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SOME MODULE COHOMOLOGICAL PROPERTIES OF BANACH ALGEBRAS

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Abstract. We find some relations between module biprojectivity and module biflatness of Banach algebras $\mathcal{A}$ and $\mathcal{B}$ and their projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{B}$. For some semigroups S, we study module biprojectivity and module biflatness of semigroup algebras $l^1(S)$.

Keywords: module amenable; module biflat; module biprojective; semigroup algebra

MSC 2010: 16E40, 46H20, 46H25

1. Introduction

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha\cdot(ab)=(\alpha\cdot a)b,\quad (ab)\cdot\alpha=a(b\cdot\alpha),\quad a,b\in\mathcal{A},\ \alpha\in\mathfrak{A}.$$

Let X be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha\cdot(a\cdot x)=(\alpha\cdot a)\cdot x,\quad a\cdot(\alpha\cdot x)=(a\cdot\alpha)\cdot x,\quad (\alpha\cdot x)\cdot a=\alpha\cdot(x\cdot a) \tag{1.1}$$

for all $a\in\mathcal{A}$, $\alpha\in\mathfrak{A}$, $x\in X$, and similarly for the right and two-sided actions. Then, we say that X is a Banach $\mathcal{A}$-$\mathfrak{A}$-bimodule. If moreover $\alpha\cdot x=x\cdot\alpha$ for $x\in X$ and $\alpha\in\mathfrak{A}$, then X is a *commutative* $\mathcal{A}$-$\mathfrak{A}$-bimodule. A bounded map $D\colon\mathcal{A}\to X$ is a *module derivation* if

$$D(a\pm b)=D(a)\pm D(b),\quad D(ab)=D(a)\cdot b+a\cdot D(b),\quad a,b\in\mathcal{A}$$

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and

$$D(\alpha \cdot a) = \alpha \cdot D(a), \quad D(a \cdot \alpha) = D(a) \cdot \alpha, \quad a \in \mathcal{A},\ \alpha \in \mathfrak{A}.$$

Note that D is not necessarily linear, but its boundedness (defined as the existence of $M > 0$ such that $\|D(a)\| \leqslant M\|a\|$ for all $a \in \mathcal{A}$) still implies its continuity, as it preserves subtraction. When X is a commutative $\mathcal{A}$-$\mathfrak{A}$-module, each $x \in X$ defines a module derivation

$$D_x(a) = a \cdot x - x \cdot a, \quad a \in \mathcal{A}.$$

These are *inner* module derivations. A Banach algebra $\mathcal{A}$ is *module amenable* (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module X, each module derivation $D\colon\ \mathcal{A} \to X^*$ is inner.

The concept of module amenability for a class of Banach algebras which is in fact a generalization of the classical amenability has been developed by Amini in [1]. Indeed, he defined the module amenability of a Banach algebra $\mathcal{A}$ in the case that there is an extra $\mathfrak{A}$-module structure on $\mathcal{A}$ and showed that for every inverse semigroup S with subsemigroup E of idempotents, the $l^1(E)$-module amenability of $l^1(S)$ is equivalent to the amenability of S (module version of Johnson's theorem, see [14]). Also, module amenability of the projective tensor product $l^1(S) \mathbin{\widehat{\otimes}} l^1(S)$ is investigated by the third author in [3]. Other notions of module amenability such as module super amenability, module approximate amenability and module character amenability were introduced by other authors (cf. [4], [7], [17] and [19]).

Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule with compatible actions. We write $I_{\mathcal{A}}$ for the closed ideal of the projective tensor product of $\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}$ generated by all elements of the form $a \cdot \alpha \otimes b - a \otimes \alpha \cdot b$, $\alpha \in \mathfrak{A}$, and $a, b \in \mathcal{A}$. We also denote by $J_{\mathcal{A}}$ the closed ideal of $\mathcal{A}$ generated by the elements of the form $(a \cdot \alpha)b - a(\alpha \cdot b)$ for $\alpha \in \mathfrak{A}$, and $a, b \in \mathcal{A}$, see [22]. If there is no risk of confusion, we may write I and J instead of $I_{\mathcal{A}}$ and $J_{\mathcal{A}}$, respectively. Then both of the quotients $(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A})/I$ and $\mathcal{A}/J$ are $\mathcal{A}$-bimodules and $\mathfrak{A}$-bimodules. Also, $\mathcal{A}/J$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module whenever $\mathcal{A}$ acts on $\mathcal{A}/J$ canonically. Let $\pi\colon\ (\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}) \to \mathcal{A}$ be the bounded linear map defined by $\pi(a \otimes b) = ab$, and let $\widetilde{\pi}\colon\ (\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A})/I \to \mathcal{A}/J$ be its induced product map, that is, $\widetilde{\pi}(a \otimes b + I) = ab + J$.

The notions of module biprojectivity and module biflatness were introduced by Bodaghi and Amini in [5]. These are the module versions of the concepts biprojectivity and biflatness for Banach algebras introduced by Helemskii in [15]. A Banach algebra $\mathcal{A}$ is *module biprojective* (as an $\mathfrak{A}$-module) if $\widetilde{\pi}$ has a bounded right inverse which is an $\mathcal{A}/J$-$\mathfrak{A}$-module homomorphism. We say that $\mathcal{A}$ is *module biflat* (as an $\mathfrak{A}$-module) if $\widetilde{\pi}^*$ has a bounded left inverse which is an $\mathcal{A}/J$ -$\mathfrak{A}$-module homomorphism. Module biflatness for the second dual of a Banach algebra is also studied in [8].

In this paper, motivated by [5], we investigate some more facts and ideas concerning module biprojectivity and module biflatness of Banach algebras.

In Section 2, among other things, we show that under certain conditions module biprojectivity (biflatness) of Banach algebras $\mathcal{A}$ and $\mathcal{B}$ implies module biprojectivity (biflatness) of $\mathcal{A} \widehat{\otimes} \mathcal{B}$ (Theorems 2.3 and 2.4), and we also study the converse (Theorem 2.6). We discuss the relation between module amenability of $\mathcal{A} \widehat{\otimes} \mathcal{B}$ and amenability of $(\mathcal{A}/J_{\mathcal{A}}) \widehat{\otimes} (\mathcal{B}/J_{\mathcal{B}})$ (Proposition 2.2).

Section 3 is devoted to module biprojectivity and module biflatness of semigroup algebras $l^1(S)$ for some specified semigroups such as zero semigroups and rectangular band semigroups (Proposition 3.4), and inverse semigroups (Theorem 3.8). As a result, we show that $l^1(S) \widehat{\otimes} l^1(S)$ is module biflat whenever S is either the bicyclic inverse semigroup or the semigroup of positive integers $\mathbb{N}$ equipped with the maximum operation (Example 3.2).

2. Module biprojectivity and module biflatness of Banach algebras

Throughout, $\mathcal{A}$ and $\mathfrak{A}$ are Banach algebras for which $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions. We say $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from left (right) if there is a continuous linear functional f on $\mathfrak{A}$ such that $\alpha \cdot a = f(\alpha)a$ ($a \cdot \alpha = f(\alpha)a$), for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$ (see [1]).

