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Czechoslovak Mathematical Journal, Vol. 69 (2019), No. 4, 969–981

Persistent URL: `http://dml.cz/dmlcz/147907`

Czechoslovak Mathematical Journal, 69 (144) (2019), 969–981

SHARP EIGENVALUE ESTIMATES OF CLOSED H-HYPERSURFACES IN LOCALLY SYMMETRIC SPACES

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Received December 10, 2017. Published online March 28, 2019.

Abstract. The purpose of this article is to obtain sharp estimates for the first eigenvalue of the stability operator of constant mean curvature closed hypersurfaces immersed into locally symmetric Riemannian spaces satisfying suitable curvature conditions (which includes, in particular, a Riemannian space with constant sectional curvature). As an application, we derive a nonexistence result concerning strongly stable hypersurfaces in these ambient spaces.

Keywords: locally symmetric Riemannian space; closed H-hypersurface; strong stability; first stability eigenvalue

MSC 2010: 53C42, 53A10

1. Introduction

Let $\psi\colon \Sigma^n \to \overline{M}^{n+1}$ be a constant mean curvature closed orientable hypersurface immersed into an $(n+1)$-dimensional oriented Riemannian manifold and denote by $\mathrm{d}\Sigma$ the standard volume element of Σ^n with respect to the induced metric from the ambient space. Fixing a unit normal vector field N globally defined on Σ^n, we will denote by A the second fundamental form with respect to N of the immersion and by H its mean curvature, which is given by $H = (\operatorname{tr} A)/n$.

It is well known that every smooth function $f \in C^\infty(\Sigma)$ induces a normal variation ψ_t of the immersion ψ, with variational normal field fN and first variation of the area functional $\mathcal{A}(t) = \int_\Sigma \mathrm{d}\Sigma_t$, where $\mathrm{d}\Sigma_t$ stands for the volume element of Σ^n

The second and fourth authors are partially supported by CNPq, Brazil, grants 303977/2015-9 and 308757/2015-7.

DOI: 10.21136/CMJ.2019.0562-17

with respect to the metric induced by ψ_t, given by

$$\delta_f \mathcal{A} = \frac{\mathrm{d}\mathcal{A}}{\mathrm{d}t}(0) = -n \int_\Sigma H f \,\mathrm{d}\Sigma.$$

As a consequence, minimal hypersurfaces (that is, with zero mean curvature) are characterized as critical points of the area functional $\mathcal{A}$ whereas constant mean curvature hypersurfaces (shortly H-hypersurfaces) are critical points of the area functional restricted to functions $f \in C^\infty(\Sigma)$ which satisfy the additional condition $\int_\Sigma f \,\mathrm{d}\Sigma = 0$. Geometrically, this additional condition means that the variations under consideration preserve a certain volume function.

For such critical points, the stability of the corresponding variational problem is given by the second variation of the area functional

$$\delta_f^2 \mathcal{A} = \frac{\mathrm{d}^2\mathcal{A}}{\mathrm{d}t^2}(0) = -\int_\Sigma f J f \,\mathrm{d}\Sigma$$

with

$$J = \Delta + |A|^2 + \overline{\mathrm{Ric}}(N, N), \tag{1.1}$$

where Δ stands for the Laplacian operator on Σ^n and $\overline{\mathrm{Ric}}$ denotes the Ricci curvature of $\overline{M}^{n+1}$. Let us recall that an H-hypersurface Σ^n is said to be *strongly stable* if $\delta_f^2 \mathcal{A} \geqslant 0$ for every $f \in C^\infty(\Sigma)$. The operator J is called the *Jacobi* or *stability operator* of Σ^n and it is a Schrödinger operator. As it is well known, the spectrum of J,

$$\mathrm{Spec}(J) = \{\lambda_1 < \lambda_2 < \lambda_3 < \ldots\},$$

consists of an increasing sequence of eigenvalues λ_k (with our notation, a real number λ is an eigenvalue of J if and only if $Jf + \lambda f = 0$ on Σ^n for some nonzero smooth function $f \in C^\infty(\Sigma)$). Moreover, the *first stability eigenvalue* λ_1 of Σ^n is simple and satisfies the following min-max characterization:

$$\lambda_1 = \min\left\{-\int_\Sigma f J f \,\mathrm{d}\Sigma \Big/ \int_\Sigma f^2 \,\mathrm{d}\Sigma \colon\ f \in C^\infty(\Sigma),\ f \neq 0\right\}. \tag{1.2}$$

We observe that, in terms of the first stability eigenvalue, a closed H-hypersurface Σ^n is strongly stable if and only if $\lambda_1 \geqslant 0$.

