Xinfeng Liang; Feng Wei; Ajda Fošner

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CENTRALIZING TRACES AND LIE-TYPE ISOMORPHISMS ON GENERALIZED MATRIX ALGEBRAS: A NEW PERSPECTIVE

Xinfeng Liang, Huainan, Feng Wei, Beijing, Ajda Fošner, Koper

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Abstract. Let $\mathcal{R}$ be a commutative ring, $\mathcal{G}$ be a generalized matrix algebra over $\mathcal{R}$ with weakly loyal bimodule and $\mathcal{Z}(\mathcal{G})$ be the center of $\mathcal{G}$. Suppose that $\mathfrak{q}\colon \mathcal{G}\times\mathcal{G}\to\mathcal{G}$ is an $\mathcal{R}$-bilinear mapping and that $\mathfrak{T}_{\mathfrak{q}}\colon \mathcal{G}\to\mathcal{G}$ is a trace of $\mathfrak{q}$. The aim of this article is to describe the form of $\mathfrak{T}_{\mathfrak{q}}$ satisfying the centralizing condition $[\mathfrak{T}_{\mathfrak{q}}(x),x]\in\mathcal{Z}(\mathcal{G})$ (and commuting condition $[\mathfrak{T}_{\mathfrak{q}}(x),x]=0$) for all $x\in\mathcal{G}$. More precisely, we will revisit the question of when the centralizing trace (and commuting trace) $\mathfrak{T}_{\mathfrak{q}}$ has the so-called proper form from a new perspective. Using the aforementioned trace function, we establish sufficient conditions for each Lie-type isomorphism of $\mathcal{G}$ to be almost standard. As applications, centralizing (commuting) traces of bilinear mappings and Lie-type isomorphisms on full matrix algebras and those on upper triangular matrix algebras are totally determined.

Keywords: generalized matrix algebra; commuting trace; centralizing trace; Lie isomorphism; Lie triple isomorphism

MSC 2010: 16R60, 16W10, 15A78

1. Introduction

Let $\mathcal{R}$ be a commutative ring with identity, $\mathcal{A}$ be a unital algebra over $\mathcal{R}$ and $\mathcal{Z}(\mathcal{A})$ be the center of $\mathcal{A}$. Let us denote the commutator or the Lie product of the elements $a,b\in\mathcal{A}$ by $[a,b]=ab-ba$. Recall that an $\mathcal{R}$-linear mapping $\mathfrak{f}\colon \mathcal{A}\to\mathcal{A}$ is said to be *centralizing* if $[\mathfrak{f}(a),a]\in\mathcal{Z}(\mathcal{A})$ for all $a\in\mathcal{A}$. In particular, the mapping $\mathfrak{f}$ is called *commuting* if $[\mathfrak{f}(a),a]=0$ for all $a\in\mathcal{A}$. When dealing with a centralizing (or commuting) mapping, the usual goal is to provide a precise description of its form. The identity mapping and every mapping which has its range in $\mathcal{Z}(\mathcal{A})$ are two classical examples of commuting mappings. Furthermore, the sum and the pointwise

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product of commuting mappings are also commuting mappings. For instance, the mapping

$$(\lozenge)\quad \mathfrak{f}(x) = \lambda_0(x)x^n + \lambda_1(x)x^{n-1} + \ldots\ldots + \lambda_{n-1}(x)x + \lambda_n(x), \quad \lambda_i\colon \mathcal{A} \to \mathcal{Z}(\mathcal{A})$$

is commuting for any choice of central maps λ_i. Of course, there are other examples, namely, elements commuting with x may not necessarily be equal to a polynomial in x (with central coefficients) and so in most rings there is a variety of possibilities of how to find commuting maps different from those described in ($\lozenge$). We encourage the reader to read the well-written monograph (see [11]), in which the author presented the development of the theory of commuting mappings and their applications in details.

Let n be a positive integer and $\mathfrak{q}\colon \mathcal{A}^n \to \mathcal{A}$. We say that $\mathfrak{q}$ is *n-linear* if $\mathfrak{q}(a_1, \ldots, a_n)$ is $\mathcal{R}$-linear in each variable a_i, that is, $\mathfrak{q}(a_1, \ldots, ra_i + sb_i, \ldots, a_n) = r\mathfrak{q}(a_1, \ldots, a_i, \ldots, a_n) + s\mathfrak{q}(a_1, \ldots, b_i, \ldots, a_n)$ for all $r, s \in \mathcal{R}$, $a_i, b_i \in \mathcal{A}$ and $i = 1, 2, \ldots, n$. The mapping $\mathfrak{T}_\mathfrak{q}\colon \mathcal{A} \to \mathcal{A}$ defined by $\mathfrak{T}_\mathfrak{q}(a) = \mathfrak{q}(a, a, \ldots, a)$ is called a *trace* of $\mathfrak{q}$. We say that a centralizing trace $\mathfrak{T}_\mathfrak{q}$ is *proper* if it can be written as