The following lemma is proved in [6], Lemma 3.13.

Lemma 2.1. *If $\mathfrak{A}$ acts on $\mathcal{A}$ trivially from the left or right and $\mathcal{A}/J$ has a right bounded approximate identity, then for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$ we have $f(\alpha)a - a \cdot \alpha \in J$.*

The following result is the main key to achieve our purpose of this section.

Proposition 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-modules with trivial left action. If $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is module amenable and $\mathcal{A}/J_{\mathcal{A}}$, $\mathcal{B}/J_{\mathcal{B}}$ have identity, then $(\mathcal{A}/J_{\mathcal{A}}) \widehat{\otimes} (\mathcal{B}/J_{\mathcal{B}})$ is amenable. The converse is true if $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$.*

Proof. Let X be a unital $(\mathcal{A}/J_{\mathcal{A}}) \widehat{\otimes} (\mathcal{B}/J_{\mathcal{B}})$-bimodule and

$$D\colon (\mathcal{A}/J_{\mathcal{A}}) \widehat{\otimes} (\mathcal{B}/J_{\mathcal{B}}) \to X^*$$

a bounded derivation. Then X is an $\mathcal{A} \widehat{\otimes} \mathcal{B}$-bimodule with module actions given by

$$(a \otimes b) \cdot x := ((a + J_{\mathcal{A}}) \otimes (b + J_{\mathcal{B}})) \cdot x, \quad \text{and} \quad x \cdot (a \otimes b) := x \cdot ((a + J_{\mathcal{A}}) \otimes (b + J_{\mathcal{B}}))$$

for $x \in X$, $a \in \mathcal{A}$, and $b \in \mathcal{B}$. In addition, X is an $\mathfrak{A}$-bimodule with trivial actions. In the light of Lemma 2.1 and by assumptions, the actions of $\mathfrak{A}$ and $\mathcal{A} \widehat{\otimes} \mathcal{B}$ on X are compatible, so that X is a commutative Banach $\mathcal{A} \widehat{\otimes} \mathcal{B}$-$\mathfrak{A}$-module. Define $T\colon (\mathcal{A} \widehat{\otimes} \mathcal{B})/J_{\mathcal{A}\otimes\mathcal{B}} \to (\mathcal{A}/J_{\mathcal{A}}) \widehat{\otimes} (\mathcal{B}/J_{\mathcal{B}})$ via

$$T(a \otimes b) + J_{\mathcal{A}\otimes\mathcal{B}} = (a + J_{\mathcal{A}}) \otimes (b + J_{\mathcal{B}}).$$

For $a, c \in \mathcal{A}$, $b, d \in \mathcal{B}$ and $\alpha \in \mathfrak{A}$, we have

$$\begin{aligned}[(a \otimes b) \cdot \alpha](c \otimes d) - (a \otimes b)[\alpha \cdot (c \otimes d)] &= (ac \otimes (b \cdot \alpha)d) - (f(\alpha)ac \otimes bd) \\ &= ac \otimes [(b \cdot \alpha)d - f(\alpha)bd]\end{aligned}$$

showing that T is well-defined. Clearly, T is $\mathfrak{A}$-bimodule morphism. Putting $\overline{D} := D \circ T \circ \pi\colon (\mathcal{A} \widehat{\otimes} \mathcal{B}) \to X^*$, where $\pi\colon (\mathcal{A} \widehat{\otimes} \mathcal{B}) \to (\mathcal{A} \widehat{\otimes} \mathcal{B})/J_{\mathcal{A}\otimes\mathcal{B}}$ is the projection map, we observe that $\overline{D}$ is a module derivation. Since $(\mathcal{A}/J_{\mathcal{A}}) \widehat{\otimes} (\mathcal{B}/J_{\mathcal{B}})$ is an $\mathfrak{A}$-bimodule, $\overline{D}((a \otimes b) \cdot \alpha) = \overline{D}(a \otimes b) \cdot \alpha$ for all $\alpha \in \mathfrak{A}$. On the other hand, $\overline{D}(\alpha \cdot (a \otimes b)) = \overline{D}(f(\alpha)(a \otimes b)) = \alpha \cdot \overline{D}(a \otimes b)$, because the left $\mathfrak{A}$-module actions on $\mathcal{A}$ and X are trivial. Therefore there exists $x^* \in X^*$ such that $\overline{D}(a \otimes b) = (a \otimes b) \cdot x^* - x^* \cdot (a \otimes b)$, hence $D((a + J_{\mathcal{A}}) \otimes (b + J_{\mathcal{B}})) = ((a + J_{\mathcal{A}}) \otimes (b + J_{\mathcal{B}}))x^* - x^* \cdot ((a + J_{\mathcal{A}}) \otimes (b + J_{\mathcal{B}}))$, and so D is inner.

Now, suppose that X is a commutative Banach $\mathcal{A} \widehat{\otimes} \mathcal{B}$-$\mathfrak{A}$- module. We consider the following module actions of $(\mathcal{A}/J_{\mathcal{A}}) \widehat{\otimes} (\mathcal{B}/J_{\mathcal{B}})$ on X:

$$((a + J_{\mathcal{A}}) \otimes (b + J_{\mathcal{B}})) \cdot x := (a \otimes b) \cdot x, \quad \text{and} \quad x \cdot ((a + J_{\mathcal{A}}) \otimes (b + J_{\mathcal{B}})) := x \cdot (a \otimes b)$$

for all $x \in X$, $a \in \mathcal{A}$, and $b \in \mathcal{B}$. A simple calculation shows that

$$[((a \cdot \alpha)b - a(\alpha \cdot b)) \otimes ((c \cdot \beta)d - c(\beta \cdot d))] \cdot x = 0, \quad a, b \in \mathcal{A},\ c, d \in \mathcal{B},\ x \in X,\ \alpha, \beta \in \mathfrak{A}.$$

We also see that $(a \otimes b) \cdot x = 0$ and $x \cdot (a \otimes b) = 0$, if $a \in J_{\mathcal{A}}$ or $b \in J_{\mathcal{B}}$. Therefore X is a Banach $(\mathcal{A}/J_{\mathcal{A}}) \widehat{\otimes} (\mathcal{B}/J_{\mathcal{B}})$-bimodule. Suppose that $D\colon (\mathcal{A} \widehat{\otimes} \mathcal{B}) \to X^*$ is a module derivation, and consider $\widetilde{D}((a + J_{\mathcal{A}}) \otimes (b + J_{\mathcal{B}})) := D(a \otimes b)$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. Then we have $D(a \otimes ((c \cdot \alpha)d - c(\alpha \cdot d))) = 0$, and

$$D(a \otimes cd) = \overline{D}((a + J_{\mathcal{A}}) \otimes (cd + J_{\mathcal{B}})) = \overline{D}((ae_{\mathcal{A}} + J_{\mathcal{A}}) \otimes (cd + J_{\mathcal{B}})) = D(ae_{\mathcal{A}} \otimes cd),$$

hence $\overline{D}$ is well-defined. Suppose that $\mathfrak{A}$ has a bounded approximate identity (ξ_i) for $\mathcal{A}$. Since f is bounded, without loss of generality, we may assume that $f(\xi_i) \to 1$, as $i \to \infty$. Then for each $\lambda \in \mathbb{C}$ we have

$$e_{\mathcal{A}} \cdot (\lambda \xi_i) - f(\xi_i)e_{\mathcal{A}} = (\lambda e_{\mathcal{A}}) \cdot \xi_i - f(\xi_i)e_{\mathcal{A}} \to \lambda e_{\mathcal{A}} - e_{\mathcal{A}}.$$