In this setting, in his seminal work (see [14]), Simons established an estimate for the first eigenvalue of J on any compact minimal hypersurface Σ^n in the standard sphere. Specifically, he proved that either $\lambda_1 = -n$ and Σ^n is a totally geodesic sphere, or $\lambda_1 \leqslant -2n$. Later on, Wu in [16] characterized the equality $\lambda_1 = -2n$ by showing that it holds only for the minimal Clifford torus. In the last decade, Perdomo in [13] gave a new proof of this spectral characterization by the first stability eigenvalue. Afterwards, Alías et al. in [2] extended Wu and Perdomo's results to the case of H-hypersurfaces in the standard sphere.

More recently, Velásquez et al. in [15] obtained upper bounds for the first eigenvalue of the stability operator of a closed H-hypersurfaces immersed either in the Euclidean space or in the hyperbolic space. On the other hand, many authors have studied estimates for the first stability eigenvalue in various types of ambient spaces. For instance, Alías et al. in [6] studied this problem in homogeneous Riemannian 3-manifolds and, in particular, in Berger spheres, finding out sharp upper bounds of the first stability eigenvalue λ_1 of compact orientable H-surfaces immersed into such manifolds. In [10], [11], Meroño and Ortiz gave sharp estimates for the first eigenvalue of the stability operator of compact orientable H-surfaces immersed into certain warped products.

Proceeding with the picture described above, in this paper we consider as the ambient spaces the so-called locally symmetric spaces, which are a wide class of Riemannian manifolds and a natural generalization of constant sectional curvature spaces. We recall that a Riemannian manifold is said to be *locally symmetric* when all the covariant derivative components of its curvature tensor vanish identically. In this direction, here we deal with compact orientable H-hypersurfaces immersed into a locally symmetric Riemannian manifold obeying standard curvature constraints. Then, by extending techniques developed in the aforementioned works, our purpose is to obtain sharp estimates for the first stability eigenvalue of such hypersurfaces.

This manuscript is organized in the following way: in Section 2 we introduce some basic facts and notations that will appear in the proof of our results. Finally, in Section 3 we establish our main results concerning upper bounds for the first stability eigenvalue, characterizing the equality by showing that if it holds, then the hypersurface must be either totally umbilical or isometric to an isoparametric hypersurface having two distinct principal curvatures, in the first case, one of them being simple (see Theorem 1) and, in the second case, with multiplicities p and $n-p$ (see Theorem 2).

2. Preliminaries

In this work, we will deal with n-dimensional, orientable and connected hypersurface $\psi\colon \Sigma^n \to \overline{M}^{n+1}$ immersed into an $(n+1)$-dimensional Riemannian manifold $\overline{M}^{n+1}$. We choose a local field of orthonormal frame $\{e_1, \dots, e_{n+1}\}$ in $\overline{M}^{n+1}$ with dual coframe $\{\omega_1, \dots, \omega_{n+1}\}$ such that at each point of Σ^n, $e_1, \dots, e_n$ are tangent to

Σ^n and e_{n+1} is normal to Σ^n. We will use the following convention for the indices:

$$1 \leqslant A, B, C, \ldots \leqslant n+1 \quad \text{and} \quad 1 \leqslant i, j, k, \ldots \leqslant n.$$

In this setting, $\overline{R}_{ABCD}$ and $\overline{R}_{AC}$ denote, respectively, the Riemannian curvature tensor and the Ricci tensor of the Riemannian manifold $\overline{M}^{n+1}$. So, we have

$$\overline{R}_{AC} = \sum_B \overline{R}_{ABCB}.$$

Now, restricting the tensor to Σ^n, we see that $\omega_{n+1} = 0$ on Σ^n. Hence, $0 = \mathrm{d}\omega_{n+1} = -\sum_i \omega_{n+1i} \wedge \omega_i$ and as it is well known, we get

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}.$$

This gives the second fundamental form of Σ^n, $A = \sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1}$ and its square length $|A|^2 = \sum_{i,j} h_{ij}^2$. Furthermore, the mean curvature H of Σ^n is defined by $H = \sum_i h_{ii}/n$.

