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NOTES ON COMMUTATOR ON THE VARIABLE EXPONENT LEBESGUE SPACES

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Abstract. We obtain the factorization theorem for Hardy space via the variable exponent Lebesgue spaces. As an application, it is proved that if the commutator of Coifman, Rochberg and Weiss $[b,T]$ is bounded on the variable exponent Lebesgue spaces, then b is a bounded mean oscillation (BMO) function.

Keywords: bounded mean oscillation; commutator; Hardy space; variable exponent Lebesgue space

MSC 2010: 42B20, 47B07

1. Introduction

Let T be a Calderón-Zygmund operator defined by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y) f(y) \, \mathrm{d}y,$$

where the kernel $K(x) = \Omega(x)/|x|^n$ satisfies the following conditions:

(i) Ω is homogeneous of degree zero on $\mathbb{R}^n$, i.e., $\Omega(\lambda x) = \Omega(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$;

(ii) $\Omega \in C^\infty(\mathbb{S}^{n-1})$ and $\int_{\mathbb{S}^{n-1}} \Omega(x) \, \mathrm{d}x = 0$.

A locally integrable function b belongs to the BMO space if b satisfies

$$\|b\|_* := \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q| \, \mathrm{d}x < \infty,$$

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where $b_Q := |Q|^{-1}\int_Q b(x)\,\mathrm{d}x$ and the supremum is taken over all cubes Q in $\mathbb{R}^n$. A well known result of Coifman, Rochberg and Weiss [3] states that the commutator

$$[b,T](f) := bT(f) - T(bf)$$

is bounded on some L^p, $1 < p < \infty$, if and only if $b \in \mathrm{BMO}$.

In 2003, Cruz-Uribe, Fiorenza, Martell and Pérez in [4] showed that if $b \in \mathrm{BMO}$, then $[b,T]$ is bounded on a variable exponent Lebesgue space. It is natural to ask whether the converse of their theorem is true.

In this paper we give an affirmative answer to this problem. After we finished writing this paper, we found that Chaffee and Cruz-Uribe in [2] had solved this problem. Morevover, they obtained the results using much weaker hypotheses on the exponent $p(\cdot)$ and on the operator T, and they hold for a large family of Banach function spaces and not just variable Lebesgue spaces. However, one advantage of the approach taken in this paper is that it provides a constructive algorithm producing the weak factorization of Hardy spaces in terms of the variable exponent Lebesgue spaces.

2. Preliminaries and main results

Given a measurable function $p\colon \mathbb{R}^n \to [1,\infty)$, $L^{p(\cdot)}(\mathbb{R}^n)$ denotes the set of measurable functions f such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n}\Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)}\,\mathrm{d}x < \infty.$$

$L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space when equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\Big\{\lambda > 0\colon \int_{\mathbb{R}^n}\Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)}\,\mathrm{d}x \leqslant 1\Big\}.$$

Such space is called a *variable Lebesgue space*, since it is generalized from the standard Lebesgue space. Variable Lebesgue spaces have become one of the most important function spaces due to the fundamental paper by Kováčik and Rákosník, see [8]. Recently, Cruz-Uribe, Fiorenza, Martell and Pérez in [4] proved that many classical operators in harmonic analysis, such as maximal operators, singular integrals, commutators and fractional integrals are bounded on the variable Lebesgue space.

Define $\mathcal{P}(\mathbb{R}^n)$ to be the set of $p(\cdot)\colon \mathbb{R}^n \to [1,\infty)$ such that

$$p^- = \operatorname{ess\,inf}\{p(x)\colon x \in \mathbb{R}^n\} > 1, \quad p^+ = \operatorname{ess\,sup}\{p(x)\colon x \in \mathbb{R}^n\} < \infty.$$

Denote $p'(x) = p(x)/(p(x)-1)$. Let $\mathcal{B}(\mathbb{R}^n)$ be the set of $p(\cdot) \in \mathbb{R}^n$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

For variable Lebesgue spaces there are some important lemmas as follows.

