-extension. Therefore, $S \subseteq \mathrm{qf}(R)$ is a λ-extension, by (C). Thus, R is a maximal non λ-domain. □

Example 3.1. Let $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $V = F[[X]] = F + M$, where $M = XV$. Let $D = \mathbb{Q}$ and set $R = D + M$. Clearly D is a maximal non λ-subring of F. Then by Theorem 3.3, R is a maximal non λ-domain.

Gilbert in [13] proved that if R is a λ-subring of $T = \prod\limits_{i \in \Delta} T_i$, where T_i's are rings for all $i \in \Delta$, then $|\Delta| = 2$ under some conditions. Retaining the same conditions, we obtain a similar result on maximal non λ-subrings.

Proposition 3.5. *Let R be a maximal non λ-subring of $T = \prod\limits_{i \in \Delta} T_i$, where T_i's are rings for all $i \in \Delta$ and $|\Delta| \geqslant 2$. Let $\pi_i \colon T \to T_i$ be the canonical projection and let $I_i = \mathrm{Ker}(\pi_i) \cap R$ for all $i \in \Delta$. Assume that $I_i + I_j$ is a proper ideal of R for all pairs $i, j \in \Delta$. Then $|\Delta| = 2$.*

P r o o f. Let i, j, k be distinct elements in Δ. Set

$$\begin{aligned} A &= \{t \in T \colon \text{there exists } r \in R \text{ such that } \pi_i(t) = \pi_i(r) \text{ and } \pi_j(t) = \pi_j(r)\}, \\ B &= \{t \in T \colon \text{there exists } r \in R \text{ such that } \pi_i(t) = \pi_i(r) \text{ and } \pi_k(t) = \pi_k(r)\}, \\ S &= \{t \in T \colon \text{there exists } r \in R \text{ such that } \pi_i(t) = \pi_i(r), \pi_j(t) = \pi_j(r) \text{ and} \\ &\qquad \pi_k(t) = \pi_k(r)\}. \end{aligned}$$

Clearly, $R \subseteq S$ and hence the following cases arise:

Case (i): $R \subset S$. Then $S \subseteq T$ is a λ-extension. Therefore, $A \subseteq B$ or $B \subseteq A$. Suppose that $A \subseteq B$. We assert that $I_k + I_i = R$. Let $s \in R$. Consider the element $t \in T$ such that $\pi_k(t) = \pi_k(s)$ and $\pi_l(t) = 0$ for all $l \neq k$. Since $\pi_i(t) = \pi_j(t) = 0$, we have $t \in A$, and so $t \in B$. Thus, there is an element r of R such that $\pi_i(t) = \pi_i(r)$ and $\pi_k(t) = \pi_k(r)$, that is, $\pi_i(r) = 0$ and $\pi_k(r) = \pi_k(s)$. Hence, $r \in I_i$ and $s - r \in I_k$ and so $s = (s - r) + r \in I_k + I_i$. Since $s \in R$ was arbitrary, $I_k + I_i = R$, which is a contradiction. Similarly, $B \nsubseteq A$.

Case (ii): $R = S$. Let $P_i = \mathrm{Ker}(\pi_i) \cap A$ for all $i \in \Delta$. Now, if $R = A$, then $A \subseteq B$, which is a contradiction by case (i). We may now assume that $R \subset A$. Then $A \subseteq T$ is a λ-extension. Now, by (D), it is enough to show that $P_i + P_j$ is a proper ideal of A for all $i, j \in \Delta$. Suppose that $P_i + P_j = A$. Then $x + y = 1$ for some $x \in P_i$ and $y \in P_j$. Since $I_i + I_j$ is a proper ideal of R, $x \in A \setminus R$ or $y \in A \setminus R$. Let $x \in A \setminus R$. Then there exists $r \in R$ such that $0 = \pi_i(x) = \pi_i(r)$ and $\pi_j(x) = \pi_j(r)$. Therefore, $r \in I_i$ and $x - r \in P_j$. Since $1 - x \in P_j$, $x - r + 1 - x = 1 - r \in I_j$. Thus, $I_i + I_j = R$, which is a contradiction. Hence, $P_i + P_j$ is a proper ideal of A for all $i, j \in \Delta$. □

In the next corollary, we discuss the decomposable rings having a field which is a maximal non λ-subring.