A well fact known is that the covariant differential of the second fundamental form A of the hypersurface Σ^n can be described in terms of the curvature tensor of the ambient space $\overline{M}^{n+1}$ by the Codazzi equation given by

$$h_{ijk} - h_{ikj} = -\overline{R}_{n+1ijk}, \tag{2.1}$$

where h_{ijk} denote the first covariant derivatives of h_{ij}.

Taking a local orthonormal frame $\{e_1, \ldots, e_n\}$ on Σ^n such that $h_{ij} = \mu_i\delta_{ij}$, the following Simons type formula is well known (see, for instance, equation (2.10) in [8]):

$$\begin{aligned} \frac{1}{2}\Delta|A|^2 = {} & |\nabla A|^2 + \sum_i \mu_i (nH)_{ii} + nH \sum_i \mu_i^3 - |A|^4 \\ & - \sum_{i,j,k} h_{ij}(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j}) \\ & + \sum_i \overline{R}_{(n+1)i(n+1)i}(nH\mu_i - |A|^2) + \sum_{i,j} (\mu_i - \mu_j)^2 \overline{R}_{ijij}. \end{aligned} \tag{2.2}$$

Proceeding within the context above, we will assume that there exist constants c_1 and c_2 such that the sectional curvature $\overline{K}$ of the ambient space $\overline{M}^{n+1}$ satisfies the following two constraints:

$$\overline{K}(\eta, v) = \frac{c_1}{n} \tag{2.3}$$

for vectors $\eta \in T^{\perp}\Sigma$ and $v \in T\Sigma$, and

$$\overline{K}(u,v) \geqslant c_2 \tag{2.4}$$

for vectors $u, v \in T\Sigma$.

From now on, we consider $\overline{M}^{n+1}$ a locally symmetric Riemannian manifold. Recall that a Riemannian manifold is said to be *locally symmetric* when all the covariant derivative components $\overline{R}_{ABCD;E}$ of its curvature tensor vanish identically.

Example 1. Obviously, when the ambient manifold $\overline{M}^{n+1}$ has constant sectional curvature $\bar{c}$, then it is locally symmetric and the curvature conditions (2.3) and (2.4) are satisfied for every hypersurface Σ^n immersed into $\overline{M}^{n+1}$ with $c_1/n = c_2 = \bar{c}$. Therefore, in some sense our assumptions are a natural generalization of the case, where the ambient space has constant sectional curvature. Moreover, when the ambient manifold is a Riemannian product of two Riemannian manifolds of constant sectional curvature, say $\overline{M} = M_1(\kappa_1) \times M_2(\kappa_2)$, then $\overline{M}$ is locally symmetric and if $\kappa_1 = 0$ and $\kappa_2 \geqslant 0$, then every hypersurface of type $\Sigma = \Sigma_1 \times M_2(\kappa_2)$, where Σ_1 is an orientable and connected hypersurface immersed into $M_1(\kappa_1)$, satisfies the curvature constraints (2.3) and (2.4) with $c_1 = c_2 = 0$ (for more details, see [4], Remark 3.1). Moreover, it is not difficult to see that the equality $\overline{R}_{n+1ijk} = 0$ holds on Σ. Then by the Codazzi equation we get that the second fundamental form A of hypersurface Σ must be a Codazzi tensor, that is, the covariant differential ∇A is symmetric in all indices. In particular, this justifies the study of geometry of hypersurfaces such that its second fundamental form is a Codazzi tensor.

