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M.S.B. Elemine Vall; A. Ahmed; A. Touzani; Abdelmoujib Benkirane

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ENTROPY SOLUTIONS FOR PARABOLIC EQUATIONS IN MUSIELAK FRAMEWORK WITHOUT SIGN CONDITION AND WITH MEASURE DATA

M.S.B. Elemine Vall, A. Ahmed, A. Touzani, and A. Benkirane

Abstract. We prove an existence result of entropy solutions for a class of strongly nonlinear parabolic problems in Musielak-Sobolev spaces, without using the sign condition on the nonlinearities and with measure data.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ satisfying the segment property, $T > 0$ and set $Q = \Omega \times]0, T[$.

We deal with boundary value problems

$$(\mathcal{P}) \begin{cases} \dfrac{\partial b(x,u)}{\partial t} + A(u) + g(x,t,u,\nabla u) = f - \operatorname{div}(F) & \text{in} \quad Q \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times [0,T] \\ b(\cdot,u)(t=0) = b(\cdot,u_0) & \text{on} \quad \Omega\,, \end{cases}$$

where $b\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ a Carathédory function (see assumptions (6.1) and (6.2)), the term $A(u) = -\operatorname{div}(a(x,t,u,\nabla u))$ is an operator of Leray-Lions type which satisfies the classical Leray Lions assumptions of Musielak type (see assumptions (6.3)–(6.5)), g is a nonlinear order term satisfying the growth condition (see (6.6)) and the datum is assumed to be in $L^1(Q) + W^{-1,x}E_\psi(Q)$.

Under these assumptions, the above problem does not admit, in general, a weak solution since the field $a(x,t,u,\nabla u)$ does not belong to $(L^1_{\text{loc}}(Q))^N$ in general. To overcome this difficulty we use in this paper the framework of entropy solutions. This notion was introduced by P. Bénilan et al. [6] for the study of nonlinear elliptic problems.

The study of the nonlinear partial differential equations in this type of spaces is strongly motivated by numerous phenomena of physics, namely the problems related to non-Newtonian fluids of strongly inhomogeneous behavior with a high

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ability of increasing their viscosity under a different stimulus, like the shear rate, magnetic or electric field (see for examples [18], [19] and [20]).

In the setting of classical Sobolev spaces, $L^p(0,T,W^{1,p}(\Omega))$, L. Boccardo and T. Gallouët in [11] have proved the existence of solutions of $(\mathcal{P})$ where $b(x,u) \equiv u$ (see also [1], [2], [10]).

In the variable exponent case, in the elliptic case the authors in [4] have studied the same problem where the nonlinearity g satisfies the sign condition and $F \equiv 0$ and in [3] the authors have studied the problem $(\mathcal{P})$ where $b(x,u) = b(u)$ and $F \equiv 0$.

In the Orlicz spaces $W^1L_M(Q)$, D. Meskine in [24] proved the existence of solutions to $(\mathcal{P})$, where $b(x,u) \equiv u$ and $g \equiv 0$, in the inhomogeneous Orlicz Sobolev spaces $W_0^{1,x}L_A(Q)$ for any $A \in \mathrm{Q}_M$ where Q_M is a special class of Orlicz functions. See also [5], [28].

Recently, in the framework of Musielak spaces, Agnieszka, Swierczewska and Gwiazda in [30] studied the existence of weak solutions of problem $(\mathcal{P})$ in the case where $g \equiv 0$ and $f \in L^\infty(Q)$, M.S.B. Elemine Vall and all in [13] have proved the existence of entropy solutions of $(\mathcal{P})$ in the case where $b(x,u) = b(u)$, $g(x,t,s,\xi) = -\mathrm{div}(\Theta(x,t,u))$ where Θ a Carathéodory function does not satisfy any growth condition and $F \equiv 0$, also in [20] proved the existence of renormalized solutions of $(\mathcal{P})$ where $a = a(x,\xi)$ and $g \equiv 0$ with the right hand side $f \in L^1(Q)$.

Our novelty in the present paper is to give an existence result of entropy solutions of the problem $(\mathcal{P})$ in the setting of inhomogeneous Musielak- Orlicz-Sobolev spaces $W_0^{1,x}L_\varphi(Q)$ for which Δ_2-conditions are not imposed, losing the reflexivity of the spaces $L_\varphi(Q)$ and $W_0^1L\varphi(Q)$. The difficulty encountered during the proof of the existence of the solution is that the lower order term g does not check the sign condition and the fact that the second term is a bounded measure.

A large number of papers was devoted to the study the existence of solutions of elliptic and parabolic problems under various assumptions and in different contexts for a review on classical results see [9], [17], [18], [19], [21], [22], [26], [27].

This article is organized as follows. In the second section we are going to recall some important definitions and results of Musielak Orlicz Sobolev spaces. The third section contains some important lemmas useful to prove our main results. In the fourth section we introduce some new approximations results in inhomogeneous Musielak-Orlicz-Sobolev spaces, and trace results. The fifth section consecrate to the compactness results used in this paper. We introduce in the final section some assumptions on $b(x,s)$, $a(x,t,s,\xi)$ and $g(x,t,s,\xi)$ for which our problem has a solution, and will be state and proved our main results.

2. Preliminary

In this section we give some well-known preliminaries properties and results of the framework of Musielak-Orlicz-Sobolev spaces.

2.1. **Musielak-Orlicz-Sobolev spaces.** Let Ω be an open set in $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$, and satisfiying the following conditions:

a) $\varphi(x,\cdot)$ is an N-function (convex, increasing, continous, $\varphi(x,0)=0$, $\varphi(x,t)>0$, $\forall t>0$, $\lim\limits_{t\longrightarrow 0}\sup\limits_{x\in\Omega}\dfrac{\varphi(x,t)}{t}=0$, $\lim\limits_{t\longrightarrow\infty}\inf\limits_{x\in\Omega}\dfrac{\varphi(x,t)}{t}=\infty$).

b) $\varphi(\cdot,t)$ is a measurable function.

A function φ, which satisfies the conditions **a)** and **b)** is called Musielak-Orlicz function.

For a Musielak-Orlicz function φ we put $\varphi_x(t)=\varphi(x,t)$ and we associate its nonnegative reciprocal function φ_x^{-1}, with respect to t that is

$$\varphi_x^{-1}(\varphi(x,t))=\varphi(x,\varphi_x^{-1}(t))=t\,.$$

The Musielak-Orlicz function φ is said to satisfy the Δ_2-condition if for some $k>0$; and a non negative function h; integrable in Ω we have

$$\varphi(x,2t)\le k\varphi(x,t)+h(x)\quad\text{for all}\quad x\in\Omega\quad\text{and}\quad t\ge 0\,. \tag{2.1}$$

When (2.1) holds only for $t\ge t_0>0$; then φ said satisfies Δ_2 near infinity.

Let φ and γ be two Musielak-Orlicz functions, we say that φ dominate γ, and we write $\gamma\prec\varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x\in\Omega$

$$\gamma(x,t)\le\varphi(x,ct)\quad\text{for all}\quad t\ge t_0\,,\quad(\text{resp. for all } t\ge 0\ \text{ i.e. }\ t_0=0)\,.$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity), and we write $\gamma\prec\prec\varphi$, If for every positive constant c we have

$$\lim_{t\longrightarrow 0}\Big(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\Big)=0\,,\quad\Big(\text{resp. }\lim_{t\longrightarrow\infty}\Big(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\Big)=0\Big)\,.$$

Remark 2.1 ([8])**.** If $\gamma\prec\prec\varphi$ near infinity, then $\forall\varepsilon>0$ there exist $k(\varepsilon)>0$ such that for almost all $x\in\Omega$ we have

$$\gamma(x,t)\le k(\varepsilon)\varphi(x,\varepsilon t)\,,\quad\text{for all}\quad t\ge 0\,. \tag{2.2}$$

We define the functional

$$\rho_{\varphi,\Omega}(u)=\int_\Omega\varphi(x,|u(x)|)\,dx\,,$$

where $u\colon\Omega\longrightarrow\mathbb{R}$ a Lebesgue measurable function. In the following the measurability of a function $u\colon\Omega\longrightarrow\mathbb{R}$ means the Lebesgue measurability.

The set

$$K_\varphi(\Omega)=\big\{u\colon\Omega\longrightarrow\mathbb{R}\text{ measurable }:\ \rho_{\varphi,\Omega}(u)<+\infty\big\}$$

is called the generalized Orlicz class.

The Musielak-Orlicz space (or the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega)=\Big\{u:\Omega\longrightarrow\mathbb{R}\ \text{ measurable }:\ \rho_{\varphi,\Omega}\Big(\frac{|u(x)|}{\lambda}\Big)<+\infty,,\ \text{for some }\lambda>0\Big\}\,.$$

Let

$$\psi(x,s)=\sup_{t\ge 0}\{st-\varphi(x,t)\}$$

that is, ψ is the Musielak-Orlicz function complementary to φ in the sense of Young with respect to the variable s.
We define in the space $L_\varphi(\Omega)$ the following two norms:

$$\|u\|_{\varphi,\Omega} = \inf\Big\{\lambda > 0/ \int_\Omega \varphi\Big(x, \frac{|u(x)|}{\lambda}\Big)\,dx \le 1\Big\}$$

which is called the Luxemburg norm and the so called Orlicz norm by:

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \le 1} \int_\Omega |u(x)v(x)|\,dx$$

where ψ is the Musielak Orlicz function complementary to φ. There two norms are equivalent [25].
The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space.
We say that sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0\,.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \big\{u \in L_\varphi(\Omega) : \forall|\alpha| \le m,\ D^\alpha u \in L_\varphi(\Omega)\big\}$$

and

$$W^m E_\varphi(\Omega) = \big\{u \in E_\varphi(\Omega) : \forall|\alpha| \le m,\ D^\alpha u \in E_\varphi(\Omega)\big\}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak Orlicz Sobolev space.
Let

$$\overline{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\le m} \rho_{\varphi,\Omega}(D^\alpha u) \quad \text{and} \quad \|u\|^m_{\varphi,\Omega} = \inf\Big\{\lambda > 0 : \overline{\rho}_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) \le 1\Big\}.$$

For $u \in W^m L_\varphi(\Omega)$, these functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$ respectively, and the pair $\big(W^m L_\varphi(\Omega), \|u\|^m_{\varphi,\Omega}\big)$ is a Banach space if φ satisfies the following condition [25]:

$$\text{there exist a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x, 1) \ge c\,. \tag{2.3}$$

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\le m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed.
We denote by $\mathcal{D}(\Omega)$ the space of infinitely smooth functions with compact support in Ω and by $\mathcal{D}(\overline{\Omega}))$ the restriction of $\mathcal{D}(\mathbb{R}^N)$ on Ω.
Let $W^m_0 L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.
Let $W^m E_\varphi(\Omega)$ the space of functions u such that u and its distribution derivatives up to order m lie in $E_\varphi(\Omega)$, and $W^m_0 E_\varphi(\Omega)$ is the (norm) closure of $\mathcal{D}(\Omega)$ in

$W^m L_\varphi(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-m}L_\psi(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega);\ f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega) \Big\}$$

and

$$W^{-m}E_\psi(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega);\ f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega) \Big\}.$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \overline{\rho}_{\varphi,\Omega}\Big(\frac{u_n - u}{k}\Big) = 0\,.$$

For φ and her complementary function ψ the following inequality is called the Young inequality [25]:

$$ts \leq \varphi(x,t) + \psi(x,s)\,, \quad \forall t,\ s \geq 0,\ x \in \Omega\,. \tag{2.4}$$

This inequality implies that

$$|||u|||_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) + 1\,. \tag{2.5}$$

In $L_\varphi(\Omega)$ we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) \quad \text{if} \quad \|u\|_{\varphi,\Omega} > 1\,. \tag{2.6}$$

$$\|u\|_{\varphi,\Omega} \geq \rho_{\varphi,\Omega}(u) \quad \text{if} \quad \|u\|_{\varphi,\Omega} \leq 1\,. \tag{2.7}$$

For two complementary Musielak Orlicz functions φ and ψ, let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$ then we have the following Hölder inequality [25]

$$\Big| \int_\Omega u(x)v(x)dx \Big| \leq \|u\|_{\varphi,\Omega} |||v|||_{\psi,\Omega}\,. \tag{2.8}$$

2.2. **Inhomogeneous Musielak-Orlicz-Sobolev spaces.** Let Ω be a bounded open subset of $\mathbb{R}^N$, $T > 0$ and set $Q = \Omega \times [0, T]$. Let $m \geq 1$ be an integer and let $\varphi$and ψ be two complementary Musielak Orlicz function. For each $\alpha \in \mathbb{N}^N$, denote by D^α_x the distributional derivative on Q of order α with respect to $x \in \mathbb{R}^N$. The inhomogeneous Musielak-Orlicz-Sobolev spaces are defined as follows

$$W^{m,x}L_\varphi(Q) = \big\{ u \in L_\varphi(Q) \ :\ D^\alpha_x u \in L_\varphi(Q), \forall |\alpha| \leq m \big\}\,,$$

and

$$W^{m,x}E_\varphi(Q) = \big\{ u \in E_\varphi(Q) \ :\ D^\alpha_x u \in E_\varphi(Q), \forall |\alpha| \leq m \big\}\,.$$

This second space is a subspace of the first one, and both are Banach spaces with the norm

$$\|u\|_{m,x} = \sum_{|\alpha| \leq m} \|D^\alpha_x u\|_{\varphi,Q}\,.$$

These spaces constitute a complementary system since Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_\varphi(Q)$,

which have as many copies as there is α order derivatives, $|\alpha| \leq m$. We shall also consider the weak topologies $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and $\sigma(\Pi L_\varphi, \Pi L_\psi)$.
If $u \in W^{m,x}L_\varphi(Q)$ then the function $t \longrightarrow u(t) = u(\cdot, t)$ is define on $[0,T]$ with values in $W^m L_\varphi(\Omega)$. If $u \in W^{m,x}E_\varphi(Q)$ the concerned function is a $W^m E_\varphi(\Omega)$-valued and is strongly measurable.

Furthermore, the embedding $W^{m,x}E_\varphi(Q) \subset L^1(0,T,W^m E_\varphi(\Omega))$ holds. The space $W^{m,x}L_\varphi(Q)$ is not in general separable, for $u \in W^{m,x}L_\varphi(Q)$, we cannot conclude that the function $u(t)$ is measurable on $[0,T]$.
However, the scalar function $t \longrightarrow \|u(t)\|_{\varphi,\Omega} \in L^1(0,T)$. the space $W_0^{m,x}E_\varphi(Q)$ is defined as the norm closure of $\mathcal{D}(Q)$ in $W^{m,x}E_\varphi(Q)$. We can easily show as in [16] that when Ω has the segment property then each element u of the closure of $\mathcal{D}(Q)$ with respect to the weak * topology $\sigma(\Pi L_\varphi, \Pi E_\psi)$ is limit in $W^{m,x}L_\varphi(Q)$ of some subsequence $(v_j) \in \mathcal{D}(Q)$ for the modular convergence .i.e there exist $\lambda > 0$ such that for all $|\alpha| \leq m$

$$\int_Q \varphi\Big(x, \frac{D_x^\alpha v_j - D_x^\alpha u}{\lambda}\Big)\, dx\, dt \longrightarrow 0 \quad \text{as} \quad j \longrightarrow +\infty\,,$$

which gives that (v_j) converges to u in $W^{m,x}L_\varphi(Q)$ for the weak topology $\sigma(\Pi L_\varphi, \Pi L_\psi)$. Consequently

$$\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi, \Pi E_\psi)} = \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi, \Pi L_\psi)}\,.$$

The space of functions satisfying such property will be denoted by $W_0^{m,x}L_\varphi(Q)$. Furthermore $W_0^{m,x}E_\varphi(Q) = W_0^{m,x}L_\varphi(Q) \cap \Pi E_\varphi(Q)$.
Thus both sides of the last inequality are equivalent norms on $W_0^{m,x}L_\varphi(Q)$. We then have the following complementary system

$$\begin{pmatrix} W_0^{m,x}L_\varphi(Q) & F \\ W_0^{m,x}E_\varphi(Q) & F_0 \end{pmatrix}\,.$$

F states for the dual space of $W_0^{m,x}E_\varphi(Q)$ and can be defined, except for an isomorphism, as the quotient of ΠL_ψ by the polar set $W_0^{m,x}E_\varphi(Q)^\perp$. It will be denoted by $F = W_0^{-m,x}L_\psi(Q)$ with

$$W^{-m,x}L_\psi(Q) = \Big\{ f = \sum_{|\alpha| \leq m} D_x^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(Q) \Big\}\,.$$

This space will be equipped with the usual quotient norm

$$\|u\|_F = \inf \sum_{|\alpha| \leq m} \|f_\alpha\|_{\psi,Q}$$

where the infimum is taken over all possible decompositions

$$f = \sum_{|\alpha| \leq m} D_x^\alpha f_\alpha \quad f_\alpha \in L_\psi(Q)\,.$$

The space F_0 is then given by

$$F_0 = \Big\{ f = \sum_{|\alpha| \leq m} D_x^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(Q) \Big\}$$

and is denoted by $W^{-m,x}E_\psi(Q)$.