Corollary 3.2. *Let K be a field. Assume that K is a maximal non λ-subring of $T = \prod_{i\in\Delta} T_i$, where T_i's are rings for all $i \in \Delta$ and $|\Delta| \geqslant 2$. Then $|\Delta| = 2$.*

Proof. Let $\pi_i\colon T \to T_i$ be the canonical projection and let $I_i = \operatorname{Ker}(\pi_i) \cap K$ for all $i \in \Delta$. Then $I_i = 0$ for all $i \in \Delta$. Now, the result follows from Proposition 3.5. □

Remark 3.1. Note that the condition $I_i + I_j$ is a proper ideal of R for all pairs $i, j \in \Delta$ is necessary in Proposition 3.5. For example, take $R = \mathbb{Z}_6$ and $T = \mathbb{Z}_6 \times K_1 \times K_2$, where $K_1 = \mathbb{Z}_6/2\mathbb{Z}_6$ and $K_2 = \mathbb{Z}_6/3\mathbb{Z}_6$. Then $I_2 + I_3 = \mathbb{Z}_6$, where I_i is the same as defined in Proposition 3.5. Also, we have $[R, T] = \{R, \mathbb{Z}_6 \times K_1, \mathbb{Z}_6 \times K_2, T\}$. Thus, R is a maximal non λ-subring of T.

4. When R is integrally closed in T

In this section, we will study both R and T under the assumption that R is a maximal non λ-subring of T such that R is integrally closed in T. We start this section with Theorem 4.1, where we prove that T is an overring of R. First, we establish that if R is a maximal non λ-subring of T, then $R \subset T$ is a P-extension. Recall from [16] that a ring extension $R \subseteq T$ is called a P-extension if each $s \in T$ is a root of some $f(X) \in R[X]$ such that at least one of coefficients of f is a unit of R. A ring extension $R \subseteq T$ is said to be an INC *extension* (see [18]) if for any two prime ideals $Q_1, Q_2 \in T$ such that $Q_1 \cap R = Q_2 \cap R$, we have Q_1, Q_2 are incomparable.

Lemma 4.1. *Let R be a maximal non λ-subring of T. Then $R \subset T$ is a P-extension.*

Proof. Let $x \in T \setminus R$. We may assume that $x^6 \notin R$. Then $R[x^6] \subseteq T$ is a λ-extension. Therefore, $R[x^2] \subseteq R[x^3]$ or $R[x^3] \subseteq R[x^2]$. Thus, $R \subseteq T$ is a P-extension. □

Theorem 4.1. *Let R be a maximal non λ-subring of an integral domain T such that R is integrally closed in T. Then T is an overring of R.*

Proof. Let K be the quotient field of R. Note that $R \subset T$ is a P-extension, by Lemma 4.1. Let $t \in T \setminus R$. Now, by [9], Corollary 4, $R \subset R[t]$ satisfies INC. Therefore, if Q is any prime ideal of $R[t]$ and $P = Q \cap R$, then by [12], Theorem, there exists $s \in R \setminus P$ such that $R[t]_s = R_s \subseteq K$. Thus, $t \in K$ and hence T is an overring of R. □

Now, we have the following immediate corollary of Theorem 4.1.

Corollary 4.1. *Let R be a maximal non λ-subring of a field K such that R is integrally closed in K. Then K is the quotient field of R.*

Remark 4.1. The integrally closed condition in the above corollary is necessary. For, if $R = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, then R is the maximal non λ-subring of K.

In Proposition 4.1, we will show that R cannot be local if R is a maximal non λ-subring of T which is integrally closed in T. First, we need the following lemma which is a direct consequence of Lemma 4.1 and [10], Lemma 3.8.

Lemma 4.2. *Let R be a maximal non λ-subring of T such that R is integrally closed in T, and let $u \in T$ and $P \in \operatorname{Spec}(R)$. Then u satisfies at least one of the following two conditions:*

(i) $u/1 \in R_P$,

(ii) $u/1$ *is a unit in* $T_{R\setminus P}$ *and* $(u/1)^{-1} \in R_P$.