Next, given $\Phi_{ij} = h_{ij} - H\delta_{ij}$ we will also consider the following symmetric tensor

$$\Phi = \sum_{i,j} \Phi_{ij}\omega_i \otimes \omega_j.$$

Let $|\Phi|^2 = \sum_{i,j} \Phi_{ij}^2$ be the square of the length of Φ. It is not difficult to check that Φ is traceless and $|\Phi|^2 = |A|^2 - nH^2 \geqslant 0$ with equality if and only if Σ^n is totally umbilical. For that reason, Φ is called the total umbilicity tensor of Σ^n. Moreover, from curvature condition (2.3) we can see that $\overline{R}_{n+1n+1} = c_1$. Then it follows from (1.1) that

$$J = \Delta + |\Phi|^2 + nH^2 + c_1. \tag{2.5}$$

In order to establish our upper bounds for the first stability eigenvalue, we recall two classic algebraic lemmas. The first one is the well known Okumura's lemma due to Okumura in [12], and completed with the equality case proved by Alencar and do Carmo in [1].

Lemma 1. *Let $\kappa_1, \ldots, \kappa_n$, $n \geqslant 3$, be real numbers such that* $\sum\limits_i \kappa_i = 0$ *and* $\sum\limits_i \kappa_i^2 = \beta^2$, *where* $\beta \geqslant 0$. *Then*

$$-\frac{(n-2)}{\sqrt{n(n-1)}}\beta^3 \leqslant \sum_i \kappa_i^3 \leqslant \frac{(n-2)}{\sqrt{n(n-1)}}\beta^3,$$

and the equality holds if and only if at least $n-1$ of the numbers κ_i are equal.

The last auxiliary result established a suitable inequality for the square length of the covariant differential of a symmetric tensor (for more details, see [7], Lemma 1).

Lemma 2. *Let Σ^n be an n-dimensional Riemannian manifold and let $T\colon \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ be a Codazzi tensor on Σ^n such that* $\operatorname{tr}(T) = 0$. *Then*

$$\big|\nabla |T|^2\big|^2 \leqslant \frac{4n}{n+2}|T|^2|\nabla T|^2.$$

3. Upper bounds for the first stability eigenvalue

This section is devoted to establish our main results concerning upper bounds for the first stability eigenvalue of a closed H-hypersurface immersed into a locally symmetric Riemannian manifold. The first one is the following:

Theorem 1. *Let $\psi\colon \Sigma^n \to \overline{M}^{n+1}$ be a closed H-hypersurface immersed in a locally symmetric Riemannian manifold $\overline{M}^{n+1}$, $n \geqslant 2$, satisfying curvature conditions* (2.3) *and* (2.4), *such that its second fundamental form is a Codazzi tensor and $H^2 + c > 0$, where $c = 2c_2 - c_1/n$. Then*
(i) *either $\lambda_1 = -n(H^2 + c_1/n)$ and Σ^n is a totally umbilical hypersurface or*
(ii)

$$\lambda_1 \leqslant -2n(H^2 + c_2) + \frac{n(n-2)}{\sqrt{n(n-1)}}|H| \max_\Sigma |\Phi|.$$

Moreover, if the equality holds (and, in the case $c > 0$, $H \neq 0$), then Σ^n is an isoparametric hypersurface with two distinct principal curvatures one of which is simple.

P r o o f. Firstly, using the constant function $f = 1$, it follows from (1.2) and (2.5) that

$$\lambda_1 \leqslant -n\Big(H^2 + \frac{c_1}{n}\Big) - \frac{1}{\operatorname{vol}(\Sigma)} \int_\Sigma |\Phi|^2 \, d\Sigma \leqslant -n\Big(H^2 + \frac{c_1}{n}\Big)$$

with equality $\lambda_1 = -n(H^2 + c_1/n)$ if and only if Σ^n is a totally umbilical hypersurface.