3. Some technical lemmas

We list here some technical lemmas which will be used in the proof of our main result. We start by the following approximation result.

Lemma 3.1 ([7]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

i) *There exists a constant $c > 0$ such that $\inf_{x\in\Omega} \varphi(x, 1) \geq c$.*

ii) *There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$ we have*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\frac{A}{\log\left(\frac{1}{|x-y|}\right)}}, \quad \forall t \geq 1\,. \tag{3.1}$$

iii)

$$\text{If} \quad D \subset \Omega \text{ is a bounded measurable set, then } \int_D \varphi(x,1)\,dx < \infty\,. \tag{3.2}$$

iv) *There exists a constant $C > 0$ such that $\psi(x, 1) \leq C$ a.e. in Ω.*

Under this assumptions, $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W^1L_\varphi(\Omega)$ the modular convergence.

Consequently, the action of a distribution S in $W^{-1}L_\psi(\Omega)$ on an element u of $W_0^1L_\varphi(\Omega)$ is well defined. It will be denoted by $< S, u >$.
Truncation Operator. For $k > 0$ we define the truncation at height k: $T_k : \mathbb{R} \longrightarrow \mathbb{R}$ by:

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k\,, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k\,. \end{cases} \tag{3.3}$$

Lemma 3.2 ([21]). *Let (f_n), $f \in L^1(\Omega)$ such that*

i) $f_n \geq 0$ *a.e. in* Ω.

ii) $f_n \longrightarrow f$ *a.e. in* Ω.

iii) $\int_\Omega f_n(x)\,dx \longrightarrow \int_\Omega f(x)\,dx$

then $f_n \longrightarrow f$ strongly in $L^1(\Omega)$.

Now, we give the modular Poincaré's inequality in Musielak-Orlicz spaces in the following lemma.

Lemma 3.3 ([14]). *Under the assumptions of Lemma 3.1, and by assuming that $\varphi(x,t)$ decreases with respect to one of coordinates of x, there exists a constant $c>0$ which depends only on Ω such that*

$$\int_\Omega \varphi(x,|u(x)|)\,dx \le \int_\Omega \varphi(x,c|\nabla u(x)|)\,dx \quad \forall u\in W_0^1L_\varphi(\Omega)\,. \tag{3.4}$$

Proof. Since $\varphi(x,t)$ decreases with respect to one of coordinates of x, there exists $i_0\in\{1,\dots,N\}$ such that the function $\sigma\longrightarrow\varphi(x_1,\dots,x_{i_0-1},\sigma,x_{i_0+1},\dots,x_N,t)$ is decreasing for every $x_1,\dots,x_{i_0-1},x_{i_0+1},\dots,x_N\in\mathbb{R}$ and $\forall t>0$.
To prove our result, it suffices to show that

$$\int_\Omega \varphi(x,|u(x)|)\,dx \le \int_\Omega \varphi\Big(x,2d\Big|\frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)\,dx\,,\quad \forall u\in W_0^1L_\varphi(\Omega) \tag{3.5}$$

with $d=\max(\mathrm{diam}(\Omega),1)$ and $\mathrm{diam}(\Omega)$ is the diameter of Ω.
First, suppose that $u\in\mathcal{D}(\Omega)$, then

$$\begin{aligned}
&\varphi(x,|u(x_1,\dots,x_N)|)\\
&\quad\le \varphi\Big(x,\int_{-\infty}^{x_{i_0}}\Big|\frac{\partial u}{\partial x_{i_0}}\Big|(x_1,\dots,x_{i_0-1},\sigma,x_{i_0+1},\dots,x_N)d\sigma\Big)\\
&\quad\le \frac{1}{d}\int_{-\infty}^{+\infty}\varphi\Big(x,d\Big|\frac{\partial u}{\partial x_{i_0}}\Big|(x_1,\dots,x_{i_0-1},\sigma,x_{i_0+1},\dots,x_N)\Big)d\sigma\\
&\quad\le \frac{1}{d}\int_{-\infty}^{+\infty}\varphi\Big(x_1,\dots,x_{i_0-1},\sigma,x_{i_0+1},\dots,x_N,d\Big|\frac{\partial u}{\partial x_{i_0}}\Big|(x_1,\dots,x_{i_0-1},\sigma,x_{i_0+1},\dots,x_N)\Big)d\sigma\,.
\end{aligned}$$

By integrating with respect to x, we get

$$\begin{aligned}
&\int_\Omega \varphi(x,|u(x_1,\dots,x_N)|)\,dx\\
&\qquad\le \int_\Omega\frac{1}{d}\int_{-\infty}^{+\infty}\varphi\Big(x_1,\dots,x_{i_0-1},\sigma,x_{i_0+1},\dots,x_N,\\
&\qquad\qquad d\Big|\frac{\partial u}{\partial x_{i_0}}\Big|(x_1,\dots,x_{i_0-1},\sigma,x_{i_0+1},\dots,x_N)\Big)d\sigma\,dx\,,
\end{aligned}$$

since $\varphi\big(x_1,\dots,x_{i_0-1},\sigma,x_{i_0+1},\dots,x_N,d\big|\frac{\partial u}{\partial x_{i_0}}\big|(x_1,\dots,x_{i_0-1},\sigma,x_{i_0+1},\dots,x_N)\big)$ independent of x_{i_0}, we can get it out of the integral to respect of x_{i_0} and by the fact that σ is arbitrary, then by Fubini's Theorem we get

$$\int_\Omega \varphi(x,|u(x)|)\,dx \le \int_\Omega \varphi\Big(x,d\Big|\frac{\partial u}{\partial x_{i_0}}\Big|(x)\Big)\,dx\,,\quad \forall\ u\in\mathcal{D}(\Omega)\,. \tag{3.6}$$

For $u\in W_0^1L_\varphi(\Omega)$ according to Lemma 3.1, we have the existence of $u_n\in\mathcal{D}(\Omega)$ and $\lambda>0$ such that

$$\overline{\varrho}_{\varphi,\Omega}\Big(\frac{u_n-u}{\lambda}\Big)=0\,,\quad \text{as}\quad n\longrightarrow+\infty\,,$$

hence

$$\begin{cases} \displaystyle\int_\Omega \varphi\Big(x, \frac{|u_n - u|}{\lambda}\Big) dx \longrightarrow 0\,, & \text{as} \quad n \longrightarrow +\infty\,, \\ \displaystyle\int_\Omega \varphi\Big(x, \frac{|\nabla u_n - \nabla u|}{\lambda}\Big) dx \longrightarrow 0\,, & \text{as} \quad n \longrightarrow +\infty\,, \\ u_n \longrightarrow u & \text{a.e. in } \Omega\,,\ \text{(for a subsequence still denote } u_n)\,. \end{cases}$$

Then, we have

$$\begin{aligned} \int_\Omega \varphi\Big(x, \frac{|u(x)|}{2d\lambda}\Big)\, dx &\le \liminf_{n\longrightarrow+\infty} \int_\Omega \varphi\Big(x, \frac{|u_n(x)|}{2d\lambda}\Big)\, dx \\ &\le \liminf_{n\longrightarrow+\infty} \int_\Omega \varphi\Big(x, \frac{1}{2\lambda}\Big|\frac{\partial u_n}{\partial x_{i_0}}(x)\Big|\Big)\, dx \\ &= \liminf_{n\longrightarrow+\infty} \int_\Omega \varphi\Big(x, \frac{1}{2\lambda}\Big|\frac{\partial u_n}{\partial x_{i_0}}(x) - \frac{\partial u}{\partial x_{i_0}}(x) + \frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)\, dx \\ &\le \frac{1}{2} \liminf_{n\longrightarrow+\infty} \int_\Omega \varphi\Big(x, \frac{1}{\lambda}\Big|\frac{\partial u_n}{\partial x_{i_0}}(x) - \frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)\, dx \\ &\quad + \frac{1}{2} \int_\Omega \varphi\Big(x, \frac{1}{\lambda}\Big|\frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)\, dx \\ &\le \int_\Omega \varphi\Big(x, \frac{1}{\lambda}\Big|\frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)\, dx\,. \end{aligned}$$

Hence

$$\int_\Omega \varphi(x, |u(x)|)\, dx \le \int_\Omega \varphi\Big(x, 2d\Big|\frac{\partial u}{\partial x_{i_0}}(x)\Big|\Big)\, dx\,, \quad \forall\ u \in W_0^1 L_\varphi(\Omega)\,.$$

□

Lemma 3.4 (The Nemytskii Operator [21]). *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak Orlicz functions. Let $f\colon \Omega\times\mathbb{R}^p \longrightarrow \mathbb{R}^q$ be a Carathodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^p$:*

$$|f(x,s)| \le c(x) + k_1 \psi_x^{-1} \varphi(x, k_2|s|)\,, \tag{3.7}$$

where k_1 and k_2 are real positives constants and $c(\cdot) \in E_\psi(\Omega)$.
Then the Nemytskii Operator N_f defined by $N_f(u)(x) = f(x,u(x))$ is continuous from

$$\Big(\mathcal{P}(E_\varphi(\Omega), \frac{1}{k_2}\Big)^p = \prod \Big\{u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < \frac{1}{k_2}\Big\}$$

into $(L_\psi(\Omega))^q$ for the modular convergence.
Furthermore if $c(\cdot) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$ then N_f is strongly continuous from $\big(\mathcal{P}(E_\varphi(\Omega), \frac{1}{k_2}\big)^p$ to $(E_\gamma(\Omega))^q$.

Lemma 3.5. *Assume that (6.3)–(6.5) are satisfies and let $(z_n)_n$ be a sequence in $W_0^1 L_\varphi(\Omega)$ such that*

i) *$z_n \rightharpoonup z$ in $W_0^1 L_\varphi(\Omega)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$.*

ii) *$(a(\cdot, t, z_n, \nabla z_n))_n$ is bounded in $(L_\psi(\Omega))^N$.*

iii) $\int_\Omega \Big(a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)\Big)(\nabla z_n - \nabla z\chi_s)dx \longrightarrow 0$ *as* $n,s \longrightarrow \infty$*. where* χ_s *is the characteristic function of* $\Omega_s = \{x \in \Omega : |\nabla z| \le s\}$.

Then, we have

$$z_n \longrightarrow z \quad \textit{for the modular convergence in} \quad W_0^1 L_\varphi(\Omega)\,.$$

Proof. Let $s > 0$ and $\Omega_s = \{x \in \Omega : |\nabla z| \le s\}$ and denote by χ_s the characteristic function of Ω_s.
Fix $r > 0$ and let $s > r$, we have

$$\begin{aligned}
0 &\le \int_{\Omega_r} \big(a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z)\big)(\nabla z_n - \nabla z)\,dx \\
&\le \int_{\Omega_s} \big(a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z)\big)(\nabla z_n - \nabla z)\,dx \\
&= \int_{\Omega_s} \big(a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)\big)(\nabla z_n - \nabla z\chi_s)\,dx \\
&\le \int_{\Omega} \big(a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)\big)(\nabla z_n - \nabla z\chi_s)\,dx\,.
\end{aligned}$$

By iii), we obtain

$$\lim_{n\longrightarrow\infty} \int_{\Omega_r} \big(a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z)\big)(\nabla z_n - \nabla z)\,dx = 0\,.$$

So as in [16], we have

$$\nabla z_n \longrightarrow \nabla z \quad \text{a.e. in } \ \Omega\,. \tag{3.8}$$

On the other hand, we have

$$\begin{aligned}
\int_\Omega a(x,t,z_n,\nabla z_n)\nabla z_n\,dx = &\int_\Omega \big(a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)\big)(\nabla z_n - \nabla z\chi_s)\,dx \\
&+ \int_\Omega a(x,t,z_n,\nabla z\chi_s)(\nabla z_n - \nabla z\chi_s)\,dx \\
&+ \int_\Omega a(x,t,z_n,\nabla z_n)\nabla z\chi_s\,dx\,. \qquad (3.9)
\end{aligned}$$

Since $(a(\cdot,t,z_n,\nabla z_n))_n$ is bounded in $(L_\psi(\Omega))^N$ and using the almost every where convergence of the gradients we obtain

$$a(x,t,z_n,\nabla z_n) \rightharpoonup a(x,t,z,\nabla z) \ \text{ weakly in } \ \big(L_\psi(\Omega)\big)^N \ \text{ for } \ \sigma(\Pi L_\psi, \Pi E_\varphi)\,,$$

which implies that

$$\int_\Omega a(x,t,z_n,\nabla z_n)\nabla z\chi_s\,dx \longrightarrow \int_\Omega a(x,t,z,\nabla z)\nabla z\chi_s\,dx\,. \tag{3.10}$$

Letting $s \longrightarrow \infty$, we obtain

$$\int_\Omega a(x,t,z,\nabla z)\nabla z\chi_s dx \longrightarrow \int_\Omega a(x,t,z,\nabla z)\nabla z dx. \tag{3.11}$$

On the other hand, it is easy to see that second term of the right hand side of (3.9) tends to 0, as $n \longrightarrow \infty$, consequently, from iii), (3.10) and (3.11), we have

$$\int_\Omega a(x,t,z_n,\nabla z_n)\nabla z_n\,dx \longrightarrow \int_\Omega a(x,t,z,\nabla z)\nabla z\,dx\,. \tag{3.12}$$

Using (6.5) and the convexity of φ, we have

$$\alpha\varphi\Big(x,\frac{|\nabla z_n-\nabla z|}{2}\Big)\le \frac{1}{2}a(x,z_n,\nabla z_n)\cdot\nabla z_n+\frac{1}{2}a(x,z,\nabla z)\cdot\nabla z\,.$$

Then by (3.12) we get

$$\lim_{\mathrm{meas}(E)\to 0}\sup_{n\in\mathbb{N}}\int_E\varphi\Big(x,\frac{|\nabla z_n-\nabla z|}{2}\Big)\,dx=0\,.$$

Then by using Vitali's theorem one has

$$z_n \longrightarrow z \quad \text{for the modular convergence in} \quad W_0^1L_\varphi(\Omega)\,.$$

□

4. Approximation and trace results

In this section, Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ with the segment property and I is a subinterval of $\mathbb{R}$ (both possibly unbounded) and $Q=\Omega\times I$. It is easy to see that Q also satisfies Lipschitz domain.