Proposition 4.1. *Let R be a maximal non λ-subring of T such that R is integrally closed in T. Then R is not a local ring.*

P r o o f. Suppose R is local. Let $u \in T$. Then by Lemma 4.2, either (i) $u \in R$ or (ii) u is a unit in T and $u^{-1} \in R$. It follows that if I is any proper ideal of T, then $I \subset R$, that is, I is an ideal of R. Let Q be any maximal ideal of T. Then $Q \in \operatorname{Spec}(R)$. Therefore, R/Q is a maximal non λ-subring of the field T/Q, by Proposition 3.3. Note that R/Q is integrally closed in T/Q. Thus, the quotient field of R/Q is T/Q, by Corollary 4.1. Now, if $x + Q \in T/Q$, then by Lemma 4.2, either (i) $x + Q \in R/Q$ or (ii) $x + Q$ is a unit in T/Q and $(x + Q)^{-1} \in R/Q$. Therefore, R/Q is a valuation domain. Thus, by (B), $R/Q \subset T/Q$ is a λ-extension, which is a contradiction. Hence, R is not local. $\square$

Remark 4.2. It is easily seen that if $R \subseteq T$ is a λ-extension, then so is $R_P \subseteq T_P$ for all $P \in \operatorname{Spec}(R)$. Now, if R is a maximal non λ-subring of T, then for any $P \in \operatorname{Spec}(R)$, either $R_P \subseteq T_P$ is a λ-extension or R_P is a maximal non λ-subring of T_P. For, if $R_P \subset T_P$ is not a λ-extension, then for any subring E, $R_P \subset E \subseteq T_P$, we have $E = S_P$ for some subring S, $R \subset S \subseteq T$. Thus, $S \subseteq T$ is a λ-extension and hence $E \subseteq T_P$ is a λ-extension. However, if R is integrally closed in T, then $R_P \subseteq T_P$ is a λ-extension for all $P \in \operatorname{Spec}(R)$ as we have the next proposition.

Proposition 4.2. *Let R be a maximal non λ-subring of T such that R is integrally closed in T. Then $R_P \subseteq T_P$ is a λ-extension for all $P \in \operatorname{Spec}(R)$.*

P r o o f. If $R_P \subseteq T_P$ is not a λ-extension for some $P \in \operatorname{Spec}(R)$, then R_P is a maximal non λ-subring of T_P, by Remark 4.2. Therefore, R_P is not local, by Proposition 4.1, which is absurd. Thus, $R_P \subseteq T_P$ is a λ-extension for all $P \in \operatorname{Spec}(R)$. □

A pair of rings (R, T) is said to be a *normal pair* (see [8]) if $R \subseteq T$ and every intermediate ring S is integrally closed in T. In the next theorem, we show that if R is a maximal non λ-subring of T and is integrally closed in T, then (R, T) is a normal pair.

Theorem 4.2. *Let R be a maximal non λ-subring of T such that R is integrally closed in T. Then (R, T) is a normal pair.*

P r o o f. By Proposition 4.2, $R_P \subseteq T_P$ is a λ-extension for all $P \in \operatorname{Spec}(R)$. Therefore, by [19], Corollary 2.5, we have (R_P, T_P) is a normal pair for all $P \in \operatorname{Spec}(R)$. Now, the result follows from [11], Proposition 3.1. □

The next theorem is a characterization of integrally closed maximal non λ-domains.

Theorem 4.3. *Let R be an integrally closed domain. Then the following statements are equivalent:*

(i) *R is a maximal non λ-domain.*

(ii) *R is a semi-local Prüfer domain with exactly two maximal ideals M and N such that $[(0), M[= [(0), N[$, where $[(0), M[$ is the set of all prime ideals of R properly contained in M.*

P r o o f. (i)⇒(ii) By Proposition 3.1 and Proposition 4.1, R has exactly two maximal ideals, say M and N. Thus, $R_M \subset \operatorname{qf}(R)$ is a λ-extension. Since R is integrally closed, R_M is integrally closed. Therefore, by (A), R_M is a valuation domain. Similarly, R_N is a valuation domain. Thus, R is a Prüfer domain. Now, suppose $P \in [(0), M[$. If $P \notin [(0), N[$, then take $T = R_P \cap R_N$. Since R_P and R_N are not comparable, therefore $T \subset \operatorname{qf}(R)$ is not a λ- extension, hence $R \subset T \subset \operatorname{qf}(R)$ contradicts the maximality of R. Thus, $[(0), M[\subseteq [(0), N[$. Similarly, $[(0), N[\subseteq [(0), M[$.

(ii)⇒(i) If R is a λ-domain, then R is a valuation domain, by (A). Thus, R is not a λ-domain. Let $R \subset S \subseteq \operatorname{qf}(R)$. Then by assumption, S must be local and hence S is a valuation domain, as R is a Prüfer domain. Now, the result follows from (B). □

The following corollary discusses the integral closures of maximal non λ-domains.