Now, let us assume that Σ^n is not totally umbilical. For any arbitrary $\varepsilon > 0$ we consider the positive smooth function $f_\varepsilon \in C^\infty(\Sigma)$ defined by

$$f_\varepsilon = \sqrt{\varepsilon + |\Phi|^2}.$$

A straightforward computation shows that

$$f_\varepsilon \Delta f_\varepsilon = \frac{1}{2}\Delta|\Phi|^2 - \frac{\big|\nabla|\Phi|^2\big|^2}{4(\varepsilon + |\Phi|^2)}. \tag{3.1}$$

On the other hand, taking a local orthonormal frame field $\{e_1, \ldots, e_n\}$ in Σ^n such that

$$h_{ij} = \mu_i \delta_{ij} \quad \text{and} \quad \Phi_{ij} = \kappa_i \delta_{ij},$$

we can check that

$$\sum_i \kappa_i = 0, \quad \sum_i \kappa_i^2 = |\Phi|^2 \quad \text{and} \quad \sum_i \mu_i^3 = \sum_i \kappa_i^3 + 3H|\Phi|^2 + nH^3.$$

Since $\overline{M}^{n+1}$ is locally symmetric and Σ^n has constant mean curvature, it follows from (2.2) that

$$\begin{aligned} \frac{1}{2}\Delta|\Phi|^2 = |\nabla\Phi|^2 + nH\sum_i \mu_i^3 - |A|^4 + \sum_i \overline{R}_{(n+1)i(n+1)i}(nH\mu_i - |A|^2) \\ + \sum_{i,j} (\mu_i - \mu_j)^2 \overline{R}_{ijij}. \end{aligned} \tag{3.2}$$

From curvature conditions (2.3) and (2.4) we get

$$\sum_i \overline{R}_{(n+1)i(n+1)i}(nH\mu_i - |A|^2) = c_1(nH^2 - |A|^2) = -c_1|\Phi|^2 \tag{3.3}$$

and

$$\sum_{i,j} \overline{R}_{ijij}(\mu_i - \mu_j)^2 \geqslant c_2 \sum_{i,j} (\mu_i - \mu_j)^2 = 2nc_2(|A|^2 - nH^2) = 2nc_2|\Phi|^2. \tag{3.4}$$

Moreover, when $n \geqslant 3$, we can apply Lemma 1 to the real numbers $\kappa_1, \ldots, \kappa_n$ and obtain

$$\begin{aligned} nH \sum_i \mu_i^3 - |A|^4 &= n^2 H^4 + 3nH^2|\Phi|^2 + nH \sum_i \kappa_i^3 - (|\Phi|^2 + nH^2)^2 \\ &\geqslant n^2 H^4 + 3nH^2|\Phi|^2 - n|H| \Big|\sum_i \kappa_i^3\Big| - |\Phi|^4 \\ &\quad - 2nH^2|\Phi|^2 - n^2 H^4 \\ &\geqslant -|\Phi|^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|^3 + nH^2|\Phi|^2. \end{aligned} \tag{3.5}$$

In the case that $n = 2$, a straightforward computation gives

$$2H \sum_i \mu_i^3 - |A|^4 = 2H(3H|\Phi|^2 + 2H^3) - (|\Phi|^2 + 2H^2)^2 = -|\Phi|^4 + 2H^2|\Phi|^2. \tag{3.6}$$

In any case, since $c = 2c_2 - c_1/n$, inserting (3.3), (3.4), (3.5) and (3.6) into (3.2) we obtain that

$$\begin{aligned} \frac{1}{2}\Delta|\Phi|^2 &\geqslant |\nabla\Phi|^2 - |\Phi|^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|^3 + n(H^2 + c)|\Phi|^2 \\ &= |\nabla\Phi|^2 - |\Phi|^2 P_{|H|,c}(|\Phi|), \end{aligned} \tag{3.7}$$

where

$$P_{|H|,c}(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}|H|x - n(H^2 + c).$$

Hence, from (3.1) and (3.7) we get

$$f_\varepsilon \Delta f_\varepsilon \geqslant |\nabla\Phi|^2 - |\Phi|^2 P_{|H|,c}(|\Phi|) - \frac{\big|\nabla|\Phi|^2\big|^2}{4(\varepsilon + |\Phi|^2)}.$$

Moreover, from Lemma 2 applied to Φ we have

$$f_\varepsilon \Delta f_\varepsilon \geqslant |\nabla\Phi|^2 - |\Phi|^2 P_{|H|,c}(|\Phi|) - \frac{n}{n+2}|\nabla\Phi|^2 = \frac{2}{n+2}|\nabla\Phi|^2 - |\Phi|^2 P_{|H|,c}(|\Phi|). \tag{3.8}$$