Definition 4.1. We say that $u_n \longrightarrow u$ in $W^{-1,x}L_\psi(Q)+L^1(Q)$ for the modular convergence if we can write

$$u_n=\sum_{|\alpha|\le 1}D_x^\alpha u_n^\alpha+u_n^0 \quad\text{and}\quad u=\sum_{|\alpha|\le 1}D_x^\alpha u^\alpha+u^0\,,$$

with $u_n^\alpha \longrightarrow u^\alpha$ in $L_\psi(Q)$ for the modular convergence for all $|\alpha|\le 1$, and $u_n^0 \longrightarrow u^0$ strongly in $L^1(Q)$.

We shall prove the following approximation theorem, which plays a fundamental role when the existence of solutions for parabolic problems is proved.

Theorem 4.1 ([27]). *Let φ be an Musielak-Orlicz function satisfies the assumption* (3.1).
If $u\in W^{1,x}L_\varphi(Q)$ (respectively $u\in W_0^{1,x}L_\varphi(Q)$) and $\frac{\partial u}{\partial t}\in W^{-1,x}L_\psi(Q)+L^1(Q)$, then there exists a sequence $(v_j)\in\mathcal{D}(\overline{Q})$ (respectively $\mathcal{D}(\overline{I},\mathcal{D}(\Omega))$) such that $v_j\longrightarrow u$ in $W^{1,x}L_\varphi(Q)$ and $\frac{\partial v_j}{\partial t}\longrightarrow\frac{\partial u}{\partial t}$ in $W^{-1,x}L_\psi(Q)+L^1(Q)$ for the modular convergence.

Lemma 4.1 ([27]). *Let $a<b\in\mathbb{R}$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then*

$$\Big\{u\in W_0^{1,x}L_\varphi(\Omega\times]a,b[)\ :\ \frac{\partial u}{\partial t}\in W^{-1,x}L_\psi(\Omega\times]a,b[)+L^1(\Omega\times]a,b[)\Big\}$$

is a subset of $\mathcal{C}(]a,b[,L^1(\Omega))$.

In order to deal with the time derivative, we introduce a time mollification of a function $u \in W_0^{1,x} L_\varphi(Q)$.
Thus we define, for all $\mu > 0$ and all $(x,t) \in Q$

$$u_\mu(x,t) = \int_{-\infty}^{t} \tilde{u}(x,\sigma) \exp(\mu(\sigma - t)) d\sigma \tag{4.1}$$

where $\tilde{u}(x,t) = u(x,t)\chi_{[0,T]}(t)$.
Throughout the paper the index ì always indicates this mollification.

Lemma 4.2 ([27]). *If $u \in L_\varphi(Q)$ then u_μ is measurable in Q and $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ and if $u \in K_\varphi(Q)$ then*

$$\int_Q \varphi(x, u_\mu)\, dx\, dt \le \int_Q \varphi(x, u)\, dx\, dt\,.$$

Lemma 4.3.

(1) *If $u \in L_\varphi(Q)$ then $u_\mu \longrightarrow u$ for the modular convergence in $L_\varphi(Q)$ as $\mu \longrightarrow \infty$.*
(2) *If $u \in W_0^{1,x} L_\varphi(Q)$ then $u_\mu \longrightarrow u$ for the modular convergence in $W_0^{1,x} L_\varphi(Q)$ as $\mu \longrightarrow \infty$.*

Proof.

(1) Let $(v_k)_k \subset \mathcal{D}(Q)$ such that $v_k \longrightarrow u$ in $L_\varphi(Q)$ for the modular convergence. Let $\lambda > 0$ large enough such that

$$\frac{u}{\lambda} \in K_\varphi(Q)\,, \quad \int_Q \varphi\Big(x, \frac{v_k - u}{\lambda}\Big)\, dx\, dt \longrightarrow 0 \quad \text{as} \quad k \longrightarrow +\infty\,.$$

On the one hand, for a.e. $(x,t) \in Q$, we have

$$\big|(v_k)_\mu(x,t) - v_k(x,t)\big| = \frac{1}{\mu}\Big|\frac{\partial v_k}{\partial t}(x,t)\Big| \le \Big\|\frac{\partial v_k}{\partial t}\Big\|_{L^\infty(Q)}\,.$$

On the other hand, one has

$$\begin{aligned}\int_Q \varphi\Big(x, \frac{u_\mu - u}{3\lambda}\Big)\, dx\, dt &\le \frac{1}{3}\int_Q \varphi\Big(x, \frac{u_\mu - (v_k)_\mu}{\lambda}\Big)\, dx\, dt \\ &\quad + \frac{1}{3}\int_Q \varphi\Big(x, \frac{(v_k)_\mu - v_k}{\lambda}\Big)\, dx\, dt + \frac{1}{3}\int_Q \varphi\Big(x, \frac{v_k - u}{\lambda}\Big)\, dx\, dt \\ &\le \frac{1}{3}\int_Q \varphi\Big(x, \frac{(u - v_k)_\mu}{\lambda}\Big)\, dx\, dt \\ &\quad + \frac{1}{3}\int_Q \varphi\Big(x, \frac{(v_k)_\mu - v_k}{\lambda}\Big)\, dx\, dt + \frac{1}{3}\int_Q \varphi\Big(x, \frac{v_k - u}{\lambda}\Big)\, dx\, dt\,.\end{aligned}$$

This implies that

$$\int_Q \varphi\Big(x, \frac{u_\mu - u}{3\lambda}\Big)\, dx\, dt \le \frac{2}{3}\int_Q \varphi\Big(x, \frac{v_k - u}{\lambda}\Big)\, dx\, dt + \int_Q \varphi\Big(x, \frac{1}{\lambda\mu}\Big\|\frac{\partial v_k}{\partial t}\Big\|_{L^\infty(Q)}\Big)\, dx\, dt\,.$$

Let $\varepsilon > 0$ there exists $k_0 > 0$ such that $\forall k > k_0$, we have

$$\int_Q \varphi\Big(x, \frac{v_k - u}{\lambda}\Big)\, dx\, dt < \varepsilon$$

and there exists $\mu_0 > 0$ such that $\forall \mu > \mu_0$ and for all $k > k_0$

$$\frac{1}{\lambda\mu}\Big\|\frac{\partial v_k}{\partial t}\Big\|_{L^\infty(Q)} \le 1\,.$$

Then, we get

$$\int_Q \varphi\Big(x, \frac{u_\mu - u}{3\lambda}\Big)\, dx\, dt \le \varepsilon + \frac{1}{\lambda\mu}\Big\|\frac{\partial v_k}{\partial t}\Big\|_{L^\infty(Q)} T \int_\Omega \varphi(x,1)\, dx\, dt\,.$$

Finaly, by using (iii) of Lemma 3.1 and by letting $\mu \longrightarrow +\infty$, there exits $\mu_1 > 0$ such that

$$\int_Q \varphi\Big(x, \frac{u_\mu - u}{3\lambda}\Big)\, dx\, dt \le \varepsilon, \quad \text{for all} \quad \mu > \mu_1\,.$$

(2) Since for all indice α such that $|\alpha| \le 1$, we have $D_x^\alpha(u_\mu) = (D_x^\alpha u)_\mu$, consequently, the first part above applied on each $D_x^\alpha u$, gives the result.

□

Remark 4.1. If $u \in E_\varphi(Q)$, we can choose λ arbitrary small since $\mathcal{D}(Q)$ is (norm) dense in $E_\varphi(Q)$.
Thus, for all $\lambda > 0$, we have

$$\int_Q \varphi\Big(x, \frac{u_\mu - u}{\lambda}\Big)\, dx\, dt \longrightarrow 0 \quad \text{as} \quad \mu \longrightarrow +\infty\,.$$

and $u_\mu \longrightarrow u$ strongly in $E_\varphi(Q)$. Idem for $W^{1,x}E_\varphi(Q)$.

Lemma 4.4. *If $u_n \longrightarrow u$ in $W_0^{1,x}L_\varphi(Q)$ strongly (resp., for the modular convergence), then $(u_n)_\mu \longrightarrow u_\mu$ strongly (resp., for the modular convergence).*

Proof. For all $\lambda > 0$ (resp., for some $\lambda > 0$),

$$\int_Q \varphi\Big(x, \frac{D_x^\alpha((u_n))_\mu - D_x^\alpha(u)_\mu}{\lambda}\Big)\, dx\, dt \longrightarrow \int_Q \varphi\Big(x, \frac{D_x^\alpha u_n - D_x^\alpha u}{\lambda}\Big)\, dx\, dt \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty$$

then $(u_n)_\mu \longrightarrow u_\mu$ in $W^{1,x}L_\varphi(Q)$ strongly (resp., for the modular convergence). □

5. Compactness results

For each $h > 0$, define the usual translated $\tau_h f$ of the function f by $\tau_h f(t) = f(t+h)$.
If f is defined on $[0,T]$ then $\tau_h f$ is defined on $[-h, T-h]$.
First of all, recall the following compactness results proved by the authors in [27].

Lemma 5.1. *Let φ be a Musielak function and ψ the complementary function of φ, we assume that there exists $c > 0$ such that $\psi(x,1) \leq c$ a.e. in Ω.*

Let Y be a Banach space such that the following continuous embedding holds $L^1(\Omega) \subset Y$. Then for all $\varepsilon > 0$ and all $\lambda > 0$, there is $C_\varepsilon > 0$ such that for all $u \in W_0^{1,x}L_\varphi(Q)$ with $\dfrac{|\nabla u|}{\lambda} \in K_\varphi(Q)$, we have

$$\|u\|_1 \leq \varepsilon\lambda\Big(\int_Q \varphi\Big(x, \frac{|\nabla u|}{\lambda}\Big)\,dx\,dt + T\Big) + C_\varepsilon \|u\|_{L^1(0,T,Y)}\,.$$

Proof. Since $W_0^1L_\varphi(\Omega) \subset L^1(\Omega)$ with compact embedding, then for all $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that for all $v \in W_0^1L_\varphi(\Omega)$

$$\|v\|_{L^1(\Omega)} \leq \varepsilon\|\nabla u\|_{L_\varphi(\Omega)} + C_\varepsilon\|v\|_Y\,. \tag{5.1}$$

Indeed, if the above assertion holds false, there is $\varepsilon_0 > 0$ and $v_n \in W_0^1L_\varphi(\Omega)$ such that

$$\|v_n\|_{L^1(\Omega)} \geq \varepsilon_0\|\nabla v_n\|_{L_\varphi(\Omega)} + n\|v_n\|_Y\,.$$

This gives, by setting $w_n = \dfrac{v_n}{\|\nabla v_n\|_{L_\varphi(\Omega)}}$,

$$\|w_n\|_{L^1(\Omega)} \geq \varepsilon_0 + n\|w_n\|_Y, \quad \|\nabla w_n\|_{L_\varphi(\Omega)} = 1\,.$$

Since $(w_n)_n$ is bounded in $W_0^1L_\varphi(\Omega)$ then for a subsequence

$$w_n \rightharpoonup w \text{ in } W_0^1L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \text{ and strongly in } L^1(\Omega)\,.$$

Thus, $\|w_n\|_{L^1(\Omega)}$ is bounded and $\|w_n\|_Y \to 0$ as $n \to +\infty$.
We conclude $w_n \to 0$ in Y and that $w = 0$ implying that $\varepsilon_0 \leq \|w_n\|_{L^1(\Omega)} \to 0$, a contradiction.
Using $v = u(t)$ in (5.1) for all $u \in W_0^{1,x}L_\varphi(Q)$ with $\dfrac{|\nabla u|}{\lambda} \in K_\varphi(Q)$ and a.e. $t \in [0,T]$, we have

$$\|u(t)\|_{L^1(\Omega)} \leq \varepsilon\|\nabla u(t)\|_{L_\varphi(\Omega)} + C_\varepsilon\|u(t)\|_Y\,.$$

Since $\displaystyle\int_Q \varphi\big(x, |\frac{\nabla u(x,t)}{\lambda}|\big)\,dx\,dt < \infty$, we have thanks to Fubini's theorem $\displaystyle\int_\Omega \varphi\big(x, |\frac{\nabla u(x,t)}{\lambda}|\big)\,dx < \infty$ for a.e. $t \in [0,T]$ and then

$$\|\nabla u(t)\|_{L^1(\Omega)} \leq \lambda\Big(\int_\Omega \varphi\Big(x, \Big|\frac{\nabla u(x,t)}{\lambda}\Big|\Big)\,dx + 1\Big),$$

which yields

$$\|u(t)\|_{L^1(\Omega)} \leq \varepsilon\lambda\Big(\int_\Omega \varphi\Big(x, \Big|\frac{\nabla u(x,t)}{\lambda}\Big|\Big)\,dx + 1\Big) + C_\varepsilon\|u(t)\|_Y\,.$$

Integrating this over $[0,T]$ yields

$$\|u\|_1 \leq \varepsilon\lambda\Big(\int_Q \varphi\Big(x, \frac{|\nabla u|}{\lambda}\Big)\,dx\,dt + T\Big) + C_\varepsilon\|u\|_{L^1(0,T,Y)}\,.$$

$\square$

We also prove the following lemma which allows us to enlarge the space Y whenever necessary.

Lemma 5.2. *Let φ be a Musielak function and ψ the complementary function of φ, we assume that there exists $c > 0$ such that $\psi(x, 1) \leq c$ a.e. in Ω.*

If F is bounded in $W_0^{1,x}L_\varphi(Q)$ and is relatively compact in $L^1(0, T, Y)$ then F is relatively compact in $L^1(Q)$ (and also in $E_\gamma(Q)$ for all Musielak function $\gamma \ll \varphi$).

Proof. Let $\varepsilon > 0$ be given. Let $C > 0$ be such that $\displaystyle\int_Q \varphi\Big(x, \frac{|\nabla f|}{C}\Big)\,dx\,dt \leq 1$ for all $f \in F$.
By the previous lemma, there exists $C_\varepsilon > 0$ such that for all $u \in W_0^{1,x}L_\varphi(Q)$ with $\dfrac{|\nabla u|}{C} \in K_\varphi(Q)$,

$$\|u\|_{L^1(Q)} \leq \frac{2\varepsilon C}{4C(1+T)}\Big(\int_Q \varphi\Big(x, \frac{|\nabla u|}{2C}\Big)\,dx + T\Big) + C_\varepsilon \|u\|_{L^1(0,T,Y)}\,.$$

Moreover, there exists a finite sequence $(f_i)_i$ in F satisfying

$$\forall f \in F,\ \ \exists\, f_i \quad \text{such that} \quad \|f - f_i\|_{L^1(0,T,Y)} \leq \frac{\varepsilon}{2C_\varepsilon}\,.$$

So that,

$$\begin{aligned}\|f - f_i\|_{L^1(Q)} &\leq \frac{\varepsilon}{2(1+T)}\Big(\int_Q \varphi\Big(x, \frac{|\nabla f - \nabla f_i|}{2C}\Big)\,dx\,dt + T\Big) + C_\varepsilon \|f - f_i\|_{L^1(0,T,Y)}\\ &\leq \varepsilon\end{aligned}$$

and hence F is relatively compact in $L^1(Q)$.
Since $\gamma \ll \varphi$ then by using Vitali's theorem, it is easy to see that F is relatively compact in $E_\gamma(Q)$. □

Remark 5.1. If $F \subset L^1(0, T, B)$ is such that $\Big\{\dfrac{\partial f}{\partial t} : f \in F\Big\}$ is bounded in $F \subset L^1(0, T, B)$ then $\|\tau_h f - f\|_{L^1(0,T,B)} \longrightarrow 0$ as $h \longrightarrow 0$ uniformly with respect to $f \in F$.