Corollary 4.2. *Let R be a maximal non λ-subring of a field K. Then the integral closure of R in K is a Bézout domain with at most two maximal ideals.*

P r o o f. If R is integrally closed in K, then by Corollary 4.1, $K = \mathrm{qf}(R)$. Therefore, by Theorem 4.3, $R = R'$ is a Prüfer domain with exactly two maximal ideals. Thus, R' is a Bézout domain. If R is not integrally closed in K, then the result follows from (A). □

A domain R is said to be a maximal non valuation subring of $\mathrm{qf}(R)$ (see [7]) if R is not a valuation domain and every proper overring of R is a valuation domain. In the next theorem, we show that the concept of maximal non λ-domains is the same as that of maximal non valuation subrings of a field provided their integral closures are not local.

Theorem 4.4. *Let R be an integral domain. If R' is not local, then the following statements are equivalent:*
(i) *R is a maximal non λ-domain;*
(ii) *R is a maximal non valuation subring of* $\mathrm{qf}(R)$.

P r o o f. In view of Proposition 3.2 and our assumption, we must have $R = R'$, that is, R is integrally closed.

(i)⇒(ii) Note that by (B), R is not a valuation subring of $\mathrm{qf}(R)$. Now, suppose that $R \subset S \subseteq \mathrm{qf}(R)$. Then $S \subseteq \mathrm{qf}(R)$ is a λ-extension. Also, by Theorem 4.3, R is a Prüfer domain and hence S is a Prüfer domain. Thus, by (A), S is a valuation domain. Hence, R is a maximal non valuation subring of $\mathrm{qf}(R)$.

(ii)⇒(i) If R is a λ-domain, then R is a valuation domain by (B). Therefore, R is not a λ-domain. Let $R \subset S \subseteq \mathrm{qf}(R)$. Then S is a valuation ring. Now, by (B), $S \subseteq \mathrm{qf}(R)$ is a λ-extension. Thus, R is a maximal non λ-domain. □

Recall from [15] that a domain R with quotient field K is said to be
(i) an FO-*domain* if R has only finitely many overrings,
(ii) an FC-*domain* if each chain of distinct overrings of R is finite.

The next theorem shows the existence of infinitely many integrally closed maximal non λ-domains which are FO-domains as well as FC-domains.

Theorem 4.5. *Let K be an algebraic extension of the field of rational numbers. Then there exist infinitely many integrally closed maximal non λ-subrings of K which are FC-domains and FO-domains.*

P r o o f. By [2], Theorem 3.3, [3], Corollary 1.3 and [4], Proposition 1.1, K has infinitely many one dimensional valuation domains which are incomparable. Let R_1 and R_2 be any two incomparable one dimensional valuation domains with quotient field K. Take $R = R_1 \cap R_2$. Clearly R is an integrally closed domain and by [18], Theorems 107 and 105, $R_1 = R_M$ and $R_2 = R_N$ for some maximal ideals M, N of R.

Also, R is a one dimensional Prüfer domain with $\mathrm{Max}(R) = \{M, N\}$. Therefore, $[(0), M[= [(0), N[$. Thus, by Theorem 4.3 and [15], Theorem 1.5, R is an integrally closed maximal non λ-subring of K which is an FC-domain as well as an FO-domain. Note that R is unique for any pair R_1, R_2 of incomparable one dimensional valuation domains with quotient field K and hence the result holds. □

An integral domain R has (#) property (see [14]) if, for any two distinct subsets Ω_1 and Ω_2 of the set of maximal ideals of R, the intersections $\bigcap_{M\in\Omega_1} R_M$ and $\bigcap_{M\in\Omega_2} R_M$ are distinct. In the next theorem, we characterize the overrings of a one-dimensional Prüfer domain R with (#) property for which R is a maximal non λ-subring.

Theorem 4.6. *Let R be a one-dimensional Prüfer domain with (#) property. Then the following statements hold:*

(i) *The overrings of R for which R is a maximal non λ-subring are precisely those overrings which are the intersection of all but two of the valuation overrings of R.*

(ii) *Let T be a proper overring of R. Then R is a maximal non λ-subring of T if and only if $|[R, T]| = 4$.*