Since $J_{\mathcal{A}}$ is a closed ideal of $\mathcal{A}$, $\lambda e_{\mathcal{A}} - e_{\mathcal{A}} \in J_{\mathcal{A}}$. Next, for each $\lambda \in \mathbb{C}$, and $a \in \mathcal{A}$, $b \in \mathcal{B}$, we have

$$\overline{D}((\lambda a + J_{\mathcal{A}}) \otimes (b + J_{\mathcal{B}})) = \lambda \overline{D}((a + J_{\mathcal{A}}) \otimes (b + J_{\mathcal{B}}))$$

so that $\overline{D}$ is $\mathbb{C}$-linear. Therefore D is an inner module derivation. □

Theorem 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-modules with trivial left action. Let $\mathcal{A}$, $\mathcal{B}$ be module biflat and let $\mathcal{A}/J_{\mathcal{A}}$, $\mathcal{B}/J_{\mathcal{B}}$ be commutative Banach $\mathfrak{A}$-module. If $\mathcal{A}$, $\mathcal{B}$ have bounded approximate identity and $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$ and $\mathcal{B}$, then $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is module biflat.*

Proof. By [5], Theorem 2.1, $\mathcal{A}$, $\mathcal{B}$ are module amenable, and so $\mathcal{A}/J_{\mathcal{A}}$, $\mathcal{B}/J_{\mathcal{B}}$ are amenable (see also [2], Proposition 3.3). It follows from [9], Corollary 2.9.62 that $(\mathcal{A}/J_{\mathcal{A}}) \widehat{\otimes} (\mathcal{B}/J_{\mathcal{B}})$ is amenable. Applying Proposition 2.2, we see that $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is module amenable. Again, by [5], Theorem 2.1 we conclude that $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is module biflat. □

The following is the module biprojective version of Theorem 2.3.

Theorem 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-modules with trivial left action. If $\mathcal{A}$ and $\mathcal{B}$ are module biprojective, then so is $\mathcal{A} \widehat{\otimes} \mathcal{B}$.*

Proof. By the assumption, there exist an $\mathcal{A}/J_{\mathcal{A}}$-$\mathfrak{A}$-module morphism $\widetilde{\varrho}_{\mathcal{A}}\colon \mathcal{A}/J_{\mathcal{A}} \to (\mathcal{A} \widehat{\otimes} \mathcal{A})/I_{\mathcal{A}}$ with $\widetilde{\pi}_{\mathcal{A}} \circ \widetilde{\varrho}_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}/J_{\mathcal{A}}}$ and a $\mathcal{B}/J_{\mathcal{B}}$-$\mathfrak{A}$-module morphism $\widetilde{\varrho}_{\mathcal{B}}\colon \mathcal{B}/J_{\mathcal{B}} \to (\mathcal{B} \widehat{\otimes} \mathcal{B})/I_{\mathcal{B}}$ with $\widetilde{\pi}_{\mathcal{B}} \circ \widetilde{\varrho}_{\mathcal{B}} = \mathrm{id}_{\mathcal{B}/J_{\mathcal{B}}}$. Define $T\colon (\mathcal{A} \widehat{\otimes} \mathcal{B})/J_{\mathcal{A}\widehat{\otimes}\mathcal{B}} \to (\mathcal{A}/J_{\mathcal{A}}) \widehat{\otimes} (\mathcal{B}/J_{\mathcal{B}})$ via $(a \otimes b) + J_{\mathcal{A}\widehat{\otimes}\mathcal{B}} \mapsto (a + J_{\mathcal{A}}) \otimes (b + J_{\mathcal{B}})$.

Let $\tilde{\theta}\colon [(\mathcal{A} \widehat{\otimes} \mathcal{A})/I_{\mathcal{A}}] \widehat{\otimes} [(\mathcal{B} \widehat{\otimes} \mathcal{B})/I_{\mathcal{B}}] \to ((\mathcal{A} \widehat{\otimes} \mathcal{B}) \widehat{\otimes} (\mathcal{A} \widehat{\otimes} \mathcal{B}))/I_{\mathcal{A}\widehat{\otimes}\mathcal{B}}$ be the isometric isomorphism given by

$$((a_1 \otimes a_2) + I_{\mathcal{A}}) \otimes ((b_1 \otimes b_2) + I_B) \mapsto (a_1 \otimes b_1) \otimes (a_2 \otimes b_2) + I_{\mathcal{A}\widehat{\otimes}\mathcal{B}}, \quad a_1, a_2 \in \mathcal{A},\ b_1, b_2 \in \mathcal{B}.$$

Setting

$$\widetilde{\varrho} = \tilde{\theta} \circ (\widetilde{\varrho}_{\mathcal{A}} \otimes \widetilde{\varrho}_{\mathcal{B}}) \circ T\colon\ (\mathcal{A} \widehat{\otimes} \mathcal{B})/J_{\mathcal{A}\widehat{\otimes}\mathcal{B}} \to ((\mathcal{A} \widehat{\otimes} \mathcal{B}) \widehat{\otimes} (\mathcal{A} \widehat{\otimes} \mathcal{B}))/I_{\mathcal{A}\widehat{\otimes}\mathcal{B}}$$

we easily see that $\widetilde{\varrho}$ is an $(\mathcal{A} \widehat{\otimes} \mathcal{B})/J_{\mathcal{A}\otimes\mathcal{B}}$-$\mathfrak{A}$-module morphism. Since $\widetilde{\pi}_{\mathcal{A}\widehat{\otimes}\mathcal{B}} = T^{-1} \circ (\widetilde{\pi}_{\mathcal{A}} \otimes \widetilde{\pi}_{\mathcal{B}}) \circ \tilde{\theta}^{-1}$, we have $\widetilde{\pi}_{\mathcal{A}\widehat{\otimes}\mathcal{B}} \circ \widetilde{\varrho} = \mathrm{id}_{(\mathcal{A}\widehat{\otimes}\mathcal{B})/J_{\mathcal{A}\widehat{\otimes}\mathcal{B}}}$. Therefore, $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is module biprojective. □

The next result is the module version of [12], Proposition 3.3.