Lemma 2.1 ([1]). *If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfies*

$$|p(x) - p(y)| \leqslant \frac{C}{-\log|x-y|}, \quad |x-y| < \frac{1}{2} \tag{2.1}$$

and

$$|p(x) - p(y)| \leqslant \frac{C}{\log(|x| + \mathrm{e})}, \quad |y| \geqslant |x|, \tag{2.2}$$

then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$.

Lemma 2.2 ([8]). *Let $\mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,\mathrm{d}x \leqslant r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where $r_p = 1 + 1/p^- - 1/p^+$.

Lemma 2.3 ([5]). *Let $\mathcal{P}(\mathbb{R}^n)$ satisfy conditions* (2.1) *and* (2.2) *in Lemma* 2.1. *Then*

$$\|\chi_Q\|_{L^{p(\cdot)}} \approx \begin{cases} |Q|^{1/p(x)} & \text{if } |Q| \leqslant 2^n \text{ and } x \in Q, \\ |Q|^{1/p_\infty} & \text{if } |Q| \geqslant 1 \end{cases}$$

for every cube (or ball) $Q \subset \mathbb{R}^n$, where $p_\infty = \lim\limits_{x\to\infty} p(x)$.

Next, we recall a technical lemma about certain $H^1(\mathbb{R}^n)$ functions.

Lemma 2.4 ([9]). *Suppose f is a function defined on $\mathbb{R}^n$ satisfying $\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = 0$, and $|f(x)| \leqslant \chi_{B(x_0,1)}(x) + \chi_{B(y_0,1)}(x)$, where $|x_0 - y_0| = M > 10$. Then we have*

$$\|f\|_{H^1(\mathbb{R}^n)} \leqslant C \log M.$$

We also define that T is homogeneous if the kernel K satisfies

$$K(x-y) \geqslant \frac{C}{M^n} \tag{2.3}$$

for disjoint balls $B_0 = B(x,r)$ and $B_1(y,r)$ satisfying $|x-y| \approx Mr$, where $r > 0$ and $M > 1$.

Remark 2.1. If $T = 0$, it is easy to see that $[b, T]$ is bounded on the variable Lebesgue space for any locally function b, so we need to assume some homogeneous condition for T. The condition (2.3) is natural. The Riesz transforms and the Hilbert transform satisfy (2.3).

Our main result is the following factorization theorem for $H^1(\mathbb{R}^n)$.

Theorem 2.1. *Suppose that* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfies conditions* (2.1) *and* (2.2) *in Lemma* 2.1 *and* T *is a homogeneous Calderón-Zygmund operator. Then for every* $f \in H^1(\mathbb{R}^n)$*, there exist sequences* $\{\lambda_s^k\} \in l^1$ *and functions* $g_s^k \in L^{p'(\cdot)}(\mathbb{R}^n)$, $h_s^k \in L^{p(\cdot)}(\mathbb{R}^n)$ *such that*

$$f = \sum_{k=1}^{\infty}\sum_{s=1}^{\infty} \lambda_s^k (h_s^k T^*(g_s^k) - g_s^k T(h_s^k)) \tag{2.4}$$

in the sense of $H^1(\mathbb{R}^n)$*, where* T^* *is the adjoint operator of* T*. Moreover,*

$$\|f\|_{H^1(\mathbb{R}^n)} \approx \inf \sum_{k=1}^{\infty}\sum_{s=1}^{\infty} |\lambda_s^k| \|g_s^k\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|h_s^k\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where the infimum above is taken over all possible representations of f *that satisfy* (2.4).

By Theorem 2.1, we conclude:

Theorem 2.2. *Suppose that* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *satisfies conditions* (2.1) *and* (2.2) *in Lemma* 2.1 *and* T *is a homogeneous Calderón-Zygmund operator. If the commutator* $[b, T]$ *is bounded on* $L^{p(\cdot)}$*, then* $b \in \mathrm{BMO}$.

3. Proofs of Theorem 2.1 and Theorem 2.2

We now proceed with proofs of Theorem 2.1 and Theorem 2.2.