$$\mathfrak{T}_\mathfrak{q}(a) = \sum_{i=0}^{n} \mu_i(a)a^{n-i} \quad \forall\, a \in \mathcal{A},$$

where μ_i $(0 \leqslant i \leqslant n)$ is a mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ and every μ_i $(0 \leqslant i \leqslant n)$ is in fact a trace of an i-linear mapping $\mathfrak{q}_i$ from $\mathcal{A}^i$ into $\mathcal{Z}(\mathcal{A})$. Let $n = 1$ and $\mathfrak{f}\colon \mathcal{A} \to \mathcal{A}$ be an $\mathcal{R}$-linear mapping. In this case, an arbitrary trace $\mathfrak{T}_\mathfrak{f}$ of $\mathfrak{f}$ exactly equals to itself. Moreover, if a centralizing trace $\mathfrak{T}_\mathfrak{f}$ of $\mathfrak{f}$ is proper, then it has the form

$$\mathfrak{T}_\mathfrak{f}(a) = za + \eta(a) \quad \forall\, a \in \mathcal{A},$$

where $z \in \mathcal{Z}(\mathcal{A})$ and η is an $\mathcal{R}$-linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$. Let us see the case of $n = 2$. Suppose that $\mathfrak{g}\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is an $\mathcal{R}$-bilinear mapping. If a centralizing trace $\mathfrak{T}_\mathfrak{g}$ of $\mathfrak{g}$ is proper, then it is of the form

$$\mathfrak{T}_\mathfrak{g}(a) = za^2 + \mu(a)a + \nu(a) \quad \forall\, a \in \mathcal{A},$$

where $z \in \mathcal{Z}(\mathcal{A})$, μ is an $\mathcal{R}$-linear mapping from $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$ and ν is a trace of some bilinear mapping $\mathcal{A}$ into $\mathcal{Z}(\mathcal{A})$. Brešar started the study of commuting and centralizing traces of multilinear mappings in his series of works (see [9], [8], [10]), where he investigated the structure of centralizing traces of (bi-)linear mappings on prime rings. It was proved that in certain rings, in particular, prime rings of characteristic different from 2 and 3, every centralizing trace of a biadditive mapping is commuting. Moreover, every centralizing mapping of a prime ring of characteristic different from 2 is of the proper form and is actually commuting. An exciting discovery (see [63]) is that every centralizing trace of arbitrary bilinear mapping on triangular algebras is commuting in some additional conditions. It has turned out that this study is closely related to the problem of characterizing Lie isomorphisms or Lie derivations of associative rings, see [4], [39], [62], [63].

Cheung in [17] and [18] studied commuting mappings of triangular algebras (e.g. of upper triangular matrix algebras and nest algebras). He determined the class of triangular algebras for which every commuting mapping is proper. Xiao and Wei in [61] extended Cheung's result to the case of generalized matrix algebras. They established sufficient conditions for each commuting mapping of a generalized matrix algebra $\begin{bmatrix} A & M \\ N & B \end{bmatrix}$ to be proper. Benkovič and Eremita in [4] considered commuting traces of bilinear mappings on a triangular algebra $\begin{bmatrix} A & M \\ O & B \end{bmatrix}$. They gave conditions under which every commuting trace of a triangular algebra $\begin{bmatrix} A & M \\ O & B \end{bmatrix}$ is proper. More recently, Franca in [25], [26], [27], [28], [29], [30], [31], Xu and Yi in [64] independently investigated commuting mappings on subsets of matrices that are not closed under addition such as invertible matrices, singular matrices, matrices of rank k, etc. The research results demonstrate that the commuting mappings on these sets basically have the proper form. Liu in [44] and [45] immediately extended Franca's and Xu's works to the case of centralizing mappings. These works explicitly imply that functional identities can be developed to the sets that are not closed under addition. The form of commuting traces of multilinear mappings of upper triangular matrix algebras was earlier described. Simultaneously, some researchers engage in characterizing k-commuting mappings of generalized matrix algebras and those of unital algebras with notrivial idempotents, see [22], [40], [41], [52]. Motivated by Benkovič and Eremita's work (see [4]), the present authors (see [43], [62], [63]) undertook the study of centralizing (or commuting) traces of bilinear mappings on triangular algebras and generalized matrix algebras. It is shown that every centralizing trace of an arbitrary bilinear mapping on a class of generalized matrix algebra has the so-called proper from. González et al. in [49] and [54] investigated commuting mappings with automorphisms on triangular algebras. Wang in [58] and [59] revisited the centralizing (commuting) traces of bilinear mappings and Lie (triple) isomorphisms on triangular algebras via a new tool-weakly loyal module. Its advantage is to embrace those upper triangular matrix algebras over 2-torsion free commutative rings which are not recovered in the existing works.