Proposition 2.5. *Assume that $\mathfrak{A}$ acts trivially on $\mathcal{A}$, $\mathcal{B}$ from the left and assume that $I = cl((\mathcal{A}/J_{\mathcal{A}})I + I(\mathcal{A}/J_{\mathcal{A}}))$. If $\mathcal{A}$ is module biflat then $\mathcal{B}$ is module biflat.*

P r o o f. Let $\widetilde{\varrho}\colon ((\mathcal{A}\widehat{\otimes}\mathcal{A})/I_{\mathcal{A}})^* \to (\mathcal{A}/J_{\mathcal{A}})^*$ be a weak splitting of the multiplication on $\mathcal{A}$. Define $\upsilon_{\mathcal{B}}\colon (\mathcal{B}/J_{\mathcal{B}})\widehat{\otimes}(\mathcal{B}/J_{\mathcal{B}}) \to (\mathcal{B}\widehat{\otimes}\mathcal{B})/I_{\mathcal{B}}$ via $\upsilon_{\mathcal{B}}((a+J_{\mathcal{B}})\otimes(b+J_{\mathcal{B}})) = (a\otimes b)+J_{\mathcal{B}}$. It was shown in the proof of Propostion 2.9 from [13] that $\upsilon_{\mathcal{B}}$ is well-defined. We define the map $\omega_{\mathcal{A}}\colon (\mathcal{A}\widehat{\otimes}\mathcal{A})/I_{\mathcal{A}} \to (\mathcal{A}/J_{\mathcal{A}})\widehat{\otimes}(\mathcal{A}/J_{\mathcal{A}})$ by the formula $\omega_{\mathcal{A}}(a\otimes b+I_{\mathcal{A}}) = (a+J_{\mathcal{A}})\otimes(b+J_{\mathcal{A}})$. For $a,b,c,d\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$, we have

$$\begin{aligned}((a\cdot\alpha)\otimes c - a\otimes(\alpha\cdot c))(b\otimes d) &= ((a\cdot\alpha)b\otimes cd) - (ab\otimes(\alpha\cdot c)d)\\ &= ((a\cdot\alpha)b\otimes cd) - (ab\otimes f(\alpha)cd)\\ &= ((a\cdot\alpha)b\otimes cd) - (a(\alpha\cdot b)\otimes cd)\\ &= ((a\cdot\alpha)b - a(\alpha\cdot b))\otimes cd.\end{aligned}$$

The above relations show that $\omega_{\mathcal{A}}$ is well-defined. Defining $\theta\colon ((\mathcal{A}\widehat{\otimes}\mathcal{A})/I_{\mathcal{A}}) \to ((\mathcal{B}\widehat{\otimes}\mathcal{B})/I_{\mathcal{B}})$ by $\theta((a_1\otimes a_2)+I_{\mathcal{A}}) = \upsilon_{\mathcal{B}}\circ(q\otimes q)\circ\omega_{\mathcal{A}}((a_1\otimes a_2)+I_{\mathcal{A}})$, we wish to complete the diagram

$$\begin{array}{ccc}((\mathcal{B}\widehat{\otimes}\mathcal{B})/I_{\mathcal{B}})^* & \xrightarrow{\theta^*} & ((\mathcal{A}\widehat{\otimes}\mathcal{A})/I_{\mathcal{A}})^*\\ \Big\downarrow{\scriptstyle\tau} & & \Big\downarrow{\scriptstyle\widetilde{\varrho}}\\ (\mathcal{B}/J_{\mathcal{B}})^* & \xrightarrow{q^*} & (\mathcal{A}/J_{\mathcal{A}})^*\end{array}$$

so that $\tau\circ\pi_{\mathcal{B}}^* = 1_{(\mathcal{B}/J_{\mathcal{B}})^*}$. Let $\varphi\in((\mathcal{B}\widehat{\otimes}\mathcal{B})/I_{\mathcal{B}})^*$ and $\psi=\varphi\circ\theta$. In order to define $\tau(\varphi)$ we must show that $\widetilde{\varrho}(\psi)(I)=0$. Let $i=(\alpha'+J_{\mathcal{A}})i'+i''(\alpha''+J_{\mathcal{A}})$ where $i',i''\in I$ and $\alpha',\alpha''\in\mathcal{A}$. Then

$$\begin{aligned}\widetilde{\varrho}(\psi)(i) &= \widetilde{\varrho}(\psi)((\alpha'+J_{\mathcal{A}})i'+i''(\alpha''+J_{\mathcal{A}}))\\ &= \widetilde{\varrho}(i'\cdot\psi)(\alpha'+J_{\mathcal{A}})+\widetilde{\varrho}(\psi\cdot i'')(\alpha''+J_{\mathcal{A}}).\end{aligned}$$

But $i'\cdot\psi((a',a'')+I_{\mathcal{A}}) = \psi((a',a''i')+I_{\mathcal{A}}) = \varphi\circ\upsilon_{\mathcal{B}}\circ(q\otimes q)\circ\omega_{\mathcal{A}}((a',a''i')+I_{\mathcal{A}}) = \varphi\circ\upsilon_{\mathcal{B}}\circ(q\otimes q)(a'+J_{\mathcal{A}},a''i'+J_{\mathcal{A}}) = \varphi\circ\upsilon_{\mathcal{B}}(q(a'+J_{\mathcal{A}}),q(a''i'+J_{\mathcal{A}}))=0$, $a',a''\in\mathcal{A}$ so $i'\cdot\psi=0$. Similarly $\psi\cdot i''=0$. Since $I=cl((\mathcal{A}/J_{\mathcal{A}})I+I(\mathcal{A}/J_{\mathcal{A}}))$, we get $\widetilde{\varrho}(\psi)(I)=\{0\}$ as desired. Hence there is a map $\tau\colon((\mathcal{B}\widehat{\otimes}\mathcal{B})/I_{\mathcal{B}})^*\to(\mathcal{B}/J_{\mathcal{B}})^*$ making the diagram commutative. By injectivity of the maps q^*, θ^* and the closed graph theorem τ is a bounded $\mathcal{B}/J_{\mathcal{B}}$-$\mathfrak{A}$-bimodule homomorphism. Finally

$$q^*\circ\tau\circ\pi_{\mathcal{B}}^* = \widetilde{\varrho}\circ\theta^*\circ\pi_{\mathcal{B}}^* = \widetilde{\varrho}\circ\pi_A^*\circ q^* = q^*$$

and then we get $\tau\circ\pi_{\mathcal{B}}^* = 1_{(\mathcal{B}/J_{\mathcal{B}})^*}$, since q^* is injective. $\square$

In the next result which is a module version of [20], Proposition 2.6, we bring the converse of Theorems 2.3 and 2.4 under some mild conditions.