Lemma 5.3. *Let φ be a Musielak function. If F is bounded in $W^{1,x}L_\varphi(Q)$ and $\Big\{\dfrac{\partial f}{\partial t} : f \in F\Big\}$ is bounded in $W^{-1,x}L_\psi(Q)$, then F is relatively compact in $L^1(Q)$.*

Proof. Let γ and θ be two locally integrables Musielak functions such that $\gamma \ll \varphi$ and $\theta \ll \psi$ near infinity.
For all $0 < t_1 < t_2 < T$ and all $f \in F$, we have

$$\begin{aligned}\Big\|\int_{t_1}^{t_2} f(t)\,dt\Big\|_{W_0^1E_\gamma(\Omega)} &\leq \int_0^T \|f(t)\|_{W_0^1E_\gamma(\Omega)}\,dt\\ &\leq C_1\|f\|_{W_0^{1,x}E_\gamma(Q)}\\ &\leq C_2\|f\|_{W_0^{1,x}E_\varphi(Q)}\\ &\leq C\,,\end{aligned}$$

where we have used the following continuous imbedding

$$W_0^{1,x}L_\varphi(Q) \subset W_0^{1,x}E_\gamma(Q) \subset L^1(0,T,W_0^1L_\varphi(\Omega))\,.$$

Since the imbedding $W_0^1L_\gamma(\Omega) \subset L^1(\Omega)$ is compact we deduce that $\big(\int_{t_1}^{t_2} f(t)\,dt\big)_{f\in F}$ is relatively compact in $L^1(\Omega)$ and $W^{-1,1}(\Omega)$ as well.
On the other hand, $\big\{\dfrac{\partial f}{\partial t} : f \in F\big\}$ is bounded in $W^{-1,x}L_\psi(Q)$ and $L^1(0,T,W^{-1,1}(\Omega))$ as well, since

$$W^{-1,x}L_\psi(Q) \subset W^{-1,x}E_\theta(Q) \subset L^1\big(0,T,W^{-1}E_\theta(\Omega)\big) \subset L^1\big(0,T,W^{-1,1}(\Omega)\big)\,,$$

with continuous imbedding. By Remark 3 of [15], we deduce that $\|\tau_h f - f\|_{L^1(0,T,W^{-1,1}(\Omega))} \longrightarrow 0$ uniformly in $f \in F$ when $h \longrightarrow +\infty$ and by using Theorem 2 of [15], F is relatively compact in $L^1(0,T,W^{-1,1}(\Omega))$.
Since $L^1(\Omega) \subset W^{-1,1}(\Omega)$ with continuous imbedding we can apply Lemma 5.2 to conclude that F is relatively compact in $L^1(Q)$. □

Lemma 5.4. *Let φ be a Musielak function.*
Let $(u_n)_n$ be a sequence of $W^{1,x}L_\varphi(Q)$ such that

$$u_n \rightharpoonup u \quad \textit{weakly in} \quad W^{1,x}L_\varphi(Q) \quad \textit{for} \quad \sigma(\Pi L_\varphi, \Pi L_\psi)$$

and

$$\frac{\partial u_n}{\partial t} = h_n + k_n \ \textit{in}\ \mathcal{D}'(Q)$$

with $(h_n)_n$ bounded in $W^{-1,x}L_\psi(Q)$ and $(k_n)_n$ bounded in the space $\mathcal{M}(Q)$ set of measures on Q.
Then $u_n \longrightarrow u$ strongly in $L^1_{\text{loc}}(Q)$.
If further $u_n \in W_0^{1,x}L_\varphi(Q)$ then $u_n \longrightarrow u$ strongly in $L^1(Q)$.

Proof. It is easily adapted from that given in [12] by using Theorem 4.4 and Remark 4.3 instead of Lemma 8 of [29]. □

6. Essential assumptions and main results

Throughout this paper, we assume that the following assumptions hold true:
Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ satisfying the segment property, $T > 0$ and set $Q = \Omega\times]0,T[$.
In the sequel, we denote by $Q_\tau = \Omega\times]0,\tau[$ for every $\tau \in [0,T]$. Let φ and γ two Musielak Orlicz functions such that $\gamma \ll \varphi$, we denote by ψ the Musielak complementary function of φ. We assume that φ and ψ satisfy the assumptions of Lemma 3.1 and that $\varphi(x,t)$ decreases with respect to one of coordinates of x.
Let

(6.1) $\qquad b\colon \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is a Carathédory function such that

for every $x \in \Omega$: $b(x,s)$ is a strictly increasing $\mathcal{C}^1$-function, with $b(x,0) = 0$.
For any $k > 0$, there exists $\lambda_k > 0$, a function A_k in $L^\infty(\Omega)$ and a function

$B_k \in L_\varphi(\Omega)$ such that

$$\lambda_k \leq \frac{\partial b(x,s)}{\partial s} \leq A_k(x) \quad \text{and} \quad \Big|\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\Big| \leq B_k(x) \tag{6.2}$$

for almost every $x \in \Omega$, for every s such that $|s| \leq k$.
Consider a second-order operator $A\colon D(A) \subset W_0^{1,x}L_\varphi(Q) \longrightarrow W^{-1,x}L_\psi(Q)$ of the form

$$A(u) = -\mathrm{div}\big(a(x,t,u,\nabla u)\big)\,,$$

where $a\colon \Omega\times]0,T[\times\mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathédory function, for almost every $(x,t) \in \Omega\times]0,T[$ and all $s \in \mathbb{R}$, $\xi \neq \xi^* \in \mathbb{R}^N$,

$$|a(x,t,s,\xi)| \leq \beta\big(h_1(x,t) + \psi_x^{-1}\gamma(x,\nu|s|) + \psi_x^{-1}\varphi(x,\nu|\xi|)\big)\,. \tag{6.3}$$

$$\big(a(x,t,s,\xi) - a(x,t,s,\xi^*)\big)(\xi-\xi^*) > 0\,. \tag{6.4}$$

$$a(x,t,s,\xi).\xi \geq \alpha\varphi(x,|\xi|) \tag{6.5}$$

with $h_1(x,t) \in E_\psi(Q), h_1 \geq 0 \in L^1(Q), \alpha, \beta, \nu > 0$.
Assume that $g\colon \Omega\times]0,T[\times\mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{R}$ be a Carathéodory function such that for a.e. $(x,t) \in \Omega\times]0,T[$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$:

$$|g(x,t,s,\xi)| \leq h_2(x,t) + d(s)\varphi(x,|\xi|) \tag{6.6}$$

with $h_2(x,t) \in L^1(Q)$ and $d\colon \mathbb{R} \longrightarrow \mathbb{R}^+$ is a bounded continuous integrable positive function.
Furthermore let

$$f \in L^1(Q)\,, \quad \text{and} \quad F \in (E_\psi(Q))^N\,, \tag{6.7}$$

$$u_0 \quad \text{is a given function in} \quad L^1(\Omega) \quad \text{such that} \quad b(\cdot,u_0) \in L^1(\Omega)\,. \tag{6.8}$$

We consider the following parabolic problem

$$\begin{cases} \dfrac{\partial b(x,u)}{\partial t} + A(u) + g(x,t,u,\nabla u) = f - \mathrm{div}(F) & \text{in} \quad Q\,, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega\times[0,T]\,, \\ b(x,u)\,|_{t=0} = b(x,u_0) & \text{on} \quad \Omega\,. \end{cases} \tag{6.9}$$

We will show that the problem (6.9) has at least one entropy solution in the following sense.

Definition 6.1. A measurable function $u\colon \Omega\times[0,T] \longmapsto \mathbb{R}$ is called entropy solution of (6.9) if, $T_k(u)$ belongs to $D(A)\cap W_0^{1,x}L_\varphi(\Omega)$ for every $k > 0$, $b(\cdot,u_0)$ belongs to $L^1(\Omega)$, and u satisfies the following inequalities

$$b(x,u) \in L^\infty\big([0,T],L^1(\Omega)\big)\,, \tag{6.10}$$

$$\lim_{m\to+\infty}\int_{\{m\leq|u|<m+1\}} a(x,t,u,\nabla u)\cdot\nabla u\,dx\,dt = 0\,, \tag{6.11}$$

and,

$$
\begin{aligned}
&\int_Q \int_0^u \frac{\partial b(x,r)}{\partial r} S'(r-v)T_k(r)\,dr\,dx\,dt \\
&\qquad + \int_0^T \int_0^t \Big\langle \frac{\partial v}{\partial \sigma}, \int_0^u \frac{\partial b(x,r)}{\partial r} S''(r-v)T_k(r)\,dr \Big\rangle \,d\sigma\,dt \\
&\qquad + \int_Q \int_0^t a(x,\sigma,u,\nabla u)\cdot \nabla T_k(u) S'(u-v)\,d\sigma\,dx\,dt \\
&\qquad + \int_Q \int_0^t S''(u-v) a(x,\sigma,u,\nabla u)\cdot (\nabla u - \nabla v) T_k(u)\,d\sigma\,dx\,dt \\
&\qquad + \int_Q \int_0^t g(x,\sigma,u,\nabla u) S'(u-v)T_k(u)\,d\sigma\,dx\,dt \\
&\quad \le \int_Q \int_0^t f S'(u-v)T_k(u)\,dx\,dt \\
&\qquad + \int_Q \int_0^t F\cdot \nabla(u-v) S''(u-v)T_k(u)\,d\sigma\,dx\,dt \\
&\qquad + \int_Q \int_0^t F\cdot \nabla T_k(u) S'(u-v)\,d\sigma\,dx\,dt \\
&\qquad + T\int_\Omega \int_0^{u_0} \frac{\partial b(x,r)}{\partial r} S'(r-v(0))T_k(r)\,dr\,dx
\end{aligned} \tag{6.12}
$$

for every $k>0$, and for all $v \in W_0^{1,x}L_\varphi(Q)\cap L^\infty(Q)$ such that $\dfrac{\partial v}{\partial t}$ belongs to $W^{-1,x}L_\psi(Q) + L^1(Q)$ (recall that T_k is the usual truncation at height k defined on $\mathbb{R}$ by $T_k(s)=\min(k,\max(s,-k))$ and for all increasing function $S\in W^{2,\infty}(\mathbb{R})$ with S' has a compact support in $\mathbb{R}$).

Inequality (6.12) is formally obtained through pointwise multiplication of equation (6.9) by $S'(u-v)T_k(u)$, and integration by parts. However, all the terms in (6.12) have a meaning in $\mathcal{D}'(Q)$.
Indeed, if $M>0$ is such that $\mathrm{supp} S' \subset [-M,M]$, the following identifications are made in (6.12)

- $S(u)$ belongs to $L^\infty(Q)$ since S is a bounded function.
- $\displaystyle\int_0^u \frac{\partial b(x,r)}{\partial r} S'(r-v)T_k(r)dr = \int_0^{T_{M+\|v\|_\infty}(u)} \frac{\partial b(x,r)}{\partial r} S'(r-v)T_k(r)dr \in L^\infty(Q)$.
- $\dfrac{\partial v}{\partial \sigma} \in W^{-1,x}L_\psi(Q)$, $\displaystyle\int_0^{T_{M+\|v\|_\infty}(u)} \frac{\partial b(x,r)}{\partial r} S'(r-v)T_k(r)dr \in W_0^{1,x}L_\varphi(Q)$.
- $S'(u-v)a(x,\sigma,u,\nabla u)\cdot\nabla T_k(u)$ identifies with $S'(u-v)a(x,\sigma,T_k(u),\nabla T_k(u))\cdot \nabla T_k(v)$ a.e. in Q. Since $S'(u-v)\in L^\infty(Q)$ and $\nabla T_k(u) \in (L_\varphi(Q))^N$, we obtain from (6.5) that $S'(u-v)a(x,\sigma,T_k(u),\nabla T_k(u))\cdot \nabla T_k(v) \in L^1(Q)$.

- We have
$S''(u-v)a(x,\sigma,u,\nabla u)\cdot\nabla(u-v)T_k(u)$
$= S''(u-v)a(x,\sigma,T_{M+\|v\|_\infty}(u),\nabla T_{M+\|v\|_\infty}(u))\cdot\nabla(T_{M+\|v\|_\infty}(u)-v)T_k(u)$
a.e. in Ω, and, $S''(u-v)a(x,\sigma,T_{M+\|v\|_\infty}(u),$
$\nabla T_{M+\|v\|_\infty}(u))\cdot\nabla(T_{M+\|v\|_\infty}(u)-v)T_k(u)\in L^1(Q)$.
- $S'(u-v)g(x,\sigma,u,\nabla u)T_k(u)$ identifies with $S'(u-v)g(x,\sigma,T_{M+\|v\|_\infty}(u),$ $\nabla T_{M+\|v\|_\infty}(u))T_k(u)$ a.e. in Q. Since $S'(u-v)T_k(u)\in L^\infty(Q)$, we obtain from (6.5) and (6.6) that
$$S'(u-v)g(x,\sigma,T_{M+\|v\|_\infty}(u),\nabla T_{M+\|v\|_\infty}(u))T_k(u)\in L^1(Q)\,.$$
- $S'(u-v)fT_k(u)$ belongs to $L^1(Q)$.
- Moreover Lemma 4.1 implies that $v\in C([0,T],L^1(\Omega))$, then (6.2) gives
$$\Big|\int_\Omega\int_0^{T_{M+\|v\|_\infty}(u_0)}\frac{\partial b(x,r)}{\partial r}S'(r-v(0))T_k(r)\,dr\,dx\Big|$$
$$\le k(M+\|v\|_\infty)\|S'\|_\infty\int_\Omega A_{M+\|v\|_\infty}(x)\,dx\,.$$

We shall prove the following existence theorem.

Theorem 6.1. *Assume that* (6.1)–(6.8) *hold true. Then the problem* (6.9) *admits at least one entropy solution solution (in the sense of Definition 6.1).*

Proof. We will use a Galerkin method due to Landes and Mustonen [23], we choose a sequence $\{w_1,w_2,\ldots\}$ in $\mathcal{D}(\Omega)$ such that $\cup_{p=0}^\infty V_p$ with $V_p=\{w_1,\ldots,w_p\}$ is dense in $H_0^m(\Omega)$ with m large enough such that $H_0^m(\Omega)$ is continuously embedded in $\mathcal{C}^1(\overline{\Omega})$. For every $v\in H_0^m(\Omega)$ there exists a sequence $(v_j)\subset\cup_{p=0}V_p$ such that $v_n\longrightarrow v$ in $H_0^m(\Omega)$ and in $\mathcal{C}^1(\overline{\Omega})$.
We denote further $\mathcal{V}_p=\mathcal{C}([0,T],V_p)$. It is easy to see that the closure of $\cup_{p=0}^\infty\mathcal{V}_p$ with respect to the norm
$$\|v\|_{\mathcal{C}^{1,0}(Q)}=\sup_{|\alpha|\le 1}\{|D_x^\alpha v(x,t)|:(x,t)\in Q\}$$
contains $\mathcal{D}(Q)$. This implies that, for any $f\in W^{-1,x}E_\psi(Q)$, there exists a sequence $(f_n)\subset\cup_{p=0}^\infty\mathcal{V}_p$ such that $f_n\longrightarrow f$ strongly in $W^{-1,x}E_\psi(Q)$.
Indeed, let $\varepsilon>0$ be given. Writing $f=\sum_{|\alpha|\le 1}D_x^\alpha f^\alpha$ there exists $g^\alpha\in\mathcal{D}(Q)$ such that $\|f^\alpha-g^\alpha\|_{\psi,Q}\le\dfrac{\varepsilon}{2N+2}$. Moreover, by setting $g=\sum_{|\alpha|\le 1}D_x^\alpha g^\alpha$, we see that $g\in\mathcal{D}(Q)$, and so there exists $v\in\cup_{p=0}^\infty\mathcal{V}_p$ such that $\|g-v\|_{\infty,Q}\le\dfrac{\varepsilon}{2\mathrm{meas}(Q)}$. We deduce that
$$\|f-v\|_{W^{-1,x}L_\psi(Q)}\le\sum_{|\alpha|\le 1}\|f^\alpha-g^\alpha\|_{\psi,Q}+\|g-v\|_{\psi,Q}\le\varepsilon\,.$$
We shall divide the theorem in several steps.