Then from (2.5) and (3.8) we see that

$$-f_\varepsilon J(f_\varepsilon) \leqslant |\Phi|^2 P_{|H|,c}(|\Phi|) - \frac{2}{n+2}|\nabla\Phi|^2 - (\varepsilon + |\Phi|^2)\Big(|\Phi|^2 + n\Big(H^2 + \frac{c_1}{n}\Big)\Big). \tag{3.9}$$

Thus, it follows from (1.2) and (3.9) that

$$\begin{aligned} \lambda_1 \int_\Sigma f_\varepsilon^2 \, d\Sigma &\leqslant -\int_\Sigma f_\varepsilon J(f_\varepsilon)\, d\Sigma \leqslant \int_\Sigma |\Phi|^2 P_{|H|,c}(|\Phi|)\, d\Sigma - \frac{2}{n+2}\int_\Sigma |\nabla\Phi|^2 \, d\Sigma \\ &\quad - \int_\Sigma (\varepsilon + |\Phi|^2)\Big(|\Phi|^2 + n\Big(H^2 + \frac{c_1}{n}\Big)\Big)\, d\Sigma. \end{aligned} \tag{3.10}$$

Since
$$\lim_{\varepsilon\to 0}\int_\Sigma f_\varepsilon^2\,\mathrm{d}\Sigma = \int_\Sigma |\Phi|^2\,\mathrm{d}\Sigma > 0,$$
letting $\varepsilon \to 0$ in (3.10) we get

$$\begin{aligned}(3.11)\qquad \lambda_1 \int_\Sigma |\Phi|^2\,\mathrm{d}\Sigma &\leqslant \int_\Sigma \Big(|\Phi|^2 P_{|H|,c}(|\Phi|) - |\Phi|^4 - n\Big(H^2 + \frac{c_1}{n}\Big)|\Phi|^2\Big)\,\mathrm{d}\Sigma \\ &\quad - \frac{2}{n+2}\int_\Sigma |\nabla\Phi|^2\,\mathrm{d}\Sigma \\ &\leqslant -2n(H^2 + c_2)\int_\Sigma |\Phi|^2\,\mathrm{d}\Sigma + \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\int_\Sigma |\Phi|^3\,\mathrm{d}\Sigma.\end{aligned}$$

Hence, from (3.11) we obtain

$$(3.12)\qquad \lambda_1 \leqslant -2n(H^2 + c_2) + \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\max_\Sigma |\Phi|.$$

Finally, let us assume that equality (3.12) holds. Thus, from (3.11) and Lemma 2 we have that $\nabla|\Phi| = 0$ and since we are assuming that Σ^n is not totally umbilical, $|\Phi|$ must be a positive constant and Σ^n is an isoparametric hypersurface. On the other hand, denoting by λ_1^Δ the first eigenvalue of the Laplacian operator of Σ^n, it follows from (2.5) that the first stability eigenvalue λ_1 satisfies

$$(3.13)\qquad \lambda_1^\Delta = \lambda_1 + \Big(|\Phi|^2 + n\Big(H^2 + \frac{c_1}{n}\Big)\Big).$$

Hence, since Σ^n is supposed to be compact, from (3.13) we have that

$$0 = \lambda_1^\Delta = |\Phi|^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi| - n(H^2 + c) = P_{|H|,c}(|\Phi|),$$

which jointly with $H^2 + c > 0$ assures that $|\Phi|$ is the unique positive root of polynomial $P_{|H|,c}(x)$ and is given by

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{n-1}}\big(\sqrt{n^2H^2 + 4(n-1)c} - (n-2)|H|\big).$$

Therefore, from (3.7) we conclude that the equality in Okumura's lemma (Lemma 1) holds, which implies that Σ^n is an isoparemetric hypersurface with two distinct principal curvatures one of which is simple. This finishes the proof of the theorem. □

Remark 1. Since there does not exist closed minimal hypersurface in $\mathbb{R}^{n+1}$ and observing that Lemma 8 of [5] guarantees that $H^2-1>0$ for a closed H-hypersurface in $\mathbb{H}^{n+1}$, we observe that the assumption $H^2+c>0$ in Theorem 1 occurs trivially when the ambient space is a space form of constant sectional curvature $c \in \{-1,0,1\}$. In this setting, the constraint $H^2+c>0$ in Theorem 1 is a mild hypothesis.