Proof. (i) Note that by (G), R is a maximal non λ-subring of those overrings which are the intersection of all but two of the valuation overrings of R. Now, suppose that T is any overring of R for which R is a maximal non λ-subring. Let Γ denotes the set of all valuation overrings of T. We assert that there are at least two valuation overrings of R which are not in Γ. If Γ contains all valuation overrings of R, then $R = T$, a contradiction. Now, assume that Γ contains all but one valuation overring of R. Then by (H), $R \subset T$ is a λ-extension, which is a contradiction. Now, we assume that there are three distinct valuation overrings V_1, V_2 and V_3 of R which are not in Γ. Set $\Gamma_i = \Gamma \cup \{V_i\}$, $\Gamma_{ij} = \Gamma \cup \{V_i, V_j\}$ for all $1 \leqslant i, j \leqslant 3$. Then for every i, j, $R \subset \bigcap_{S\in\Gamma_{ij}} S \subset T$ by (#) property. Therefore, $\bigcap_{S\in\Gamma_{ij}} S \subset T$ is a λ-extension. Now, by [14], Corollary 2 and (H), $\bigcap_{S\in\Gamma_{ij}} S \subset T$ is a minimal ring extension, which is not possible as $\bigcap_{S\in\Gamma_{ij}} S \subset \bigcap_{S\in\Gamma_i} S \subset T$. Thus, Γ contains all but two of the valuation overrings of R.

(ii) The necessity follows from part (i). For sufficiency, if $|[R, T]| = 4$, then by (H), $R \subset T$ is not a λ-extension. Let S_1, S_2 be the intermediate rings between R and T. By the proof of part (i), S_1 and S_2 are not comparable, as R has (#) property. Thus, by (H), R is a maximal non λ-subring of T. □

A proper subring R of a ring T is said to be a maximal subring of T (see [2]) or T is said to be a minimal ring extension of R if there is no ring between R and T. The next corollary gives the complete structure of the overrings of a principal ideal domain R for which R is a maximal non λ-subring.

Corollary 4.3. *Let R be a principal ideal domain not equal to its quotient field. Then the overrings of R for which R is a maximal non λ-subring are precisely the rings $R[1/pq]$, where p and q are distinct irreducible elements of R.*

Proof. Note that R is a one dimensional Prüfer domain with (#) property. Therefore, every overring of R has (#) property, by (G).

Let T be an overring of R such that R is a maximal non λ-subring of T. Then by Theorem 4.6, $[R,T] = \{R, S_1, S_2, T\}$, where S_1, S_2 are incomparable. Therefore, by (I), we have $S_1 = R[1/p]$ and $S_2 = R[1/q]$ for some distinct irreducible elements p, q of R. Thus, $T = R[1/pq]$.

Conversely, assume that p and q are distinct irreducible elements of R. Take $T = R[1/pq]$. We claim that $R[1/p]$ is a principal ideal domain. Let I be an ideal of $R[1/p]$. Then $I \cap R = rR$ for some $r \in R$. Choose the least non-negative integer j such that $r/p^j \in I$. We may assume that $j \geqslant 1$. We assert that I is generated by r/p^j in $R[1/p]$. Let $s/p^i \in I$ such that $\gcd(s,p) = 1$. Then $s = ry$ for some $y \in R$. Now, if $i > j$, then $s/p^i = (r/p^j)(y/p^{i-j})$. Otherwise, we have $s/p^i = (r/p^j)(yp^{j-i})$. Thus, our claim holds. Similarly, $R[1/q]$ is a principal ideal domain. Therefore, $R[1/p]$ and $R[1/q]$ are maximal subrings of T, by (I). Now, assume that $z = t/p^iq^j$ is an arbitrary element in T for some $t \in R$ and $i,j \geqslant 0$ such that $\gcd(t,pq) = 1$. If $i = 0$, then $z \in R[1/q]$. Similarly, if $j = 0$, then $z \in R[1/p]$. Now, assume that $i,j > 0$. Since $\gcd(t,pq) = 1$, therefore $tx + p^iq^jy = 1$ for some $x,y \in R$. This gives $1/p = (t/p)x + p^{i-1}q^jy$ and therefore $R[1/p] \subset R[z]$. Thus, by (I), we have $|[R,T]| = 4$ and hence R is a maximal non λ-subring of T. □

We now recall few definitions from [1] and [17].

(i) A graph is said to be a Y-graph if it can be drawn in the shape of the letter Y. See [1], Remark 3.5.

(ii) For any ordered set S, the *dimension* of S is the supremum of lengths n of chains $x_0 < x_1 < \ldots < x_n$ of distinct elements of S. See [17], Definition 7.