Step 1 : Approximate problem
Let us define the following approximations of the data

$$b_n(x,s) = b(x,T_n(s)) + \frac{1}{n}s\,, \qquad \text{a.e.} \quad x\in\Omega,\ \forall\ s\in\mathbb{R}\,, \tag{6.13}$$

$$g_n(x,t,s,\xi) = T_n(g(x,t,s,\xi))\,, \qquad \text{a.e.} \quad x\in\Omega,\ \forall\ s\in\mathbb{R},\ \forall\ \xi\in\mathbb{R}^N, \tag{6.14}$$

$$f_n\in\mathcal{C}_0^\infty(Q) \quad \text{such that} \quad f_n\longrightarrow f \ \text{ in } \ L^1(Q) \quad \text{and} \quad \|f_n\|_{L^1}\le\|f\|_{L^1}\,, \tag{6.15}$$

$$u_{0n}\in\mathcal{C}_0^\infty(Q)\colon b_n(x,u_{0n})\longrightarrow b(x,u_0) \quad \text{in} \quad L^1(Q)$$

and

$$\|b_n(x,u_{0n})\|_{L^1}\le\|b(x,u_0)\|_{L^1}\,. \tag{6.16}$$

We consider the approximate problem
(6.17)

$$\begin{cases} u_n\in\mathcal{V}_n,\ \dfrac{\partial u_n}{\partial t}\in L^1(0,T,V_n),\ u_n(x,0)=u_{0n}(x) \quad \text{a.e. in} \quad \Omega\,, \\ \dfrac{\partial b_n(x,u_n)}{\partial t} - \operatorname{div}(a(x,u_n,\nabla u_n)) + g_n(x,t,u_n,\nabla u_n) = f_n - \operatorname{div}(F) \quad \text{in} \quad \mathcal{D}'(Q)\,. \end{cases}$$

Since g_n is bounded for all fixed $n\in\mathbb{N}$, there exists at least one solution u_n of (6.17) (this solution u_n can be obtained from Galerkin solution (see [23]).

Step 2 : A priori estimates
In this section we denote by c_i, $i=1,2,\dots$ generic positive constants .
Let $D(s)=\dfrac{2}{\alpha}\displaystyle\int_0^s d(\sigma)\,d\sigma$ where d is the function in (6.6).
For $k>0$ taking $T_k(u_n)\exp(D(|u_n|))$ as a test function in (6.17), we get

$$\begin{aligned} &\int_Q \frac{\partial b_n(x,u_n)}{\partial t} T_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\ &\qquad + \int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\ &\qquad + \frac{2}{\alpha}\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n|T_k(u_n)|d(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\ &\qquad + \int_Q g_n(x,t,u_n,\nabla u_n)T_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\ &\quad = \int_Q f_nT_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt + \int_Q F\cdot\nabla T_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\ &\qquad + \frac{2}{\alpha}\int_Q F\cdot\nabla u_n|T_k(u_n)|d(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\,. \end{aligned} \tag{6.18}$$

For the first term of the left hand side of last inequality, we have

$$\int_Q \frac{\partial b_n(x,u_n)}{\partial t} T_k(u_n) \exp\big(D(|u_n|)\big)\,dx\,dt = \int_\Omega B_k^n\big(x,u_n(T)\big)\,dx - \int_\Omega B_k^n(x,u_{0n})\,dx\,, \tag{6.19}$$

where $B_k^n(x,s) = \int_0^s T_k(t)\frac{\partial b_n(x,t)}{\partial t}\exp(D(|t|))\,dt$.
Then, (6.18) becomes

$$\begin{aligned}
&\int_\Omega B_k^n\big(x,u_n(T)\big)\,dx + \int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\\
&\quad + \frac{2}{\alpha}\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n|T_k(u_n)|d(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\\
&\quad + \int_Q g_n(x,t,u_n,\nabla u_n)T_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\\
&= \int_Q f_nT_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt + \int_Q F\cdot\nabla T_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\\
&\quad + \frac{2}{\alpha}\int_Q F\cdot\nabla u_n|T_k(u_n)|d(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt + \int_\Omega B_k^n(x,u_{0n})\,dx\,.
\end{aligned} \tag{6.20}$$

Using now the conditions (6.5) and (6.6), we get

$$\begin{aligned}
&\int_\Omega B_k^n\big(x,u_n(T)\big)\,dx + \int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\\
&\quad + 2\int_Q \varphi(x,|\nabla u_n|)|T_k(u_n)|d(u_n)exp\big(D(|u_n|)\big)\,dx\,dt\\
&= \int_Q f_nT_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt + \int_Q F\cdot\nabla T_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\\
&\quad + \int_Q \big[h_2(x,t)+d(u_n)\varphi(x,|\nabla u_n|)\big]|T_k(u_n)|\exp\big(D(|u_n|)\big)\,dx\,dt\\
&\quad + \frac{2}{\alpha}\int_Q F\cdot\nabla u_n|T_k(u_n)|d(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt + \int_\Omega B_k^n(x,u_{0n})\,dx\,.
\end{aligned} \tag{6.21}$$

From (6.13)–(6.16), and since

$$\begin{aligned}
\int_\Omega B_k^n(x,u_{0n})\,dx &\le \exp\Big(\frac{2\|d\|_{L^1(\mathbb{R})}}{\alpha}\Big)k\|b_n(x,u_{0n})\|_{L^1(\Omega)}\\
&\le \exp\Big(\frac{2\|d\|_{L^1(\mathbb{R})}}{\alpha}\Big)k\|b(x,u_0)\|_{L^1(\Omega)}\,,
\end{aligned}$$

we have

$$\begin{aligned}
&\int_\Omega B_k^n(x,u_n(T))dx + \int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\quad + \int_Q \varphi(x,|\nabla u_n|)|T_k(u_n)|d(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\leq \exp\Big(\frac{2\|d\|_{L^1(\mathbb{R})}}{\alpha}\Big)k\big(\|f\|_{L^1(Q)} + \|h_2\|_{L^1(Q)} + \|b(x,u_0)\|_{L^1(\Omega)}\big) \\
&\quad + \int_Q F\cdot\nabla T_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\quad + \frac{2}{\alpha}\int_Q F\cdot\nabla u_n|T_k(u_n)|d(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\,. \qquad (6.22)
\end{aligned}$$

Then, by using Young's inequality on the second and third term of the last inequality, we obtain

$$\begin{aligned}
&\int_\Omega B_k^n(x,u_n(T))dx + \int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\quad + \int_Q \varphi(x,|\nabla u_n|)|T_k(u_n)|d(u_n)\exp(D(|u_n|))\,dx\,dt \\
&\leq \exp\Big(\frac{2\|d\|_{L^1(\mathbb{R})}}{\alpha}\Big)k\big(\|f\|_{L^1(\Omega)} + \|h_2\|_{L^1(\Omega)} + \|b(x,u_0)\|_{L^1(\Omega)}\big) \\
&\quad + \exp\Big(\frac{2\|d\|_{L^1(\mathbb{R})}}{\alpha}\Big)\int_Q \psi(x,\frac{2(\alpha+1)}{\alpha}|F|)\,dx\,dt \\
&\quad + \frac{\alpha}{2(\alpha+1)}\int_Q \varphi(x,|\nabla T_k(u_n)|)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\quad + \|d\|_\infty\exp\Big(\frac{2\|d\|_{L^1(\mathbb{R})}}{\alpha}\Big)k\int_Q \psi(x,c_\alpha|F|)\,dx\,dt \\
&\quad + \int_Q \varphi(x,|\nabla u_n|)|T_k(u_n)|d(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\,, \qquad (6.23)
\end{aligned}$$

where c_α is a positive constant depend only on α.
Then, by applying (6.5) and the fact that $B_k^n(x,u_n(T))\geq 0$, we get

$$\frac{2\alpha+1}{2(\alpha+1)}\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \leq c_1k + c_2\,. \qquad (6.24)$$

Then by using (6.5), we have

$$\int_Q \varphi\big(x,|\nabla T_k(u_n)|\big)\,dx\,dt \leq c_3k + c_4\,. \qquad (6.25)$$

By using Lemma 3.3, we have $(T_k(u_n))$ is bounded in $W_0^{1,x}L_\varphi(Q)$, then there exists v_k such that

$$\begin{cases} T_k(u_n) \rightharpoonup v_k & \text{in } \; W_0^{1,x}L_\varphi(Q) \;\text{ for }\; \sigma(\Pi L\varphi,\Pi E_\psi) \\ T_k(u_n) \longrightarrow v_k & \text{strongly in } \; E_\varphi(Q)\,. \end{cases} \qquad (6.26)$$

Therefore, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω, then for all $k>0$ and $\delta,\varepsilon>0$ there exists $n_0=n_0(k,\delta,\varepsilon)$ such that

$$\text{meas}\left\{|T_k(u_n)-T_k(u_m)|>\delta\right\}\le\frac{\varepsilon}{3}\,,\quad \forall m,n\ge n_0\,. \tag{6.27}$$

We have by simple calculus

$$\begin{aligned}\inf_{x\in\Omega}\varphi\Big(x,\frac{k}{c}\Big)\text{meas}\Big\{|u_n|>k\Big\} &= \int_{\{|u_n|>k\}}\inf_{x\in\Omega}\varphi\Big(x,\frac{k}{c}\Big)\,dx\,dt\\ &\le \int_\Omega \inf_{x\in\Omega}\varphi\Big(x,\frac{|T_k(u_n)|}{c}\Big)\,dx\,dt\\ &\le \int_\Omega \varphi\Big(x,\frac{|T_k(u_n)|}{c}\Big)\,dx\,dt\\ &\le \int_\Omega \varphi\Big(x,|\nabla T_k(u_n)|\Big)\,dx\,dt\,, &&\text{(using Lemma 3.3)}\\ &\le c_3k+c_4\,, &&\text{(using (6.25))}\,,\end{aligned}$$

where this c is the constant of Lemma 3.3.
Then, by the definition of φ, we get

$$\text{meas}\Big\{|u_n|>k\Big\}\le\frac{c_3k+c_4}{\inf_{x\in\Omega}\varphi\Big(x,\dfrac{k}{c}\Big)}\longrightarrow 0\,,\quad\text{as}\quad k\longrightarrow+\infty\,. \tag{6.28}$$

Since $\forall\delta>0$

$$\begin{aligned}\text{meas}\left\{|u_n-u_m|>\delta\right\}\le{}&\text{meas}\left\{|u_n|>k\right\}+\text{meas}\left\{|u_m|>k\right\}\\ &+\text{meas}\left\{|T_k(u_n)-T_k(u_m)|>\delta\right\}.\end{aligned} \tag{6.29}$$

Then, we have $\forall\varepsilon>0$, there exists $k_0>0$ such that

$$\text{meas}\left\{|u_n|>k\right\}\le\frac{\varepsilon}{3}\,,\quad \text{meas}\left\{|u_m|>k\right\}\le\frac{\varepsilon}{3}\,,\quad\forall k\ge k_0(\varepsilon)\,. \tag{6.30}$$

Combining (6.27), (6.29) and (6.30), we obtain that for all $\delta,\varepsilon>0$, there exists $n_0=n_0(\delta,\varepsilon)$ such that

$$\text{meas}\left\{|u_m-u_m|>\delta\right\}\le\varepsilon\,,\quad\forall n,m\ge n_0\,.$$

It follows that $(u_n)_n$ is a Cauchy sequence in measure, then converges in measure.
Now, we turn to prove the almost every convergence of u_n.
Consider now a $\mathcal{C}^2(\mathbb{R})$, and nondecreasing function r_k such that $r_k(s)=s$ for $|s|\le\dfrac{k}{2}$ and $r_k(s)=k\text{sign}\,(s)$ if $|s|>k$. Multiplying the approximate equation (6.17) by $r_k'(u_n)$, one has

$$\begin{aligned}\frac{\partial B_k^n(x,u_n)}{\partial t}&-\text{div}\,\big(a(x,t,u_n,\nabla u_n)r_k'(u_n)\big)+a(x,t,u_n,\nabla u_n)\cdot\nabla u_n r_k''(u_n)\\ &+g_n(x,t,u_n,\nabla u_n)r'(u_n)=f_nr'(u_n)+F\cdot\nabla u_n r_k''(u_n)\quad\text{in}\quad\mathcal{D}'(Q)\,,\end{aligned}$$

with $B_k^n(x,s)=\displaystyle\int_0^s\frac{\partial b_n(x,\sigma)}{\partial t}r'(\sigma)d\sigma$.

Which yields easily that $\dfrac{\partial B_k^n(x,u_n)}{\partial t}$ is bounded in $W^{-1,x}L_\varphi(Q)+L^1(Q)$. Due to the properties of r_k and Lemma 5.4, we conclude that $\dfrac{\partial r_k(u_n)}{\partial t}$ is bounded in $W^{-1,x}L_\varphi(Q)+L^1(Q)$.
Thanks to Lemma 5.3, we deduce that $r_k(u_n)$ is compact in $L^1(Q)$.

Due to the choice of r_k, we conclude that for each k, the sequence $T_k(u_n)$ converges almost everywhere in Q, which implies that the sequence u_n converges almost everywhere to some measurable function u in Q.

Consequently, we get

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{in } W_0^1L_\varphi(\Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi) \\ T_k(u_n) \longrightarrow T_k(u) & \text{strongly in } E_\varphi(\Omega)\,. \end{cases} \tag{6.31}$$

Step 3 : Boundness of $(a(x,t,T_k(u_n),\nabla T_k(u_n)))_n$ in $(L_\psi(\Omega))^N$
Let $w\in(E_\varphi(Q)^N$ be arbitrary such that $\|w\|_{\varphi,Q}=1$, by (6.4) we have

$$\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\frac{w}{\nu})\Big)\Big(\nabla T_k(u_n)-\frac{w}{\nu}\Big)\exp(D(|u_n|))>0\,.$$

Hence

$$\begin{aligned} &\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\frac{w}{\nu}\exp\big(D(|u_n|)\big)\,dx\,dt\\ &\qquad\le \int_Q a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\\ &\qquad\quad -\int_Q a\Big(x,t,T_k(u_n),\frac{w}{\nu}\Big)\Big(\nabla T_k(u_n)-\frac{w}{\nu}\Big)\exp\big(D(|u_n|)\big)\,dx\,dt\,, \end{aligned} \tag{6.32}$$

hence, by using (6.24)

$$\int_Q a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\le c_5k+c_6\,. \tag{6.33}$$

For μ large enough ($\mu>\beta$), we have by using (6.3)

$$\begin{aligned} &\int_Q\psi_x\Big(\frac{a(x,t,T_k(u_n),\frac{w}{\nu})}{3\mu}\Big)\,dx\,dt\\ &\le\int_Q\psi_x\Big(\frac{\beta(h_1(x,t)+\psi_x^{-1}(\gamma(x,\nu|T_k(u_n)|))+\psi_x^{-1}(\varphi(x,|w|)))}{3\mu}\Big)\,dx\,dt\\ &\le\int_Q\psi_x\Big(\frac{\beta(h_1(x,t)+\psi_x^{-1}(\gamma(x,\nu|T_k(u_n)|))+\psi_x^{-1}(\varphi(x,|w|)))}{3\mu}\Big)\,dx\,dt\\ &\le\frac{\beta}{\mu}\int_Q\psi_x\Big(\frac{h_1(x,t)+\psi_x^{-1}(\gamma(x,\nu|T_k(u_n)|))+\psi_x^{-1}(\varphi(x,|w|))}{3}\Big)\,dx\,dt\\ &\le\frac{\beta}{3\mu}\Big(\int_Q\psi_x(h_1(x,t))\,dx\,dt+\int_Q\gamma(x,\nu|T_k(u_n)|)\,dx\,dt+\int_Q\varphi(x,|w|)\,dx\,dt\Big)\\ &\le c_5(k)\,. \end{aligned}$$