It is well known that there are no strongly stable closed H-hypersurfaces immersed into the sphere $\mathbb{S}^{n+1}$ (see, for instance, [3], Section 2). More generally, if $\overline{M}^{n+1}$ is any locally symmetric space with $c_1>0$ and Σ^n is a closed H-hypersurface immersed into $\overline{M}^{n+1}$, then using $f=1$ as a test function in (1.2), it follows from (2.5) that

$$\begin{aligned}\lambda_1 &\leqslant -\frac{1}{\operatorname{vol}(\Sigma)}\int_\Sigma (|\Phi|^2+nH^2+c_1)\,\mathrm{d}\Sigma\\ &\leqslant -\frac{1}{\operatorname{vol}(\Sigma)}\int_\Sigma (nH^2+c_1)\,\mathrm{d}\Sigma = -n\Big(H^2+\frac{c_1}{n}\Big)<0,\end{aligned}$$

which means that there are no strongly stable closed H-hypersurfaces immersed into $\overline{M}^{n+1}$. When $c_1 \geqslant 0$, as a consequence of Theorem 1 we obtain an extension of this result for such a locally symmetric space.

Corollary 1. *There are no strongly stable closed H-hypersurface immersed in a locally symmetric Riemannian manifold $\overline{M}^{n+1}$, $n\geqslant 3$, satisfying curvature conditions* (2.3) *to* (2.4), *such that its second fundamental form is a Codazzi tensor, $H\neq 0$, $H^2+c>0$, $c_1\geqslant 0$ and its total umbilicity operator Φ satisfies*

$$|\Phi| < \frac{\sqrt{n(n-1)}(H^2+c)}{(n-2)|H|}.$$

Proceeding in order to prove our next result and motivated by Okumura's lemma, we will make use of the following Okumura type inequality introduced by Meléndez in [9]

$$|\operatorname{tr}(\Phi^3)| \leqslant \frac{(n-2p)}{\sqrt{np(n-p)}}|\Phi|^3, \tag{3.14}$$

where $1\leqslant p\leqslant n/2$. It is worth pointing out that since Φ is traceless, by the classical Okumura's lemma, inequality (3.14) is automatically true when $p=1$. Moreover, to suppose that inequality (3.14) holds is weaker than to assume that the hypersurface has two distinct principal curvatures with multiplicities p and $n-p$. Indeed, in this case there are real numbers μ and ν such that

$$\kappa_1=\ldots=\kappa_p=\mu,\quad \kappa_{p+1}=\ldots=\kappa_n=\nu,$$

where κ_i stands for the eigenvalues of Φ. Then it is not difficult to see that

$$0 = \sum_i \kappa_i = p\mu + (n-p)\nu \quad \text{and} \quad |\Phi|^2 = \sum_i \kappa_i^2 = p\mu^2 + (n-p)\nu^2.$$

Hence, we obtain that

$$\operatorname{tr}(\Phi^3) = \sum_i \kappa_i^3 = p\mu^3 + (n-p)\nu^3 = \pm\frac{(2p-n)}{\sqrt{np(n-p)}}|\Phi|^3.$$

The next result is an Okumura type lemma obtained by Meléndez in [9], which says that the reciprocal of this fact is also true.

Lemma 3. *Let $\kappa_1, \ldots, \kappa_n$, $n \geqslant 3$, be real numbers such that $\sum_i \kappa_i = 0$ and $\sum_i \kappa_i^2 = \beta^2$, where $\beta \geqslant 0$. Then*

$$\sum_i \kappa_i^3 = \frac{(n-2p)}{\sqrt{np(n-p)}}\beta^3 \quad \left(\text{or } \sum_i \kappa_i^3 = -\frac{(n-2p)}{\sqrt{np(n-p)}}\beta^3\right), \quad 1 \leqslant p \leqslant n-1,$$

holds if and only if p of the numbers κ_i are nonnegative (or nonpositive) and equal and the rest $n-p$ of the numbers κ_i are nonpositive (or nonnegative) and equal.