Proposition 2.6. *Let $\mathcal{A}$ be unital, let $\mathcal{B}$ contain a nonzero idempotent b_0, and let $\mathfrak{A}$ act trivially on $\mathcal{A}$ and $\mathcal{B}$ from the left. Suppose that $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is module biprojective (biflat). Then $\mathcal{A}$ is module biprojective (biflat).*

P r o o f. Suppose that $\mathcal{A}\mathbin{\widehat{\otimes}}\mathcal{B}$ is module biprojective. Then there exists an $(\mathcal{A}\mathbin{\widehat{\otimes}}\mathcal{B})/J_{\mathcal{A}\widehat{\otimes}\mathcal{B}}$-$\mathfrak{A}$-module morphism $\widetilde{\varrho}\colon (\mathcal{A}\mathbin{\widehat{\otimes}}\mathcal{B})/J_{\mathcal{A}\widehat{\otimes}\mathcal{B}} \to ((\mathcal{A}\mathbin{\widehat{\otimes}}\mathcal{B})\mathbin{\widehat{\otimes}}(\mathcal{A}\mathbin{\widehat{\otimes}}\mathcal{B}))/I_{\mathcal{A}\widehat{\otimes}\mathcal{B}}$ with $\widetilde{\pi}_{\mathcal{A}\widehat{\otimes}\mathcal{B}}\circ\widetilde{\varrho} = \mathrm{id}_{(\mathcal{A}\widehat{\otimes}\mathcal{B})/J_{\mathcal{A}\widehat{\otimes}\mathcal{B}}}$. We regard $(\mathcal{A}\mathbin{\widehat{\otimes}}\mathcal{B})/J_{\mathcal{A}\otimes\mathcal{B}}$ as an $\mathcal{A}/J_{\mathcal{A}}$-$\mathfrak{A}$-module with the actions

$$\begin{aligned}
(a_1+J_{\mathcal{A}})\cdot((a_2\otimes b)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}}) &= (a_1a_2\otimes b)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}},\\
((a_2\otimes b)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}})\cdot(a_1+J_{\mathcal{A}}) &= (a_2a_1\otimes b)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}},\\
\alpha\cdot((a\otimes b)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}}) &= (\alpha\cdot a\otimes b)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}},\\
((a\otimes b)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}})\cdot\alpha &= (a\cdot\alpha\otimes b)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}},
\end{aligned}$$

where $\alpha\in\mathfrak{A}$, $a_1,a_2,a\in\mathcal{A}$, $b\in\mathcal{B}$. Then for $a_1,a_2\in\mathcal{A}$ we have

$$\begin{aligned}
&\widetilde{\varrho}((a_1+J_{\mathcal{A}})\cdot((a_2\otimes b_0)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}}))\\
&\qquad= \widetilde{\varrho}((a_1a_2\otimes b_0)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}})\\
&\qquad= \widetilde{\varrho}(((a_1\otimes b_0)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}})((a_2\otimes b_0)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}}))\\
&\qquad= ((a_1\otimes b_0)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}})\cdot\widetilde{\varrho}((a_2\otimes b_0)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}})\\
&\qquad= ((a_1+J_{\mathcal{A}})\cdot((e\otimes b_0)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}}))\cdot\widetilde{\varrho}((a_2\otimes b_0)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}})\\
&\qquad= (a_1+J_{\mathcal{A}})\cdot\widetilde{\varrho}((a_2\otimes b_0)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}}).
\end{aligned}$$

Similarly, we can obtain a right-module version of this equation. Hence

$$\begin{aligned}
\widetilde{\varrho}((a_1+J_{\mathcal{A}})\cdot((a_2\otimes b_0)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}})) &= (a_1+J_{\mathcal{A}})\cdot\widetilde{\varrho}((a_2\otimes b_0)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}})\\
&= \widetilde{\varrho}((a_2\otimes b_0)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}})\cdot(a_1+J_{\mathcal{A}})
\end{aligned} \tag{2.1}$$

for all $a_1,a_2\in\mathcal{A}$. Take $\varphi\in((\mathcal{B}/J_{\mathcal{B}})^*)_{[1]}$ with $\langle b_0+J_{\mathcal{B}},\varphi\rangle = 1$ and define

$$\begin{aligned}
\tilde{\theta}\colon\quad ((\mathcal{A}\mathbin{\widehat{\otimes}}\mathcal{B})\mathbin{\widehat{\otimes}}(\mathcal{A}\mathbin{\widehat{\otimes}}\mathcal{B}))/I_{\mathcal{A}\widehat{\otimes}\mathcal{B}} &\to (\mathcal{A}\mathbin{\widehat{\otimes}}\mathcal{A})/I_{\mathcal{A}}\\
((a_1\otimes b_1)\otimes(a_2\otimes b_2))+I_{\mathcal{A}\widehat{\otimes}\mathcal{B}} &\mapsto (\varphi(b_1b_2)a_1\otimes a_2)+I_{\mathcal{A}}.
\end{aligned}$$

Then, $\tilde{\theta}$ is an $\mathcal{A}/J_{\mathcal{A}}$-$\mathfrak{A}$-module morphism. We now define $\bar{\varrho}\colon \mathcal{A}/J_{\mathcal{A}}\to(\mathcal{A}\mathbin{\widehat{\otimes}}\mathcal{A})/I_{\mathcal{A}}$ via

$$\bar{\varrho}(a+J_{\mathcal{A}}) = \tilde{\theta}\circ\widetilde{\varrho}\circ T((a\otimes b_0)+J_{\mathcal{A}\widehat{\otimes}\mathcal{B}})$$

where $T\colon (\mathcal{A}/J_{\mathcal{A}} \widehat{\otimes} \mathcal{B}/J_{\mathcal{B}}) \to (\mathcal{A} \widehat{\otimes} \mathcal{B})/J_{\mathcal{A}\widehat{\otimes}\mathcal{B}}$ is defined by $T((a+J_{\mathcal{A}}) \otimes (b+J_{\mathcal{B}})) = a \otimes b + J_{\mathcal{A}\otimes\mathcal{B}}$. By (2.1), $\bar{\varrho}$ is an $\mathcal{A}/J_{\mathcal{A}}$-$\mathfrak{A}$-module homomorphism. The identity $\pi_{\mathcal{A}/J_{\mathcal{A}}} \circ \tilde{\theta} = (\mathrm{id}_{\mathcal{A}/J_{\mathcal{A}}} \otimes \varphi) \circ \pi_{(\mathcal{A}\widehat{\otimes}\mathcal{B})/I_{\mathcal{A}\widehat{\otimes}\mathcal{B}}}$ implies that $\pi_{\mathcal{A}/J_{\mathcal{A}}} \circ \bar{\varrho} = \pi_{\mathcal{A}/J_{\mathcal{A}}} \circ \tilde{\theta} \circ \widetilde{\varrho} \circ T = (\mathrm{id}_{\mathcal{A}/J_{\mathcal{A}}} \otimes \varphi) \circ \pi_{(\mathcal{A}\widehat{\otimes}\mathcal{B})/I_{\mathcal{A}\widehat{\otimes}\mathcal{B}}} \circ \widetilde{\varrho} \circ T$. Therefore, $\mathcal{A}$ is module biprojective. For the biflat case, we notice that for the given $\widetilde{\varrho}\colon (\mathcal{A}\widehat{\otimes}\mathcal{B})/J_{\mathcal{A}\widehat{\otimes}\mathcal{B}} \to (((\mathcal{A}\widehat{\otimes}\mathcal{B})\widehat{\otimes}(\mathcal{A}\widehat{\otimes}\mathcal{B}))/I_{\mathcal{A}\widehat{\otimes}\mathcal{B}})^{**}$ with $\pi^{**}_{\mathcal{A}\widehat{\otimes}\mathcal{B}} \circ \widetilde{\varrho} = \mathrm{id}_{(\mathcal{A}\widehat{\otimes}\mathcal{B})/J_{\mathcal{A}\widehat{\otimes}\mathcal{B}}}$, one may define $\bar{\varrho}\colon \mathcal{A}/J_{\mathcal{A}} \to ((\mathcal{A}\widehat{\otimes}\mathcal{A})/I_{\mathcal{A}})^{**}$ through