Now, since γ grows essentially less rapidly than φ near infinity and by using the Remark 2.1, there exists $r'(k) > 0$ such that $\gamma(x, \nu k) \leq r'(k)\varphi(x, 1)$ and so we have

$$\int_Q \psi_x\Big(\frac{a(x,t,T_k(u_n),\frac{w}{\nu})}{3\mu}\Big)\,dx\,dt$$

(6.34)

$$\leq \frac{\beta}{3\mu}\Big(\int_Q \psi_x(h_1(x,t))\,dx\,dt + r'(k)\int_Q \varphi(x,1)\,dx\,dt + \int_Q \varphi(x,|w|)\,dx\,dt\Big)\,.$$

hence $a(x,t,T_k(u_n),\dfrac{w}{\nu})$ is bounded in $(L_\psi(Q))^N$.
Which implies that second term of the right hand side of (6.32) is bounded, consequently, we obtain

$$\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))w\,dx\,dt \leq c_2(k)\,,$$

for all $w \in (L_\varphi(Q))^N$ with $\|w\|_{\varphi,Q} \leq 1$. Hence by the theorem of Banach-Steinhous, the sequence $(a(x,t,T_k(u_n),\nabla T_k(u_n)))_n$ remains bounded in $(L_\psi(Q))^N$.
Which implies that, for all $k > 0$ there exists a function $l_k \in (L_\psi(Q))^N$ such that

$$(6.35)\quad a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup l_k \quad \text{weak star in } (L_\psi(Q.))^N \text{ for } \sigma(\Pi L_\psi, \Pi E_\varphi)\,.$$

Step 4 : Modular convergence of truncations
Let $(v_j)_j$ be a sequence in $\mathcal{D}(Q)$ such that

$$(6.36)\qquad v_j \longrightarrow u \quad \text{with respect to the modular convergence in} \quad W_0^{1,x}L_\varphi(Q)$$

and let $w_i \in \mathcal{D}(\Omega)$ be a sequence which converges strongly to u_0 in $L^2(\Omega)$.
Set $w^i_{\mu,j} = T_k(v_j)_\mu + \exp(-\mu t)T_k(w_i)$ where $T_k(v_j)_\mu$ is the mollification with respect to time of $T_k(v_j)$.
Note that $w^i_{\mu,j}$ a smooth function having the following proprieties

$$\begin{cases} \dfrac{\partial}{\partial t}(w^i_{\mu,j}) = \mu(T_k(v_j) - w^i_{\mu,j}), \quad w^i_{\mu,j}(0) = T_k(v_j), \quad |w^i_{\mu,j}| \leq k\,; \\ w^i_{\mu,j} \longrightarrow T_k(u)_\mu + \exp(-\mu t)T_k(w_i) \text{ in } W_0^{1,x}L_\varphi(Q) \\ \qquad\text{for the modular convergence as } \ j \longrightarrow +\infty\,; \\ T_k(u)_\mu + \exp(-\mu t)T_k(w_i) \longrightarrow T_k(u) \text{ in } W_0^{1,x}L_\varphi(Q) \\ \qquad\text{for the modular convergence as } \ j \longrightarrow +\infty\,. \end{cases}$$

For $m > k$ we define the function ρ_m on $\mathbb{R}$ by

$$\rho_m(s) = \begin{cases} 1 & \text{if } \ |s| \leq m\,, \\ m+1-|s| & \text{if } \ m \leq |s| \leq m+1\,, \\ 0 & \text{if } \ |s| > m+1 \end{cases}$$

For the sake of simplicity, we denote by $\varepsilon(n,j,\mu,s)$ any quantity (possible different) such that

$$\lim_{s\longrightarrow\infty}\lim_{\mu\longrightarrow\infty}\lim_{j\longrightarrow\infty}\lim_{n\longrightarrow\infty}\varepsilon(n,j,\mu,s) = 0\,.$$

If the quantity we consider does not depend on one of parameters n, j, μ and s, we will omit the dependence on the corresponding parameter as an example, $\varepsilon(n)$ is any quantity such that

$$\lim_{j\longrightarrow\infty}\lim_{n\longrightarrow\infty}\varepsilon(n,j)=0\,.$$

We denote also χ_s the characteristic functions of the set

$$Q_s=\big\{(x,t,)\in Q:|\nabla T_k(u)|\le s\big\}\,.$$

Let $D(s)=\dfrac{1}{\alpha}\displaystyle\int_0^s d(t)dt$, taking $(T_k(u_n)-w^i_{\mu,j})\rho_m(u_n)\exp(D(|u_n|))$ as a test function in (6.17), one has

$$\begin{aligned}
&\int_Q\frac{\partial b_n(x,u_n)}{\partial t}(T_k(u_n)-w^i_{\mu,j})\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\\
&\quad+\int_Q a(x,t,u_n,\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla w^i_{\mu,j})\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\\
&\quad+\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\rho'_m(u_n)(T_k(u_n)-w^i_{\mu,j})\exp\big(D(|u_n|)\big)\,dx\,dt\\
&\quad+\frac{1}{\alpha}\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\mathrm{sign}\,(u_n)(T_k(u_n)-w^i_{\mu,j})d(u_n)\rho_m(u_n)\\
&\quad\times\exp\big(D(|u_n|)\big)\,dx\,dt\\
&\quad+\int_Q g_n(x,t,u_n,\nabla u_n)(T_k(u_n)-w^i_{\mu,j})\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\\
&=\int_Q f_n(T_k(u_n)-w^i_{\mu,j})\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\\
&\quad+\int_Q F\cdot(\nabla T_k(u_n)-\nabla w^i_{\mu,j})\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\\
&\quad+\int_Q F\cdot\nabla u_n(T_k(u_n)-w^i_{\mu,j})\rho'_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\\
&\quad+\frac{1}{\alpha}\int_Q F\cdot\nabla u_n\mathrm{sign}\,(u_n)(T_k(u_n)-w^i_{\mu,j})d(u_n)\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\,.
\end{aligned}\tag{6.37}$$

Firstly, for the first term of the left hand side of (6.37), by the definition of $w^i_{\mu,j}$ and Lemma 5.6 of [28], we get

$$\int_Q\frac{\partial b_n(x,u_n)}{\partial t}(T_k(u_n)-w^i_{\mu,j})\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\ge\varepsilon(n,\mu,j,i)\,.\tag{6.38}$$

Secondly, for the third term of the left hand side of (6.37), we get

$$\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\rho'_m(u_n)(T_k(u_n)-w^i_{\mu,j})\exp\big(D(|u_n|)\big)\,dx\,dt\\
&\qquad\le 2k\exp\Big(\frac{\|d\|_{L^1(\mathbb{R})}}{\alpha}\Big)\int_{m\le|u_n|\le m+1}a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt\,.
\end{aligned}$$

Hence by Lemma 5.1 of [3], we get

$$\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\rho_m'(u_n)(T_k(u_n)-w_{\mu,j}^i)\exp\big(D(|u_n|)\big)\,dx\,dt$$
$$= \varepsilon(n,j,\mu,i,m)\,. \tag{6.39}$$

Thirdly, for the fourth term of the right hand side, we get

$$\Big|\frac{1}{\alpha}\int_Q F\cdot\nabla u_n\mathrm{sign}\,(u_n)(T_k(u_n)-w_{\mu,j}^i)d(u_n)\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\Big|$$
$$\le \frac{\|d\|_\infty}{\alpha}\exp\Big(\frac{\|d\|_{L^1(\mathbb{R})}}{\alpha}\Big)\int_Q |F|\cdot|\nabla T_{m+1}(u_n)||T_k(u_n)-w_{\mu,j}^i|\,dx\,dt\,.$$

Then, by using the fact that $T_k(u_n)-w_{\mu,j}^i$ converges to $T_k(u)-w_{\mu,j}^i$ strongly in $E_\varphi(Q)$ and $\nabla T_{m+1}(u_n)$ converges weakly to $\nabla T_{m+1}(u)$ in $(L_\varphi(Q))^N$ as $n\longrightarrow+\infty$, then by using the modular convergence on μ and j, we get

$$\frac{1}{\alpha}\int_Q F\cdot\nabla u_n\mathrm{sign}(u_n)(T_k(u_n)-w_{\mu,j}^i)d(u_n)\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt$$
$$= \varepsilon(n,j,\mu,i)\,. \tag{6.40}$$

By a similar calculus, we get

$$\int_Q f_n(T_k(u_n)-w_{\mu,j}^i)\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt = \varepsilon(n,j,\mu,i)\,, \tag{6.41}$$

$$\int_Q F\cdot(\nabla T_k(u_n)-\nabla w_{\mu,j}^i)\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt = \varepsilon(n,j,\mu,i)\,, \tag{6.42}$$

and

$$\int_Q F\cdot\nabla u_n(T_k(u_n)-w_{\mu,j}^i)\rho_m'(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt = \varepsilon(n,j,\mu,i)\,. \tag{6.43}$$

Now, combining (6.37)–(6.43) and using (6.6), we obtain

$$\int_Q a(x,t,u_n,\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla w_{\mu,j}^i)\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt$$
$$+\frac{1}{\alpha}\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\mathrm{sign}\,(u_n)(T_k(u_n)-w_{\mu,j}^i)$$
$$\times\, d(u_n)\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt$$
$$\le \varepsilon(n,j,\mu,i,m)+\int_Q h_2(x,t)|T_k(u_n)-w_{\mu,j}^i|\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt$$
$$+\int_Q d(u_n)\varphi(x,|\nabla u_n|)|T_k(u_n)-w_{\mu,j}^i|\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \tag{6.44}$$

Splitting the second term of the left hand side and the third term of the right hand side of (6.44) on $\{|u_n|\le k\}$ and $\{|u_n|>k\}$, and using (6.5) and the fact that

$(T_k(u_n) - w^i_{\mu,j})u_n \geq 0$ on $\{|u_n| > k\}$, one has

$$\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)\cdot(\nabla T_k(u_n) - \nabla w^i_{\mu,j})\rho_m(u_n))\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\qquad - \frac{1}{\alpha}\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)|T_k(u_n) - w^i_{\mu,j}| \\
&\qquad \times d(u_n)\rho_m(u_n))\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\qquad + \frac{1}{\alpha}\int_{\{|u_n|>k\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n|T_k(u_n) - w^i_{\mu,j}|d(u_n) \\
&\qquad \times \rho_m(u_n))\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\quad \leq \varepsilon(n,j,\mu,i,m) + \int_Q h_2(x,t)|T_k(u_n) - w^i_{\mu,j}|\rho_m(u_n))\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\qquad + \int_Q d(u_n)a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)|T_k(u_n) - w^i_{\mu,j}| \\
&\qquad \times \rho_m(u_n))\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\qquad + \frac{1}{\alpha}\int_{\{|u_n|>k\}} d(u_n)a(x,t,u_n,\nabla u_n)\cdot\nabla u_n|T_k(u_n) - w^i_{\mu,j}| \\
&\qquad \times \rho_m(u_n))\exp\big(D(|u_n|)\big)\,dx\,dt\,. \qquad (6.45)
\end{aligned}$$

Then, by simplification, we have

$$\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)\cdot(\nabla T_k(u_n) - \nabla w^i_{\mu,j})\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\quad \leq \varepsilon(n,j,\mu,i,m) + \int_Q h_2(x,t)|T_k(u_n) - w^i_{\mu,j}|\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\qquad + \frac{2}{\alpha}\int_Q d(u_n)a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)|T_k(u_n) - w^i_{\mu,j}| \\
&\qquad \times \rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \qquad (6.46)
\end{aligned}$$

Similarly, like in (6.41) and (6.39), we get

$$\begin{aligned}
&\int_Q h_2(x,t)|T_k(u_n) - w^i_{\mu,j}|\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\quad = \varepsilon(n,j,\mu,i)\,, \qquad (6.47)
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\frac{2}{\alpha}\int_Q d(u_n)a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)|T_k(u_n) - w^i_{\mu,j}| \\
&\qquad \times \rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\Big|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{4\|d\|_\infty}{\alpha} \exp\Big(\frac{\|d\|_{L^1(\mathbb{R})}}{\alpha}\Big) \int_{m\leq |u_n|\leq m+1} a(x,t,T_k(u_n),\nabla T_k(u_n)) \\
&\quad \cdot \nabla T_k(u_n)\,dx\,dt \\
&= \varepsilon(n,\mu,j,i,m)\,. \qquad (6.48)
\end{aligned}
$$

Thus, by combining (6.46), (6.48) and (6.47), one has

$$
\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)\cdot(\nabla T_k(u_n)-\nabla w^i_{\mu,j})\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\qquad\qquad \leq \varepsilon(n,j,\mu,i,m)\,. \qquad (6.49)
\end{aligned}
$$

Since $\rho_m(u_n)=0$ if $|u_n|>m+1$, one has

$$
\begin{aligned}
&\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla w^i_{\mu,j})\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\quad -\int_{\{|u_n|>k\}} a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\cdot\nabla w^i_{\mu,j}\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\quad \leq \varepsilon(n,j,\mu,i,m)\,. \qquad (6.50)
\end{aligned}
$$

Since $a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))$ converges weak star to l_{m+1} in $(L_\psi(Q))^N$ and ρ_m is continuous, we get

$$
\begin{aligned}
&\int_{\{|u_n|>k\}} a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\cdot\nabla w^i_{\mu,j}\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\quad =\int_{\{|u|>k\}} l_{m+1}\cdot\nabla w^i_{\mu,j}\rho_m(T_{m+1}(u))\exp\big(D(|T_{m+1}(u)|)\big)\,dx\,dt+\varepsilon(n)\,. \qquad (6.51)
\end{aligned}
$$

Then, by passing to the limit on j, μ and i, we get

$$
\begin{aligned}
&\int_{\{|u_n|>k\}} a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\cdot\nabla w^i_{\mu,j}\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\quad =\varepsilon(n,j,\mu,i)\,. \qquad (6.52)
\end{aligned}
$$

Thus, we deduce that,

$$
\begin{aligned}
&\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla w^i_{\mu,j})\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\quad \leq \varepsilon(n,j,\mu,i,m)\,. \qquad (6.53)
\end{aligned}
$$

Remark that,

$$\begin{aligned}
&\int_Q \big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)\big] \\
&\qquad \times \big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big)\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\quad \le -\int_Q a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)(\nabla T_k(u_n) - \nabla T_k(u)\chi_s) \\
&\qquad \times \rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\qquad - \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u)\chi_s - \nabla w^i_{\mu,j}) \\
&\qquad \times \rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\qquad + \varepsilon(n,j,\mu,i,m) \\
&\quad = J_1 + J_2 + \varepsilon(n,j,\mu,i,m)\,. \qquad (6.54)
\end{aligned}$$

We shall go to the limit as n, μ, j, i and s to infinity in the integrals of the right-hand side.
Starting by J_1, we have

$$\begin{aligned}
J_1 &= \int_Q a(x,t,T_k(u),\nabla T_k(u)\chi_s)(\nabla T_k(u) - \nabla T_k(u)\chi_s) \\
&\qquad \times \rho_m(u)\exp\big(D(|u|)\big)\,dx\,dt + \varepsilon(n) \\
&= \varepsilon(n,j,\mu,i,m,s)\,. \qquad (6.55)
\end{aligned}$$

Concerning J_2, one has

$$\begin{aligned}
J_2 &= \int_Q l_k(\nabla T_k(u)\chi_s - \nabla T_k(u))\rho_m(u)\exp\big(D(|u|)\big)\,dx\,dt + \varepsilon(n,j,\mu,i) \\
&= \varepsilon(n,j,\mu,i,m,s), \qquad (6.56)
\end{aligned}$$

Combining (6.54), (6.55) and (6.56), follows

$$\begin{aligned}
&\int_Q \big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)\big] \\
&\qquad \times \big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big)\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\quad \le \varepsilon(n,j,\mu,i,m,s)\,. \qquad (6.57)
\end{aligned}$$