Proof of Theorem 2.1. We apply the method due to Coifman, Rochberg and Weiss in [3], see also [7] or [10], which is different from that applied by Janson in [6]. Note that for any $g \in L^{p'(\cdot)}(\mathbb{R}^n)$ and $h \in L^{p(\cdot)}(\mathbb{R}^n)$, we have $hT^*(g) - gT(h) \in L^1(\mathbb{R}^n)$ by Lemma 2.2 and the boundedness of T on the variable Lebesgue space, see [4]. Moreover, we get

$$\int_{\mathbb{R}^n} (h(x)T^*(g)(x) - g(x)T(h)(x))\,\mathrm{d}x = 0.$$

For any $b \in \mathrm{BMO}$, we conclude that

$$\left|\int_{\mathbb{R}^n} b(x)(h(x)T^*(g)(x) - g(x)T(h)(x))\,\mathrm{d}x\right|$$
$$= \left|\int_{\mathbb{R}^n} g(x)[b,T](h)(x)\,\mathrm{d}x\right| \leqslant C\|h\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|b\|_*.$$

Therefore, $(hT^*(g) - gT(h))$ is in $H^1(\mathbb{R}^n)$ with

$$\|hT^*(g) - gT(h)\|_{H^1(\mathbb{R}^n)} \leqslant C\|h\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

It is immediate that for any representation of f,

$$f = \sum_{k=1}^{\infty}\sum_{s=1}^{\infty} \lambda_s^k(h_s^k T^*(g_s^k) - g_s^k T(h_s^k)),$$

we have that $\|f\|_{H^1(\mathbb{R}^n)}$ is bounded by

$$C \inf \sum_{k=1}^{\infty}\sum_{s=1}^{\infty} |\lambda_s^k|\|h_s^k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|g_s^k\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

We turn to showing that the reverse inequality holds and that it is possible to obtain such a decomposition for any $f \in H^1(\mathbb{R}^n)$.

We first show that for every $H^1(\mathbb{R}^n)$-atom $a(x)$ and for all $\varepsilon > 0$ and for all $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying conditions (2.1) and (2.2), there exist $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, $h \in L^{p(\cdot)}(\mathbb{R}^n)$ and a large positive number M (depending only on ε) such that

$$\|a - \big(hT^*(g) - gT(h)\big)\|_{H^1(\mathbb{R}^n)} < \varepsilon$$

and that $\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|h\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant CM^n$, where C is an absolute positive constant.

Let $a(x)$ be an $H^1(\mathbb{R}^n)$-atom supported in $B(x_0, r)$, satisfying

$$\int_{\mathbb{R}^n} a(x)\,\mathrm{d}x = 0 \quad \text{and} \quad \|a\|_{L^\infty(\mathbb{R}^n)} \leqslant r^n.$$

Fix $\varepsilon > 0$, choose M, a large integer which we shall determine later. Select $y_0 \in \mathbb{R}^n$ so that $y_{0,i} - x_{0,i} = Mr/\sqrt{n}$ with $|y_{0,i}| \geqslant |x_{0,i}|$, where $x_{0,i}$, $y_{0,i}$ are, respectively, the ith coordinates of x_0, y_0 for $i = 1, 2, \ldots, n$. Then

$$|x_0 - y_0| = Mr \quad \text{and} \quad |y_0| \geqslant |x_0|. \tag{3.1}$$

We then set

$$g(x) = \chi_{B(y_0,r)}(x), \quad h(x) = \frac{a(x)}{T^*(g)(x_0)}.$$

By (2.3), we have that there exists a positive constant C such that

$$|T^*(g)(x_0)| \geqslant CM^{-n}.$$

From the definitions of functions g and f we obtain that $\operatorname{supp} g \subset B(y_0, r)$ and $\operatorname{supp} h \subset B(x_0, r)$. Moreover,

$$\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)} = \|\chi_{B(y_0,r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \quad \text{and} \quad \|h\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant CM^n r^{-n} \|\chi_{B(x_0,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

If $|B(y_0, r)| = |B(x_0, r)| > 1$, by Lemma 2.3 we have

$$\begin{aligned}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|h\|_{L^{p(\cdot)}(\mathbb{R}^n)} &= \|\chi_{B(y_0,r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x_0,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leqslant CM^n r^{-n}|B(y_0,r)|^{1/p'_\infty}|B(x_0,r)|^{1/p_\infty} \leqslant CM^n,\end{aligned}$$

where

$$p'_\infty = \lim_{x\to\infty} p'(x) = \lim_{x\to\infty}\Big(1 - \frac{1}{p(x)-1}\Big) = 1 - \frac{1}{p_\infty - 1}.$$