$$\bar{\varrho}(a+J_{\mathcal{A}}) = \tilde{\theta}^{**} \circ \widetilde{\varrho} \circ T((a \otimes b_0) + J_{\mathcal{A}\otimes\mathcal{B}}), \quad a \in \mathcal{A}.$$

Then it is easily checked that $\bar{\varrho}$ has the required properties. □

For a Banach algebra $\mathcal{A}$ and a nonempty set Λ, we denote by $\mathbb{M}_\Lambda(\mathcal{A})$, the Banach algebra of $\Lambda \times \Lambda$ matrices (a_{ij}) with entries in $\mathcal{A}$ such that $\|(a_{ij})\| = \sum_{i,j\in\Lambda} \|a_{ij}\| < \infty$.

Corollary 2.7. *Suppose that $\mathfrak{A}$ acts trivially on a unital Banach algebra $\mathcal{A}$ from the left and that it is a nonempty set. Then $\mathbb{M}_\Lambda(\mathcal{A})$ is module biflat (module biprojective) if and only if $\mathcal{A}$ is module biflat (module biprojective).*

P r o o f. Using Proposition 2.6, the proof is similar to that of [20], Proposition 2.7. □

3. Applications to semigroup algebras

Let S be a semigroup. An element $p \in S$ is an *idempotent* if $p^2 = p$. We write $E(S)$ for the set of all idempotents of S. We say S is a *band semigroup* if $S = E(S)$ or briefly E, and it is a *semilattice* if S is a commutative band semigroup. We also say S is an *inverse semigroup* if for each $s \in S$ there exists a unique element $s^* \in S$ with $ss^*s = s$ and $s^*ss^* = s^*$. Let S be an inverse semigroup with the set of idempotents E, where the order of E is defined by

$$e \leqslant d \Leftrightarrow ed = e, \quad e, d \in E.$$

It is standard that the *semigroup algebra* $l^1(S)$ is a Banach algebra and a Banach $l^1(E)$-module with compatible actions (see [1]). Here, for a technical reason we let $l^1(E)$ act on $l^1(S)$ by multiplication from right and trivially from left, that is,

$$\delta_e \cdot \delta_s = \delta_s, \quad \delta_s \cdot \delta_e = \delta_{se} = \delta_s * \delta_e, \quad s \in S,\ e \in E.$$

In this case, the ideal J (see Section 2) is the closed linear span of $\{\delta_{set} - \delta_{st}\colon s, t \in S, e \in E\}$. We consider an equivalence relation on S as follows:

$$s \approx t \Leftrightarrow \delta_s - \delta_t \in J, \quad s, t \in S.$$

For an inverse semigroup S, the quotient $\mathcal{G}_S \cong S/{\approx}$ is a discrete group (see [2] and [17]). Indeed, $\mathcal{G}_S$ is homomorphic to the maximal group homomorphic image of S (see [18]). In particular, S is amenable if and only if $\mathcal{G}_S$ is amenable (see [10]). As in [21], Theorem 3.3, we may observe that $\ell^1(S)/J \cong \ell^1(\mathcal{G}_S)$. With the notation of Section 2, $\ell^1(S)/J$ is a commutative $\ell^1(E)$-bimodule with the following actions:

$$\delta_e \cdot (\delta_s + J) = \delta_s + J, \quad (\delta_s + J) \cdot \delta_e = \delta_{se} + J, \quad s \in S, \ e \in E.$$

Let $k \in \mathbb{N}$. Recall that E satisfies condition D_k (see [10]) if given $f_1, f_2, \ldots, f_{k+1} \in E$ there exist $e \in E$ and i, j such that

$$1 \leqslant i < j \leqslant k+1, \quad f_i e = f_i, \quad f_j e = f_j.$$

Duncan and Namioka in [10], Theorem 16 proved that for any inverse semigroup S, $l^1(S)$ has a bounded approximate identity if and only if E satisfies condition D_k for some k.

Theorem 3.1. *Let S be an inverse semigroup with the nonempty set of idempotents E. If E satisfies condition D_k for some k, then $l^1(S) \widehat{\otimes} l^1(S)$ is module biflat as an $l^1(E)$-module (with trivial left action) if and only if S is amenable.*

P r o o f. We first note that $l^1(S)/J \cong \ell^1(\mathcal{G}_{\mathcal{C}})$ has an identity. If S is amenable, then $l^1(S)$ is module biflat by [5], Theorem 3.2, and so $l^1(S) \widehat{\otimes} l^1(S)$ is module biflat by Theorem 2.3. The converse holds by Proposition 2.6 and [5], Theorem 3.2. $\square$

E x a m p l e 3.2. Let $\mathcal{C}$ be the bicyclic inverse semigroup generated by p and q, that is,

$$\mathcal{C} = \{p^a q^b \colon a, b \geqslant 0\}, \quad (p^a q^b)^* = p^b q^a.$$

The multiplication operation is defined by

$$(p^a q^b)(p^c q^d) = p^{a-b+\max\{b,c\}} q^{d-c+\max\{b,c\}}.$$

The set of idempotents of $\mathcal{C}$ is $E_{\mathcal{C}} = \{p^a q^a \colon a = 0, 1, \ldots\}$, which is also totally ordered with the order

$$p^a q^a \leqslant p^b q^b \Leftrightarrow a \leqslant b.$$

Therefore, E satisfies condition D_1. It is shown in [5] that $l^1(\mathcal{C})$ is module biflat. Furthermore, consider $(\mathbb{N}, \vee)$ with maximum operation $m \vee n = \max(m, n)$, then each element of $\mathbb{N}$ is an idempotent. It is also shown in [5] that $l^1(\mathbb{N})$ is module biflat. Now, if $l^1(S)$ is either the Banach algebra $l^1(\mathcal{C})$ or $l^1(\mathbb{N})$, then $l^1(S) \widehat{\otimes} l^1(S)$ is module biflat by Theorem 2.3.