Since $\rho_m(u_n) = 1$ in $\{|u_n| \le m\}$ and $\{|u_n| \le k\} \subset \{|u_n| \le m\}$, for m large enough, we get

$$\begin{aligned}
&\int_Q \big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)\big] \\
&\qquad \times \big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big)\exp\big(D(|u_n|)\big)\,dx\,dt \\
&\quad = \int_Q \big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)\big]
\end{aligned}$$

$$\begin{aligned}
&\times \big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big)\rho_m(u_n)\exp\big(D(|u_n|)\big)\,dx\,dt\\
&+\int_{\{|u_n|>k\}} \big[a(x,t,T_k(u_n),0) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)\big]\\
&\times \big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big)(1-\rho_m(u_n))\exp\big(D(|u_n|)\big)\,dx\,dt\,. \qquad (6.58)
\end{aligned}$$

It is easy to see that the last terms of the last equality tend to zero as n tends to infinity.
Which yields

$$\begin{aligned}
\exp\big(D(-\infty)\big)\int_Q &\big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)\big]\\
&\times \big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big)\,dx\,dt\\
&\le \varepsilon(n,j,\mu,i,m,s)\,. \qquad (6.59)
\end{aligned}$$

Passing to the limit in (6.59) as n and s tends infinity, we get

$$\begin{aligned}
\lim_{n,s\to+\infty}\int_Q &\big[a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)\big]\\
&\times \big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s\big)\,dx\,dt = 0\,. \qquad (6.60)
\end{aligned}$$

Using Lemma 3.5, we have

$$T_k(u_n) \longrightarrow T_k(u) \text{ for the modular convergence in } W_0^{1,x}L_\varphi(Q)\,. \qquad (6.61)$$

Step 5 : Equi-integrability of the nonlinearity sequence
This follows by the same method as in First, note that thanks to (6.61), we obtain that ∇u_n converges to ∇u a.e. in Q (for a subsequence).
Now, we will show that

$$g_n(x,t,u_n,\nabla u_n) \longrightarrow g(x,t,u,\nabla u) \quad \text{strongly in } L^1(Q)\,. \qquad (6.62)$$

Considering for $h>0$ the function $v_n^h = \Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}ds\Big)\exp\big(D(u_n)\big)$ as a test function in the approximate problem (6.17), where $d(s) = \dfrac{2}{\alpha}\displaystyle\int_0^s d(t)\,dt$, we obtain

$$\begin{aligned}
\int_Q &\frac{\partial b_n(x,u_n)}{\partial t}\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}\,ds\Big)\exp(D(u_n))\,dx\,dt\\
&+\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n d(u_n)\chi_{\{u_n>h\}}\exp\big(D(u_n)\big)\,dx\,dt\\
&+\frac{2}{\alpha}\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n d(u_n)\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}ds\Big)\exp\big(D(u_n)\big)\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_Q g_n(x,t,u_n,\nabla u_n)\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}\,ds\Big)\exp\big(D(u_n)\big)\,dx\,dt\\
&\le \int_Q f_n\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}ds\Big)\exp\big(D(u_n)\big)\,dx\,dt\\
&+\int_Q F\cdot\nabla u_n d(u_n)\chi_{\{u_n>h\}}\exp\big(D(u_n)\big)\,dx\,dt\\
&+\frac{2}{\alpha}\int_Q F\cdot\nabla u_n d(u_n)\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}ds\Big)\exp\big(D(u_n)\big)\,dx\,dt.
\end{aligned}
$$

Then, by noting $B_n^h(x,r)=\int_0^r \frac{\partial b_n(x,\tau)}{\partial t}\Big(\int_0^\tau d(\sigma)\chi_{\{\sigma>\tau\}}\,d\sigma\Big)\exp\big(D(\tau)\big)\,d\tau$, one has

$$
\begin{aligned}
&\int_\Omega B_n^h(x,u_n(T))\,dx+\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n d(u_n)\chi_{\{u_n>h\}}\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad+\frac{2}{\alpha}\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n d(u_n)\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}ds\Big)\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad+\int_Q g_n(x,t,u_n,\nabla u_n)\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}\,ds\Big)\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad\le \int_Q f_n\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}ds\Big)\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad+\int_Q F\cdot\nabla u_n d(u_n)\chi_{\{u_n>h\}}\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad+\frac{2}{\alpha}\int_Q F\cdot\nabla u_n d(u_n)\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}\,ds\Big)\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad+\int_\Omega B_n^h(x,u_{0n})\,dx\,.
\end{aligned}
$$

By a simple calculus and by using (6.6) and the fact that $B_n^h(x,u_n(T))\ge 0$, we get

$$
\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n d(u_n)\chi_{\{u_n>h\}}\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad+\frac{2}{\alpha}\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n d(u_n)\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}\,ds\Big)\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad\le \int_Q f_n\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}ds\Big)\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad+\int_Q h_2(x,t)\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}ds\Big)\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad+\int_Q \varphi(x,|\nabla u_n|)d(u_n)\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}\,ds\Big)\exp\big(D(u_n)\big)\,dx\,dt
\end{aligned}
$$

$$+\int_Q F\cdot\nabla u_n d(u_n)\chi_{\{u_n>h\}}\exp\big(D(u_n)\big)\,dx\,dt$$
$$+\frac{2}{\alpha}\int_Q F\cdot\nabla u_n d(u_n)\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}\,ds\Big)\exp\big(D(u_n)\big)\,dx\,dt$$
$$+\int_\Omega B_n^h(x,u_{0n})\,dx\,.$$

Then, by (6.5), one has

$$\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n d(u_n)\chi_{\{u_n>h\}}\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad+\frac{1}{\alpha}\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n d(u_n)\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}ds\Big)\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad\le\Big(\int_h^{+\infty} d(s)\,ds\Big)\exp\Big(\frac{\|d\|_{L^1(\mathbb{R})}}{\alpha}\Big)\big(\|f_n\|_{L^1(\mathbb{R})}+\|h_2\|_{L^1(\mathbb{R})}+\|b(\cdot,u_0\|_{L^1(\mathbb{R})}\big)\\
&\quad+\int_Q F\cdot\nabla u_n d(u_n)\chi_{\{u_n>h\}}\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad+\frac{2}{\alpha}\int_Q F\cdot\nabla u_n d(u_n)\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}\,ds\Big)\exp\big(D(u_n)\big)\,dx\,dt\,.
\end{aligned}$$

Thus, by applying Young's inequality on the second and third of the right hand side of last inequality, we get

$$\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n d(u_n)\chi_{\{u_n>h\}}\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad+\frac{1}{\alpha}\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n d(u_n)\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}ds\Big)\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad\le\Big(\int_h^{+\infty} d(s)ds\Big)\exp\Big(\frac{\|d\|_{L^1(\mathbb{R})}}{\alpha}\Big)\big(\|f_n\|_{L^1(\mathbb{R})}+\|h_2\|_{L^1(\mathbb{R})}+\|b(\cdot,u_0\|_{L^1(\mathbb{R})}\big)\\
&\quad+\int_Q \psi\Big(x,\frac{2(\alpha+1)|F|}{\alpha}\Big)d(u_n)\chi_{\{u_n>h\}}\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad+\frac{\alpha}{2(\alpha+1)}\int_Q \varphi(x,|\nabla u_n|)d(u_n)\chi_{\{u_n>h\}}\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad+\int_Q \psi\Big(x,\frac{2|F|}{\alpha}\Big)d(u_n)\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}\,ds\Big)\exp\big(D(u_n)\big)\,dx\,dt\\
&\quad+\int_Q \varphi(x,\nabla u_n)d(u_n)\Big(\int_0^{u_n} d(s)\chi_{\{s>h\}}\,ds\Big)\exp\big(D(u_n)\big)\,dx\,dt\,.
\end{aligned}$$

Hence, by (6.5), we have

$$\alpha\int_Q \varphi(x,|\nabla u_n|)d(u_n)\chi_{\{u_n>h\}}\exp\big(D(u_n)\big)\,dx\,dt$$

$$\leq \Big(\int_h^{+\infty} d(s)ds\Big)\exp\Big(\frac{\|d\|_{L^1(\mathbb{R})}}{\alpha}\Big)$$
$$\times \Big(\|f_n\|_{L^1(\mathbb{R})} + \|h_2\|_{L^1(\mathbb{R})} + \|b(\cdot, u_0\|_{L^1(\mathbb{R})} + \|d\|_\infty \int_Q \psi\Big(x, \frac{2|F|}{\alpha}\Big)\, dx\, dt\Big)$$
$$+ \exp\Big(\frac{\|d\|_{L^1(\mathbb{R})}}{\alpha}\Big)\|d\|_\infty \int_Q \psi\Big(x, \frac{2(\alpha+1)|F|}{\alpha}\Big)\chi_{\{u_n>h\}}\, dx\, dt$$
$$+ \frac{\alpha}{2(\alpha+1)}\int_Q \varphi(x, |\nabla u_n|)d(u_n)\chi_{\{u_n>h\}} \exp\big(D(u_n)\big)\, dx\, dt\,,$$

which yields,

$$\frac{\alpha(2\alpha+1)}{2(\alpha+1)}\int_Q \varphi(x, |\nabla u_n|)d(u_n)\chi_{\{u_n>h\}} \exp\big(D(u_n)\big)\, dx\, dt$$
$$\leq \Big(\int_h^{+\infty} d(s)ds\Big)\exp\Big(\frac{\|d\|_{L^1(\mathbb{R})}}{\alpha}\Big)$$
$$\times \Big(\|f_n\|_{L^1(\mathbb{R})} + \|h_2\|_{L^1(\mathbb{R})} + \|b(\cdot, u_0\|_{L^1(\mathbb{R})} + \|d\|_\infty$$
$$\times \int_Q \psi\Big(x, \frac{2|F|}{\alpha}\Big)\, dx\, dt\Big)$$
$$+ \exp\Big(\frac{\|d\|_{L^1(\mathbb{R})}}{\alpha}\Big)\|d\|_\infty \int_Q \psi\Big(x, \frac{2(\alpha+1)|F|}{\alpha}\Big)\chi_{\{u_n>h\}}\, dx\, dt\,.$$

Since d continuous on Q and Q bounded, then $d \in L^1(Q)$, then we have

$$\lim_{h\to+\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}} \varphi(x, |\nabla u_n|)d(u_n)\exp\big(D(u_n)\big)\, dx\, dt = 0\,. \tag{6.63}$$

Similarly, let $w_n^h = \Big(\int_0^{u_n} d(s)\chi_{\{s<-h\}}ds\Big)\exp(-D(u_n))$ as a test function in the approximate problem (6.17), we conclude that

$$\lim_{h\to+\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}} \varphi(x, |\nabla u_n|)d(u_n)\exp\big(D(u_n)\big)\, dx\, dt = 0\,. \tag{6.64}$$

Consequently, combining (6.63) and (6.64), we conclude that

$$\lim_{h\to+\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}} \varphi(x, |\nabla u_n|)d(u_n)\exp\big(D(u_n)\big)\, dx\, dt = 0\,. \tag{6.65}$$

which, for h large enough and for a subset E of Q, yields

$$\begin{aligned}\lim_{\text{meas(E)}\to 0}\int_E \varphi(x,|\nabla u_n|)d(u_n)\exp\big(D(u_n)\big)\,dx\,dt \\ \le \|d\|_\infty \lim_{\text{meas(E)}\to 0}\int_E \varphi(x,|\nabla T_h(u_n)|)\exp\big(D(u_n)\big)\,dx\,dt \\ + \lim_{\text{meas(E)}\to 0}\int_{\{|u_n|>h\}\cap E}\varphi(x,|\nabla u_n|)d(u_n)\exp\big(D(u_n)\big)\,dx\,dt\,.\end{aligned}$$

So, we conclude that $\varphi(\cdot,|\nabla u_n|)d(u_n)$ is equi-integrable, which implies that

$$\varphi(\cdot,|\nabla u_n|)d(u_n)\longrightarrow\varphi(\cdot,|\nabla u|)d(u)\quad\text{ strongly in } L^1(Q).$$

Consequently, using (6.6) and Vitali's Theorem, we conclude the equi-integrability of the nonlinearities.

Step 6 : Passage to the limit
In this step, we shall prove that u is an entropy solution to the problem (6.9) in the sense of Definition 6.1.

Firstly, we prove that u satisfies (6.10).
For $\tau\in[0,T]$, considering $T_k(u_n)\exp\big(D(|u_n|)\big)\chi_{[0,\tau]}$ as a test function in (6.17), then like Step 1, we get

$$\int_{Q_\tau}\frac{\partial b_n(x,u_n)}{\partial t}T_k(u_n)\exp\big(D(u_n)\big)\,dx\,dt\le c_1k+c_2\,.$$

Then, for $k\ge c_2$, we get

$$\int_{Q_\tau}\frac{\partial b_n(x,u_n)}{\partial t}T_k(u_n)\exp\big(D(u_n)\big)\,dx\,dt\le (c_1+1)k\,.$$

By passing to the limit inf with respect to n, we obtain

$$\frac{1}{k}\int_{Q_\tau}\frac{\partial b(x,u)}{\partial t}T_k(u)\exp\big(D(u_n)\big)\,dx\,dt\le c_1+1\,.$$

$$\int_0^{u(\tau)}\operatorname{sgn}(r)\frac{\partial b(x,r)}{\partial r}\exp\big(D(|r|)\big)\,dr\le c_1+1\,.$$

Observe that,

$$|b(x,u(\tau)|\le\int_0^{u(\tau)}\operatorname{sgn}(r)\frac{\partial b(x,r)}{\partial r}\exp\big(D(|r|)\big)\,dr\,,$$

which shows that $b(x,u)\in L^\infty([0,T],L^1(\Omega))$.

Secondly, we shall show that u fulfills the condition (6.11).
Indeed, since $a(x,t,u_n,\nabla u_n)\cdot\nabla u_n=a(x,t,T_{M+1}(u_n),\nabla T_{M+1}(u_n))$

$\cdot\nabla T_{M+1}(u_n)$ a.e. in Q, by a simple calculus, we get

$$\begin{aligned}
&\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt\\
&\quad=\int_{\{m\le|u|\le m+1\}}\big(a(x,t,T_{M+1}(u_n),\nabla T_{M+1}(u_n))\\
&\quad-a(x,t,T_{M+1}(u_n),\nabla T_{M+1}(u)\chi_s)\big)\big(\nabla T_{M+1}(u_n)-\nabla T_{M+1}(u)\chi_s\big)\,dx\,dt\\
&\quad+\int_{\{m\le|u|\le m+1\}}\big(a(x,t,T_{M+1}(u_n),\nabla T_{M+1}(u_n))\\
&\quad-a(x,t,T_{M+1}(u_n),\nabla T_{M+1}(u)\chi_s)\big)\dot{\nabla} T_{M+1}(u)\chi_s\,dx\,dt\\
&\quad+\int_{\{m\le|u|\le m+1\}} a(x,t,T_{M+1}(u_n),\nabla T_{M+1}(u)\chi_s)\cdot\nabla T_{M+1}(u_n)\,dx\,dt\,.
\end{aligned}$$

Then, by (6.60), (6.61) and the fact that $a(x,t,T_{M+1}(u_n),\nabla T_{M+1}(u_n))$ converges weak star to $a(x,t,T_{M+1}(u),\nabla T_{M+1}(u))$ and the strong convergence of $a(x,t,T_{M+1}(u_n),\nabla T_{M+1}(u)\chi_s)$ to $a(x,t,T_{M+1}(u),\nabla T_{M+1}(u)\chi_s)$, we get

$$\begin{aligned}
&\limsup_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt\\
&\qquad=\int_{\{m\le|u|\le m+1\}} a(x,t,u,\nabla u)\cdot\nabla u\,dx\,dt\,,
\end{aligned}\tag{6.66}$$

and then by Lemma 5.1 of [28] the condition (6.11) is fulfill.