In this context, under assumption (3.14) on the total umbilicity tensor of the hypersurface and reasoning as in the proof of Theorem 1 we are able to improve the estimate given by Theorem 1 as follows.

Theorem 2. *Let $\psi\colon \Sigma^n \to \overline{M}^{n+1}$ be a closed H-hypersurface immersed in a locally symmetric Riemannian manifold $\overline{M}^{n+1}$, $n \geqslant 4$, satisfying curvature conditions* (2.3) *and* (2.4), *such that its second fundamental form is a Codazzi tensor and $H^2 + c > 0$, where $c = 2c_2 - c_1/n$. If its total umbilicity tensor Φ satisfies* (3.14) *for some $1 < p \leqslant n/2$, then*

(i) *either $\lambda_1 = -n(H^2 + c_1/n)$ and Σ^n is a totally umbilical hypersurface or*

(ii)

$$\lambda_1 \leqslant -2n(H^2 + c_2) + \frac{n(n-2p)}{\sqrt{np(n-p)}}|H| \max_\Sigma |\Phi|.$$

Moreover, if the equality holds (and, in the case $c > 0$, $H \neq 0$), then Σ^n is an isoparametric hypersurface with two distinct principal curvatures with multiplicities p and $n-p$.

P r o o f. In what follows, we keep the notation established in the proof of the previous theorem. So, as in the proof of Theorem 1, we get that $\lambda_1 = -n(H^2+c_1/n)$ if and only if Σ^n is a totally umbilical hypersurface.

From now on, let us assume that Σ^n is not totally umbilical. It follows from our hypothesis (3.14) that

$$nH\sum_i \mu_i^3 - |A|^4 \geqslant -|\Phi|^4 - \frac{n(n-2p)}{\sqrt{np(n-p)}}|H||\Phi|^3 + nH^2|\Phi|^2. \tag{3.15}$$

On the other hand, it is clear that equations (3.3) and (3.4) also occur in this case. Then (2.2) jointly with (3.15) gives

$$\begin{aligned}\frac{1}{2}\Delta|\Phi|^2 &\geqslant |\nabla\Phi|^2 - |\Phi|^4 - \frac{n(n-2p)}{\sqrt{np(n-p)}}|H||\Phi|^3 + n(H^2+c)|\Phi|^2 \\ &= |\nabla\Phi|^2 - |\Phi|^2 P_{|H|,c,p}(|\Phi|),\end{aligned} \tag{3.16}$$

where

$$P_{|H|,c,p}(x) = x^2 + \frac{n(n-2p)}{\sqrt{np(n-p)}}|H|x - n(H^2+c).$$

Hence, we can reason in a similarly to the proof of Theorem 1 and obtain that

$$\lambda_1 \leqslant -2n(H^2+c_2) + \frac{n(n-2p)}{\sqrt{np(n-p)}}|H|\max_\Sigma|\Phi|. \tag{3.17}$$

Finally, if equality (3.17) holds, we can reason again as in the proof of Theorem 1 and conclude that $|\Phi|$ is a positive constant which is given by

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\big(\sqrt{n^2H^2+4p(n-p)c} - (n-2p)|H|\big).$$

Therefore, from (3.16) we conclude that the equality in Lemma 3 holds, which implies that Σ^n is an isoparametric hypersurface with two distinct principal curvatures of multiplicities p and $n-p$. □

Remark 2. We point out that for all $1 \leqslant p \leqslant n/2$ it holds that

$$\frac{n-2p}{\sqrt{p(n-p)}} \leqslant \frac{n-2}{\sqrt{n-1}},$$

with the equality if and only if $p = 1$. In this setting, Theorem 2 is a refinement of Theorem 1 for the case when the total umbilicity tensor Φ of the hypersurface satisfies (3.14).

Acknowledgements. The authors would like to thank the referee for his/her valuable suggestions and useful comments which improved the paper.

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