In analogue to [5], Proposition 2.1, we have the next result. Since the proof is similar, it is omitted.

Proposition 3.3. *Assume that $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from the left (right) and $\mathcal{A}/J$ has at least a left (right) identity. If $\mathcal{A}$ is biprojective, then $\mathcal{A}$ is module biprojective.*

We recall that a semigroup S is a *right* (*left*) zero semigroup if $st = t$ ($st = s$) for each $s, t \in S$. Also, an idempotent semigroup S is a *rectangular band semigroup* if $xyx = x$ for each $x, y \in S$. In the case that S is right or left zero semigroup, we have $E = S$. In particular, $l^1(E) = l^1(S)$ and so $J_{l^1(S)} = \{0\}$. Once more, for every element s in right (left) S, δ_s can be viewed as a left (right) identity for $l^1(S)$. Now, we generalize Proposition 3.1 and Proposition 3.2 of [11] by using Proposition 3.3 as the upcoming result.

Proposition 3.4. *Let S be either a right (left) zero semigroup or a rectangular band semigroup. Then, $l^1(S)$ is module biprojective.*

Let $l^1(S)$ be module biflat (as an $l^1(E)$-module with trivial left action). Then there exists an $l^1(S)/J$-$l^1(E)$-bimodule morphism $\widetilde{\varrho}: l^1(S)/J \to ((l^1(S)\widehat{\otimes} l^1(S))/I)^{**}$ with $\widetilde{\pi}^{**} \circ \widetilde{\varrho} = i_{l^1(S)/J}$. Fix $u \in S$. Suppose that $ru = vw$ for some element $r, v, w \in S$, and set $\theta = ru = vw$. We can find nets $(z_\alpha + I)$ and $(w_\alpha + I)$ in $((l^1(S) \mathbin{\widehat{\otimes}} l^1(S))/I)_{[\|\widetilde{\varrho}\|]}$ indexed by the same directed set such that $\lim_\alpha z_\alpha + I = \widetilde{\varrho}(\delta_u + J)$ and $\lim_\alpha w_\alpha + I = \widetilde{\varrho}(\delta_v + J)$ in the weak*-topology. Set $\lambda_\theta = \chi_{\{\theta\}} \in l^\infty(S) = l^1(S)^*$, and define $\Lambda_\theta \colon\ l^1(S)/J \to \mathbb{C}$ by

$$\Lambda_\theta(\delta_s + J) := \langle \delta s, \lambda_\theta \rangle = \begin{cases} 1, & \delta = s, \\ 0, & \delta \neq s. \end{cases}$$

Then we have

$$\begin{aligned} 1 = \langle \widetilde{\pi}^*(\Lambda_\theta), \widetilde{\varrho}(\delta_\theta + J) \rangle &= \lim_\alpha \langle \widetilde{\pi}^*(\Lambda_\theta), (\delta_r + J) \cdot (z_\alpha + I) \rangle \\ &= \lim_\alpha \langle \Lambda_\theta, \widetilde{\pi}((\delta_r + J) \cdot (z_\alpha + I)) \rangle. \end{aligned}$$

Since $\lim_\alpha((\delta_r + J) \cdot (z_\alpha + I) - (w_\alpha + I) \cdot (\delta_w + J)) = I$ in the weak topology on $(l^1(S) \mathbin{\widehat{\otimes}} l^1(S))/I$, we may by Mazur's theorem suppose that

$$\lim_\alpha \|(\delta_r + J) \cdot (z_\alpha + I) - (w_\alpha + I) \cdot (\delta_w + J)\|_\pi = I.$$

Similarly to the proof of [20], Lemma 3.1, we may see that

$$\lim_\alpha \sum_{(y,t) \in Z(r,w,\theta)} z^\alpha_{y,t} = 1,$$

where

$$Z(r,w,\theta)=\{(y,t)\in S\times S\colon\ t\in Sw,\ ryt=\theta\}.$$

The two next propositions are module versions of Theorem 3.2 and Proposition 3.4 of [20], respectively. Since their proofs are mainly verbatim of the classical case, we omit them.

Proposition 3.5. *Let S be a semigroup. Suppose that the Banach algebra $l^1(S)$ is module biflat (as an $l^1(E)$-module with trivial left actions). Then there is a constant $C>0$ such that the following property holds for each $u\in S$, $N\in\mathbb{N}$, and elements $(r_1,v_1,w_1),\dots,(r_n,v_n,w_n)\in S\times S\times S$ such that*

(i) $r_iu=v_iw_i$, $i=1,\dots,N$; *and,*

(ii) *the sets $Sw_1\cap[r_1^{-1}(r_1u)],\dots,Sw_N\cap[r_N^{-1}(r_Nu)]$ are pairwise disjoint.*

Then necessarily $N\leqslant C$.

Let $(\mathcal{P},\leqslant)$ be a partially ordered set. Then $\mathcal{P}$ is called *uniformly locally finite* if for some $C\geqslant 1$, $\sup\{|(x]|\colon\ x\in P\}\leqslant C$.

Proposition 3.6. *Let S be a semigroup such that $l^1(S)$ is module biflat (as an $l^1(E)$-module with trivial left action). Then E is uniformly locally finite.*

The module case of [11], Theorem 3.6 can be formulated as follows. The proof is similar but we include its proof for the sake of completeness.

Proposition 3.7. *Let $S=\bigcup\limits_{\alpha\in\tau}S_\alpha$ be a band semigroup which is a strong semilattice of rectangular band semigroups S_α on a semilattice τ and let $l^1(S)$ be module biflat (as an $l^1(E)$-module with trivial left action). Then τ is a uniformly locally finite semilattice.*

Proof. Let $\mathcal{A}=l^1(S)$, $\mathcal{A}_\alpha=l^1(S_\alpha)$ and $\varphi_\alpha\colon\ \mathcal{A}_\alpha/J_\alpha\to\mathbb{C}$ be the augmentation character on $\mathcal{A}_\alpha/J_\alpha$, that is, $\varphi_\alpha\Big(\sum\limits_{s\in S_\alpha}\beta_s\delta_s+J_\alpha\Big)=\sum\limits_{s\in S_\alpha}\beta_s$ for each $\alpha\in\tau$. We claim that $\ker\varphi_\alpha=\overline{\ker\varphi_\alpha(\mathcal{A}_\alpha/J_\alpha)+(\mathcal{A}_\alpha/J_\alpha)\ker\varphi_\alpha}$ for each $\alpha\in Y$. To see this, let $s_\alpha,t_\alpha\in S_\alpha$. Since S_α is a rectangular band semigroup we have

$$\begin{aligned}(\delta_{s_\alpha}-\delta_{t_\alpha})+J_\alpha&=((\delta_{s_\alpha}+\delta_{t_\alpha})(\delta_{t_\alpha s_\alpha}-\delta_{t_\alpha})-(\delta_{t_\alpha s_\alpha}-\delta_{s_\alpha t_\alpha}))+J_\alpha\\&=((\delta_{s_\alpha}+\delta_{t_\alpha})+J_\alpha)((\delta_{t_\alpha s_\alpha}-\delta_{t_\alpha})+J_\alpha)-((\delta_{t_\alpha s_\alpha}-\delta_{s_\alpha t_\alpha})+J_\alpha),\end{aligned}$$

and

$$(\delta_{t_\alpha s_\alpha}-\delta_{s_\alpha t_\alpha})+J_\alpha=((\delta_{t_\alpha s_\alpha}-\delta_{s_\alpha})+J_\alpha)((\delta_{s_\alpha}+\delta_{t_\alpha})+J_\alpha)-((\delta_{t_\alpha}-\delta_{t_\alpha})+J_\alpha).$$