Finally, we show that u fulfills the condition (6.12).

Let S be an increasing function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support and $M>0$ such that $\operatorname{supp}(S')\subset[-M,M]$.

Let $v\in W_0^{1,x}L_\varphi(Q)\cap L^\infty(Q)$ such that, $\dfrac{\partial v}{\partial t}\in W^{-1,x}L_\psi(Q)$. Using $S'(u_n-v)T_k(u_n)$ as test function in (6.17) the by using integration by parts, we get

$$\begin{aligned}
&\int_Q\int_0^{u_n}\frac{\partial b_n(x,r)}{\partial r}S'(r-v)T_k(r)\,dr\,dx\,dt\\
&\qquad+\int_0^T\int_0^t\Big\langle\frac{\partial v}{\partial\sigma},\int_0^{u_n}\frac{\partial b_n(x,r)}{\partial r}S''(r-v)T_k(r)dr\Big\rangle\,d\sigma\,dt\\
&\qquad+\int_Q\int_0^t a(x,\sigma,u_n,\nabla u_n)\cdot\nabla T_k(u_n)S'(u_n-v)\,d\sigma\,dx\,dt\\
&\qquad+\int_Q\int_0^t S''(u_n-v)a(x,\sigma,u_n,\nabla u_n)\cdot\nabla(u_n-v)T_k(u_n)\,d\sigma\,dx\,dt\\
&\qquad+\int_Q\int_0^t g_n(x,\sigma,u_n,\nabla u_n)S'(u_n-v)T_k(u_n)\,d\sigma\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_Q \int_0^t f_n S'(u_n - v) T_k(u_n) d\sigma\, dx\, dt + \int_Q \int_0^t F \cdot \nabla T_k(u_n) S'(u_n - v) d\sigma\, dx\, dt \\
&\quad + \int_Q \int_0^t F \cdot \nabla (u_n - v) S''(u_n - v) T_k(u_n)\, d\sigma\, dx\, dt \\
&\quad + T \int_\Omega \int_0^{u_{0n}} \frac{\partial b_n(x,r)}{\partial r} S'(r - v(0)) T_k(r)\, dr\, dx\,. \qquad (6.67)
\end{aligned}
$$

Now, we pass to the limit in each term of (6.67) as n tends to infinity.

- Since S is bounded and continuous, one has

$$
\int_Q \int_0^{u_n} \frac{\partial b_n(x,r)}{\partial r} S'(r - v) T_k(r)\, dr\, dx\, dt = \int_Q \int_0^{u} \frac{\partial b(x,r)}{\partial r} S'(r - v) T_k(r)\, dr\, dx\, dt + \varepsilon(n)\,,
$$

and, $\int_0^{u_n} \frac{\partial b_n(x,r)}{\partial r} S''(r - v) T_k(r)\, dr$ tends to $\int_0^{u} \frac{\partial b(x,r)}{\partial r} S''(r - v) T_k(r)\, dr$ a.e. in Q and weakly in $W_0^{1,x} L_\varphi(Q)$ and L^∞ weak $*$, and , $\int_0^{u_{0n}} \frac{\partial b_n(x,r)}{\partial r} S'(r - v) T_k(r)\, dr$ tends to $\int_0^{u_0} \frac{\partial b(x,r)}{\partial r} S'(r - v) T_k(r) dr + \varepsilon(n)$ a.e. in Ω and L^∞ weak $*$, then

$$
\begin{aligned}
&\int_0^T \int_0^t \Big\langle \frac{\partial v}{\partial \sigma}, \int_0^{u_n} \frac{\partial b_n(x,r)}{\partial r} S''(r - v) T_k(r) dr \Big\rangle\, d\sigma\, dt \\
&\quad = \int_0^T \int_0^t \Big\langle \frac{\partial v}{\partial \sigma}, \int_0^{u} \frac{\partial b(x,r)}{\partial r} S''(r - v) T_k(r) dr \Big\rangle\, d\sigma\, dt + \varepsilon(n)\,, \qquad (6.68)
\end{aligned}
$$

and,

$$
\begin{aligned}
&\int_\Omega \int_0^{u_{0n}} \frac{\partial b_n(x,r)}{\partial r} S'(r - v(0)) T_k(r)\, dr\, dx \\
&\quad = \int_\Omega \int_0^{u_0} \frac{\partial b(x,r)}{\partial r} S'(r - v(0)) T_k(r)\, dr\, dx + \varepsilon(n)\,. \qquad (6.69)
\end{aligned}
$$

- Since $\operatorname{supp}(S') \subset [-M, M]$ and since for $n \geq k$, one has

$$
S'(u_n - v) a(x, \sigma, u_n, \nabla u_n) \nabla T_k(u_n) = S'(u_n - v) a(x, \sigma, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n)
$$

a.e. in Q. Thus, the almost everywhere convergence of ∇u_n to ∇u and the bounded character of S' permit us to conclude that $S'(u_n - v) a(x, \sigma, u_n, \nabla u_n) \nabla T_k(u_n)$ tends to $S'(u - v) a(x, \sigma, T_k(u), \nabla T_k(u)) \nabla T_k(u)$ weak star in $(L_\psi(Q))^N$, for the topology $\sigma(\Pi L_\psi, \Pi E_\varphi)$, as n tends to infinity, which yields, by using the modular convergence of $T_k(u_n)$ in $W_0^{1,x} L_\varphi(Q)$

$$
\begin{aligned}
&\int_Q \int_0^t a(x, \sigma, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) S'(u_n - v)\, d\sigma\, dx\, dt \\
&\quad = \int_Q \int_0^t a(x, \sigma, T_k(u), \nabla T_k(u)) \cdot \nabla T_k(u) S'(u - v)\, d\sigma\, dx\, dt + \varepsilon(n)\,,
\end{aligned}
$$

and,

$$\begin{aligned}&\int_Q\int_0^t S''(u_n-v)a\big(x,\sigma,T_{M+\|v\|_\infty}(u_n),\nabla T_{M+\|v\|_\infty}(u_n)\big)\\&\qquad\cdot\nabla(T_{M+\|v\|_\infty}(u_n)-v)T_k(u_n)\,d\sigma\,dx\,dt\\&\quad=\int_Q\int_0^t S''(u-v)a\big(x,\sigma,T_{M+\|v\|_\infty}(u),\nabla T_{M+\|v\|_\infty}(u)\big)\\&\qquad\cdot\nabla(T_{M+\|v\|_\infty}(u)-v)T_k(u)\,d\sigma\,dx\,dt+\varepsilon(n)\,.\end{aligned}$$

- Since $g_n(x,\sigma,u_n,\nabla u_n)$ converges strongly to $g(x,\sigma,u,\nabla u)$ in $L^1(Q)$, and the bounded character to S', one has

$$g_n(x,\sigma,u_n,\nabla u_n)S'(u_n-v)\longrightarrow g(x,\sigma,u,\nabla u)\text{ strongly in }L^1(Q),\ \text{ as }n\longrightarrow+\infty\,.$$

and since $T_k(u_n)$ converges to $T_k(u)$ weak star in $L^\infty(Q)$, then

$$\begin{aligned}&\int_Q\int_0^t g_n(x,\sigma,u_n,\nabla u_n)S'(u_n-v)T_k(u_n)\,d\sigma\,dx\,dt\\&\quad=\int_Q\int_0^t g(x,\sigma,u,\nabla u)S'(u-v)T_k(u)\,d\sigma\,dx\,dt+\varepsilon(n)\,.\end{aligned}\tag{6.70}$$

- Due to the strong convergence of $(f_n)_n$ to f in $L^1(Q)$ and weak star convergence of $T_k(u_n)$ to $T_k(u)$ in $L^\infty(Q)$ and since S' is bounded and $(u_n)_n$ converges to u almost everywhere in Q, we get

$$\int_Q\int_0^t f_nS'(u_n-v)T_k(u_n)\,d\sigma\,dx\,dt=\int_Q\int_0^t fS'(u-v)T_k(u)\,d\sigma\,dx\,dt\,.\tag{6.71}$$

- Similarly as above, we get

$$\begin{aligned}&\int_Q\int_0^t F\cdot\nabla T_k(u_n)S'(u_n-v)\,d\sigma\,dx\,dt\\&\quad=\int_Q\int_0^t F\cdot\nabla T_k(u)S'(u-v)\,d\sigma\,dx\,dt+\varepsilon(n)\,,\end{aligned}\tag{6.72}$$

and,

$$\begin{aligned}&\int_Q\int_0^t F\cdot\nabla(u_n-v)S''(u_n-v)T_k(u_n)\,d\sigma\,dx\,dt\\&\quad=\int_Q\int_0^t F\cdot\nabla(u-v)S''(u-v)T_k(u)\,d\sigma\,dx\,dt+\varepsilon(n)\,.\end{aligned}\tag{6.73}$$

Consequently, combining (6.67)–(6.73) we conclude that (6.12) is fulfill. Which means that u is an entropy solution of (6.9) in the sense of Definition 6.1. This completes the proof of the Theorem 6.1. □

Example 6.1. Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$ and $T > 0$ we denote by $Q = \Omega \times [0, T]$.

Let φ and ψ two complementary Musielak functions which satisfy the assumptions of lemma 3.1, moreover we assume that $\varphi(x, t)$ decreases with respect to one of coordinates of x.

We set

$$b(x,s) = \begin{cases} h(x)(m(1)s - M(s)) & \text{if } |s| \le 1\,, \\ h(x)(m(1) - M(1))s & \text{if } |s| > 1\,, \end{cases}$$

where M is an Orlicz function such that $M(t) \le \operatorname{ess}\inf_{x\in\Omega} \varphi(x,t)$, m is the right derivative of M and $h \in W^{1,x}E_\varphi(\Omega) \cap L^\infty(\Omega)$ such that $\inf_{x\in\Omega} h(x) > c_1 > 0$.
Then, one has

$$\frac{\partial b(x,s)}{\partial s} = \begin{cases} h(x)(m(1) - m(s)) & \text{if } |s| \le 1\,, \\ h(x)(m(1) - M(1)) & \text{if } |s| > 1\,, \end{cases}$$

which means that $b(x, \cdot)$ is an inceasing $\mathcal{C}^1$ function with $b(x, 0) = 0$. and for all $k > 0$ and all $|s| \le k$

$$c_1 m(1) \le \frac{\partial b(x,s)}{\partial s} \le \big(m(1) + \max(m(k), M(1))\big)h(x) \in L^\infty(\Omega)\,,$$

and

$$\left|\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\right| \le \big|\big(m(1) + \max(m(k), M(1))\big)\nabla_x h(x)\big| \in E_\varphi(\Omega)$$

and we denote by

$$a(x,t,s,\zeta) = h_1(x,t)(3 + \sin^2(\varphi(x,|s|)))\psi_x^{-1}(\varphi(x,|\zeta|))\frac{\zeta}{|\zeta|}\,,$$
$$g(x,t,s,\zeta) = h_2(x,t)\sin(s)\exp(-\sigma s^2)\varphi(x,|\zeta|)\,,$$

where $h_1, h_2 \in L^\infty(Q)$ $\sigma > 0$.
Thus, for all $f \in L^1(Q)$ and all $F \in (E_\varphi(Q))^N$ the following problem has at least

one solution

$$
\begin{cases}
T_k(u) \in W_0^{1,x}L_\varphi(Q) \quad \forall k>0,\\
b(x,u) \in L^\infty([0,T],L^1(\Omega)),\\
\lim\limits_{m\to+\infty} \displaystyle\int_{\{m\le|u|\le m+1\}} (3+\sin^2(\varphi(x,|u|)))\psi_x^{-1}\varphi(x,|\nabla u|))|\nabla u|\,dx\,dt = 0,\\
\displaystyle\int_Q\int_0^u h(x)(m(1)-m(r))S'(r-v)T_k(r)\chi_{\{|r|\le 1\}}\,dr\,dx\,dt\\
+(m(1)-M(1))\displaystyle\int_Q\int_0^u h(x)S'(r-v)T_k(r)\chi_{\{|r|>1\}}\,dr\,dx\,dt\\
+\displaystyle\int_0^T\int_0^t \left\langle \frac{\partial v}{\partial\sigma}, \int_0^u h(x)(m(1)-m(r))S''(r-v)T_k(r)\chi_{\{|r|\le1\}}dx\right\rangle d\sigma\,dt\\
+\displaystyle\int_0^T\int_0^t \left\langle \frac{\partial v}{\partial\sigma}, (m(1)-M(1))\int_0^u h(x)S''(r-v)T_k(r)\chi_{\{|r|>1\}}\,dx\right\rangle d\sigma\,dt\\
+\displaystyle\int_Q\int_0^t h(x,\sigma)(3+\sin^2(\varphi(x,|u|)))\psi_x^{-1}(\varphi(x,|\nabla u|))\frac{\nabla u}{|\nabla u|}\\
\qquad\cdot\nabla T_k(u)S'(u-v)\,d\sigma\,dx\,dt\\
+\displaystyle\int_Q\int_0^t S''(u-v)h(x,\sigma)(3+\sin^2(\varphi(x,|u|)))\psi_x^{-1}(\varphi(x,|\nabla u|))\frac{\nabla u}{|\nabla u|}\\
\qquad\cdot(\nabla u-\nabla v)T_k(u)\,d\sigma\,dx\,dt\\
+\displaystyle\int_Q\int_0^t h_2(x,t)\sin(u)\exp(-\sigma u^2)\varphi(x,|\nabla u|)S'(u-v)T_k(u)\,d\sigma\,dx\,dt\\
\le \displaystyle\int_Q fS'(u-v)T_k(u)\,dx\,dt + \int_Q\int_0^t S'(u-v)F\cdot\nabla T_k(u)\,dx\,dt\\
+\displaystyle\int_q\int_0^t S''(u-v)F\cdot\nabla(u-v)T_k(u)\,dx\,dt + T\int_Q\int_\Omega\int_0^{u_0} h(x)(m(1)-m(r))\\
\qquad\times S'(r-v(0))T_k(r)\chi_{\{|r|\le1\}}\,dr\,dx\,dt\\
+T(m(1)-M(1))\displaystyle\int_Q\int_\Omega\int_0^{u_0} h(x)S'(r-v(0))T_k(r)\chi_{\{|r|\le1\}}\,dr\,dx\,dt\,,
\end{cases}
$$

for every $k>0$ and all $v \in W_0^{1,x}L_\varphi(Q)\cap L^\infty(Q)$ such that $\dfrac{\partial v}{\partial t} \in W^{-1,x}L_\psi(Q) + L^1(Q)$ and all increasing function $S \in W^{2,\infty}(\mathbb{R})$ with S' has a compact support in $\mathbb{R}$.

It remains to show that u satisfies the initial condition.

To this end, remark that $\dfrac{\partial b(x,u_n)}{\partial t}$ is bounded in $W^{-1,x}L_\psi(Q)+L^1(Q)$. As a consequence of Aubin's lemma (see, e.g, [29], Corollary 4) and (see also Lemma 4.1) $b_n(x,u_n)$ lies in $\mathcal{C}([0,T],L^1(\Omega))$. It follows that, on one hand, $b_n(x,u_n(x,0)) = b_n(x,u_{0n}(x))$ for all $x\in\Omega$ which converges to $b(x,u_0(x))$ strongly in $L^1(\Omega)$.

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Corresponding author:
Mohamed Saad Bouh ELEMINE VALL,
University of Sidi Mohamed Ibn Abdellah,
Faculty of Sciences Dhar El Mahraz,
Laboratory LAMA, Department of Mathematics,
B.P. 1796 Atlas Fez, Morocco
E-mail: saad2012bouh@gmail.com ahmedmath2001@gmail.com
atouzani07@gmail.com abd.benkirane@gmail.com