If $|B(y_0, r)| = |B(x_0, r)| < 2^n$, Lemma 2.3 and (3.1) imply that

$$\begin{aligned}\|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|h\|_{L^{p(\cdot)}(\mathbb{R}^n)} &= \|\chi_{B(y_0,r)}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_{B(x_0,r)}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leqslant CM^n r^{-n}|B(y_0,r)|^{1/p'(y_0)}|B(x_0,r)|^{1/p(x_0)} \\ &\leqslant CM^n|B(x_0,r)|^{1/p(x_0)-1/p(y_0)} \leqslant CM^n,\end{aligned}$$

where in the last inequality we use the fact that

$$\frac{1}{p(x_0)} - \frac{1}{p(y_0)} = \frac{p(y_0)-p(x_0)}{p(x_0)p(y_0)} \leqslant \frac{C}{(p^-)^2}$$

and $|B(x_0, r)| < 2^n$.

Next, we have

$$\begin{aligned}&a(x) - (h(x)T^*(g)(x) - g(x)T(h)(x)) \\ &\qquad = \frac{a(x)(T^*(g)(x_0) - T^*(g)(x))}{T^*(g)(x_0)} - g(x)T(h)(x) = I + II.\end{aligned}$$

We first estimate I. For $x \in B(x_0, r)$,

$$\begin{aligned}|I| &\leqslant \frac{|a(x)|}{T^*(g)(x_0)}\int_{B(y_0,r)} |K(x_0-y) - K(x-y)|\,\mathrm{d}y \\ &\leqslant CM^n r^{-n}\int_{B(y_0,r)} \frac{|x-x_0|}{|x_0-y|^{n+1}}\,\mathrm{d}y \leqslant CM^n r^{-n}\frac{r}{(Mr)^{n+1}}r^n \leqslant \frac{C}{Mr^n}.\end{aligned}$$

Hence we conclude that $|I| \leqslant (1/Mr^n)\chi_{B(x_0,r)}(x)$.

For II, from the definitions of g and h and the fact that $\int_{\mathbb{R}^n} a(x)\,\mathrm{d}x = 0$, it follows that for any $x \in B(y_0, r)$,

$$|T(h)(x)| \leqslant \frac{T(a)(x)}{T^*(g)(x_0)} \leqslant CM^n \int_{B(x_0,r)} \Big|K(y_0 - y) - K(x - y)\Big| |a(y)|\,\mathrm{d}y \leqslant \frac{C}{Mr^n}.$$

Thus,

$$|II| \leqslant \frac{1}{Mr^n} \chi_{B(y_0,r)}(x)$$

which yields that

$$|a(x) - (h(x)T^*(g)(x) - g(x)T(h)(x))| \leqslant \frac{C}{Mr^n}(\chi_{B(x_0,r)}(x) + \chi_{B(y_0,r)}(x)). \tag{3.2}$$

Since

$$\int_{\mathbb{R}^n} a(x) - (h(x)T^*(g)(x) - g(x)T(h)(x))\,\mathrm{d}x = 0,$$

and for M sufficiently large such that

$$\frac{C \log M}{M} < \varepsilon,$$

from Lemma 2.4 we get

$$\|a - (hT^*(g) - gT(h))\|_{H^1(\mathbb{R}^n)} < \varepsilon.$$

Now we return to the proof. For any $f \in H^1(\mathbb{R}^n)$ we can find a sequence $\{\lambda_s^1\} \in l^1$ and a sequence of $H^1(\mathbb{R}^n)$-atoms $\{a_s^1\}$ such that $f = \sum\limits_{s=1}^{\infty} \lambda_s^1 a_s^1$ and $\sum\limits_{s=1}^{\infty} |\lambda_s^1| \leqslant C_0 \|f\|_{H^1(\mathbb{R}^n)}$.