By the above relations we have

$$(\delta_{s_\alpha} - \delta_{t_\alpha}) + J_\alpha = \tfrac{1}{2}((\delta_{s_\alpha} + \delta_{t_\alpha}) + J_\alpha)((\delta_{t_\alpha s_\alpha} - \delta_{t_\alpha}) + J_\alpha) \\ - \tfrac{1}{2}((\delta_{t_\alpha s_\alpha} - \delta_{s_\alpha}) + J_\alpha)((\delta_{s_\alpha} + \delta_{t_\alpha}) + J_\alpha).$$

Since $((\delta_{t_\alpha s_\alpha} - \delta_{t_\alpha}) + J_\alpha), ((\delta_{t_\alpha s_\alpha} - \delta_{s_\alpha}) + J_\alpha) \in \ker \varphi_\alpha$, it follows that $(\delta_{s_\alpha} - \delta_{t_\alpha}) + J_\alpha \in \ker \varphi_\alpha(\mathcal{A}_\alpha/J_\alpha) + (\mathcal{A}_\alpha/J_\alpha) \ker \varphi_\alpha$. Since $\ker \varphi_\alpha$ is generated by $\{(\delta_{s_\alpha} - \delta_{t_\alpha}) + J_\alpha \colon s_\alpha, t_\alpha \in S_\alpha\}$, the claim is proved. Define $\varphi\colon \{\bigoplus \varphi_\alpha \colon \mathcal{A}/J \to l^1(\tau)/J\}$ by

$$\varphi(f + J) = \varphi\biggl(\sum_{\alpha\in\tau} (f_\alpha + J_\alpha)\biggr) = \sum_{\alpha\in\tau} \varphi_\alpha((f_\alpha + J_\alpha)\delta_\alpha)$$

for each $f = \sum\limits_{\alpha\in\tau} f_\alpha \in \mathcal{A}$. It is easy to check that φ is an epimorphism and $\ker \varphi = \overline{\ker \varphi(\mathcal{A}/J) + (\mathcal{A}/J) \ker \varphi}$. Thus the short sequence

$$0 \to \ker \varphi \xrightarrow{i} \mathcal{A}/J \xrightarrow{\varphi} l^1(\tau)/J \to 0$$

is exact. Now Proposition 2.5 yields that $l^1(\tau)$ is module biflat, and hence τ is a uniformly locally finite semilattice, by Proposition 3.6. $\square$

For an inverse semigroup S, there is a relation $\mathfrak{D}$ on S defined by $s\mathfrak{D}t$ if there exists $x \in S$ such that $s^*s = xx^*$ and $t^*t = x^*x$ (see [11]). Next, for a collection of Banach algebras $\{\mathcal{A}_\alpha \colon \alpha \in I\}$, we notice that $\bigoplus\limits_{\alpha\in I}^{1} \mathcal{A}_\alpha$, the l^1-direct sum of $\mathcal{A}_\alpha$'s, is module biflat (module biprojective) if and only if $\mathcal{A}_\alpha$ is module biflat (module biprojective) for each $\alpha \in I$. The idea of the following is taken from [11], Theorem 3.9.

Theorem 3.8. *Suppose that S is an inverse semigroup and consider $l^1(S)$ as an $l^1(E)$-module with trivial left action. Suppose that $\{\mathfrak{D}_\lambda \colon \lambda \in \Lambda\}$ is the collection of all $\mathfrak{D}$-classes of S where Λ is finite and that every $\mathfrak{D}$-class has finitely many idempotents. Then the following are equivalent:*

(i) *$l^1(S)^{**}$ is module biflat.*
(ii) *$l^1(S)$ is module biprojective.*
(iii) *$l^1(S)^{**}$ is module biprojective.*

P r o o f. We first take an idempotent $p_\lambda \in \mathfrak{D}_\lambda$ and let G_{p_λ} be the maximal subgroup of S at p_λ, for each $\lambda \in \Lambda$.

(i) $\Rightarrow$ (ii): Suppose that $l^1(S)^{**}$ is module biflat. By [8], Theorem 3.2 we conclude that $l^1(S)$ is module biflat, and so S is uniformly locally finite by virtue of Proposition 3.5. On the other hand, by [20], Theorem 2.18, we have

$$l^1(S) \cong \bigoplus_{\lambda\in\Lambda}^{1} \mathbb{M}_{E(\mathfrak{D}_\lambda)} l^1(G_{p_\lambda})$$

as Banach algebras, where $\mathbb{M}_{E(\mathfrak{D}_\lambda)}l^1(G_{p_\lambda})$ denotes the l^1-Munn algebra on $l^1(G_{p_\lambda})$. Since Λ is finite and every $\mathfrak{D}$-class has finitely many idempotents we have

$$l^1(S)^{**} \cong \bigoplus_{\lambda\in\Lambda}^{1} \mathbb{M}_{E(\mathfrak{D}_\lambda)}(l^1(G_{p_\lambda}))^{**} \cong \bigoplus_{\lambda\in\Lambda}^{1} \mathbb{M}_{E(\mathfrak{D}_\lambda)}(l^1(G_{p_\lambda})^{**}).$$

For each $\lambda \in \Lambda$, $\mathbb{M}_{E(D_\lambda)}(l^1(G_{p_\lambda})^{**})$ is module biflat. Using Proposition 2.6, we conclude that $l^1(G_{p_\lambda})^{**}$ is module biflat and by [5], Theorem 3.2, $l^1(G_{p_\lambda})^{**}$ is amenable, and by [11], Theorem 3.5, G_{p_λ} is finite for each $\lambda \in \Lambda$. The result now follows from [16], Corollary 3.5.

(ii) $\Rightarrow$ (iii): Let $l^1(S)$ be module biprojective. By [16], Corollary 3.5 every maximal subgroup of S is finite. Thus $l^1(G_{p_\lambda})^{**} = l^1(G_{p_\lambda})$ is module biprojective for each $\lambda \in \Lambda$. By using [20], Proposition 2.7, $\mathbb{M}_{E(\mathfrak{D}_\lambda)}(l^1(G_{p_\lambda})^{**})$ is module biprojective for each $\lambda \in \Lambda$. Now, it follows that $l^1(S)^{**}$ is module biprojective.

The implication (iii) $\Rightarrow$ (i) is clear. $\square$

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