(iii) Let a be a positive integer or ∞, let b be a non-negative integer or ∞. An (a,b)-Y *graph* is a graph that can be drawn in the shape of the letter Y as in Figure 1, where the subgraph enclosed between the vertex Q and the two

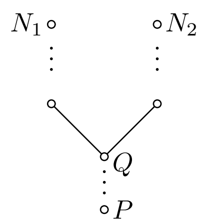

Figure 1. (a,b)-Y graph

vertices N_1 and N_2 is of dimension a, while the chain enclosed between the vertices P and Q is of dimension b. An (a,b)-Y graph is of dimension $d = a+b$. See [17], Definition 7.

The proof of the next theorem, follows mutatis mutandis from the proof of [17], Theorem 9.

Theorem 4.7. *Let R be a maximal non λ-domain. Then exactly one of the following holds:*

(i) $\mathrm{Spec}(R)$ *and* $\mathrm{Spec}(R')$ *are chains of the same dimension.*

(ii) $\mathrm{Spec}(R)$ *and* $\mathrm{Spec}(R')$ *are an* (a,b)-Y *graph.*

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References

[1] *A. Ayache, A. Jaballah*: Residually algebraic pairs of rings. Math. Z. *225* (1997), 49–65. zbl MR doi
[2] *A. Azarang*: On maximal subrings. Far East J. Math. Sci. *32* (2009), 107–118. zbl MR
[3] *A. Azarang, O. A. S. Karamzadeh*: On the existence of maximal subrings in commutative Artinian rings. J. Algebra Appl. *9* (2010), 771–778. zbl MR doi
[4] *A. Azarang, G. Oman*: Commutative rings with infinitely many maximal subrings. J. Algebra Appl. *13* (2014), Article ID 1450037, 29 pages. zbl MR doi
[5] *A. Badawi*: On 2-absorbing ideals of commutative rings. Bull. Aust. Math. Soc. *75* (2007), 417–429. zbl MR doi
[6] *E. Bastida, R. Gilmer*: Overrings and divisorial ideals of rings of the form $D+M$. Mich. Math. J. *20* (1973), 79–95. zbl MR doi
[7] *M. Ben Nasr, N. Jarboui*: On maximal non-valuation subrings. Houston J. Math. *37* (2011), 47–59. zbl MR
[8] *D. E. Davis*: Overrings of commutative rings III: Normal pairs. Trans. Am. Math. Soc. *182* (1973), 175–185. zbl MR doi
[9] *D. E. Dobbs*: On INC-extensions and polynomials with unit content. Can. Math. Bull. *23* (1980), 37–42. zbl MR doi
[10] *D. E. Dobbs, G. Picavet, M. Picavet-L'Hermitte*: Characterizing the ring extensions that satisfy FIP or FCP. J. Algebra *371* (2012), 391–429. zbl MR doi
[11] *D. E. Dobbs, J. Shapiro*: Normal pairs with zero-divisors. J. Algebra Appl. *10* (2011), 335–356. zbl MR doi
[12] *E. G. Evans, Jr.*: A generalization of Zariski's main theorem. Proc. Am. Math. Soc. *26* (1970), 45–48. zbl MR doi
[13] *M. S. Gilbert*: Extensions of Commutative Rings with Linearly Ordered Intermediate Rings. PhD Thesis. University of Tennessee, Knoxville, 1996. Available at `https://search.proquest.com/docview/304271872?accountid=8179`. MR
[14] *R. W. Gilmer, Jr.*: Overrings of Prüfer domains. J. Algebra *4* (1966), 331–340. zbl MR doi
[15] *R. Gilmer*: Some finiteness conditions on the set of overrings of an integral domain. Proc. Am. Math. Soc. *131* (2003), 2337–2346. zbl MR doi
[16] *R. W. Gilmer, Jr., J. F. Hoffmann*: A characterization of Prüfer domains in terms of polynomials. Pac. J. Math. *60* (1975), 81–85. zbl MR doi

[17] *A. Jaballah*: Maximal non-Prüfer and maximal non-integrally closed subrings of a field. J. Algebra Appl. *11* (2012), Article ID 1250041, 18 pages. zbl MR doi
[18] *I. Kaplansky*: Commutative Rings. University of Chicago Press, Chicago, 1974. zbl MR
[19] *R. Kumar, A. Gaur*: On λ-extensions of commutative rings. J. Algebra Appl. *17* (2018), Article ID 1850063, 9 pages. zbl MR doi
[20] *I. J. Papick*: Topologically defined classes of going-down domains. Trans. Am. Math. Soc. *219* (1976), 1–37. zbl MR doi

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