Let $\varepsilon \in (0, 1/C_0)$. For each atom a_s^1, there exist $g_s^1 \in L^{p'(\cdot)}(\mathbb{R}^n)$ and $h_s^1 \in L^{p'(\cdot)}(\mathbb{R}^n)$ such that

$$\|a_s^1 - (h_s^1 T^*(g_s^1) - g_s^1 T(h_s^1))\|_{H^1(\mathbb{R}^n)} < \varepsilon.$$

We have

$$\begin{aligned} f = \sum_{s=1}^{\infty} \lambda_s^1 a_s^1 &= \sum_{s=1}^{\infty} \lambda_s^1 (h_s^1 T^*(g_s^1) - g_s^1(x) T(h_s^1)) + \sum_{s=1}^{\infty} \lambda_s^1 (a_s^1 - h_s^1 T^*(g_s^1) - g_s^1 T(h_s^1)) \\ &= F_1 + G_1. \end{aligned}$$

Observe that

$$\|G_1\|_{H^1(\mathbb{R}^n)} \leqslant \sum_{s=1}^{\infty} |\lambda_s^1| \|a_s^1 - h_s^1 T^*(g_s^1)) - g_s^1 T(h_s^1)\|_{H^1(\mathbb{R}^n)} \leqslant \varepsilon C_0 \|f\|_{H^1(\mathbb{R}^n)}.$$

Repeating this construction on the function G_1 we obtain that

$$G_1 = \sum_{s=1}^{\infty} \lambda_s^2 a_s^2 = \sum_{s=1}^{\infty} \lambda_s^2 (h_s^2 T^*(g_s^2) - g_s^2 T(h_s^2)) + \sum_{s=1}^{\infty} \lambda_s^2 (a_s^2 - h_s^2 T^*(g) - g_s^2 T(h_s^2))$$
$$= F_2 + G_2$$

and $\|G_2\|_{H^1(\mathbb{R}^n)} \leqslant (\varepsilon C_0)^2 \|f\|_{H^1(\mathbb{R}^n)}$.

Continuing this process indefinitely, we obtain for any K

$$f = \sum_{k=1}^{K} F_k + E_K = \sum_{k=1}^{K} \sum_{s=1}^{\infty} \lambda_s^k (h_s^k T^*(g_s^k) - g_s^k T(h_s^k)) + G_K,$$

where $\|G_K\|_{H^1(\mathbb{R}^n)} \leqslant (\varepsilon C_0)^K \|f\|_{H^1(\mathbb{R}^n)}$. Letting $K \to \infty$ gives the desired decomposition of

$$f = \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \lambda_s^k (h_s^k T^*(g_s^k) - g_s^k T(h_s^k))$$

and

$$\sum_{k=1}^{\infty} \sum_{s=1}^{\infty} |\lambda_s^k| \leqslant \sum_{k=1}^{\infty} (\varepsilon C_0)^k \|f\|_{H^1(\mathbb{R}^n)} \leqslant \frac{C_0}{1 - \varepsilon C_0} \|f\|_{H^1(\mathbb{R}^n)}.$$

Thus, we have completed the proof of Theorem 2.1. □

P r o o f of Theorem 2.2. For $f \in H^1(\mathbb{R}^n)$, by Theorem 2.1 and the $L^{p(\cdot)}(\mathbb{R}^n)$ boundedness of $[b, T]$, we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^n} b(x) f(x) \, \mathrm{d}x \right| &= \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} |\lambda_s^k| \left| \int_{\mathbb{R}^n} b(x) h_s^k(x) T^*(g_s^k)(x) - b(x) g_s^k(x) T(h_s^k)(x) \, \mathrm{d}x \right| \\ &= \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} |\lambda_s^k| \left| \int_{\mathbb{R}^n} g_s^k(x) [b, T](h_s^k)(x) \, \mathrm{d}x \right| \\ &\leqslant C \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} |\lambda_s^k| \|g_s^k\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \|h_s^k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} |\lambda_s^k|. \end{aligned}$$

From the duality theorem between $H^1(\mathbb{R}^n)$ and BMO, it follows that $b \in \mathrm{BMO}$. □

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