Another important purpose of this article is devoted to the study of Lie-type isomorphisms problem of generalized matrix algebras. Given a commutative ring $\mathcal{R}$ with identity and two associative $\mathcal{R}$-algebras $\mathcal{A}$ and $\mathcal{B}$, one defines a *Lie triple isomorphism* from $\mathcal{A}$ into $\mathcal{B}$ to be an $\mathcal{R}$-linear bijective mapping $\mathfrak{l}$ satisfying the condition

$$\mathfrak{l}([[a,b],c]) = [[\mathfrak{l}(a),\mathfrak{l}(b)],\mathfrak{l}(c)] \quad \forall\, a,b,c \in \mathcal{A}.$$

For example, an isomorphism or the negative of an anti-isomorphism of one algebra onto another is also a Lie isomorphism. Furthermore, every Lie isomorphism and every Jordan isomorphism are Lie triple isomorphisms. One can ask whether the converse is true in some special cases. That is, does every Lie triple isomorphism between certain associative algebras arise from isomorphisms and anti-isomorphisms in the sense of modulo mappings whose range is central? If $\mathfrak{m}$ is an isomorphism or the negative of an anti-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ and $\mathfrak{n}$ is an $\mathcal{R}$-linear mapping from $\mathcal{A}$ into the center $\mathcal{Z}(\mathcal{B})$ of $\mathcal{B}$ such that $\mathfrak{n}([[a,b],c]) = 0$ for all $a,b,c \in \mathcal{A}$, then the mapping

$$(\spadesuit) \qquad \mathfrak{l} = \mathfrak{m} + \mathfrak{n}$$

is a Lie triple homomorphism. We shall say that a Lie triple isomorphism $\mathfrak{l}\colon\ A \to B$ is *standard* when it can be expressed in the preceding form ($\spadesuit$).

Hua in [33] proved that every Lie automorphism of the full matrix algebra $\mathcal{M}_n(\mathcal{D})$ ($n \geqslant 3$) over a division ring $\mathcal{D}$ is of the standard form ($\spadesuit$). This result was extended to the nonlinear case by Dolinar in [20] and Šemrl in [56] and was further refined by them. Marcoux and Sourour in [47] classified the linear mappings preserving commutativity in both directions (i.e., $[x,y] = 0$ if and only if $[\mathfrak{f}(x),\mathfrak{f}(y)] = 0$) on upper triangular matrix algebras $\mathcal{T}_n(\mathbb{F})$ over a field $\mathbb{F}$. Such a mapping is either the sum of an algebraic automorphism of $\mathcal{T}_n(\mathbb{F})$ (which is inner) and a mapping into the center $\mathbb{F}I$, or the sum of the negative of an algebraic anti-automorphism and a mapping into the center $\mathbb{F}I$. The classification of the Lie automorphisms of $\mathcal{T}_n(\mathbb{F})$ is obtained as a consequence. Benkovič and Eremita in [4] directly applied the theory of commuting traces to the study of Lie isomorphisms on a triangular algebra $\left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$. They provided sufficient conditions under which every commuting trace of $\left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ is proper. This is directly applied to the study of Lie isomorphisms of $\left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$. It turns out that under some mild assumptions, each Lie isomorphism of $\left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ has the standard form ($\spadesuit$). These results are further extended to the case of generalized matrix algebras, see [43], [62], [63]. Lie triple isomorphisms between rings and between (non-)self-adjoint operator algebras have received a fair amount of attention and have also been intensively studied, see [2], [13], [14], [15], [16], [20], [21], [32], [46], [48], [51], [52], [53], [56], [55], [57], [58], [59], [65], [66].

The aim of this paper is to revisit commuting (centralizing) traces of bilinear mappings and Lie (triple) isomorphisms on generalized matrix algebras. One significant improvement in this paper is that we proceed to prove our results under the frame of weakly loyal bimodules, which is explicitly distinguished from our previous works in this vein. Its advantage is that it permits us to embrace two prevailing examples—full matrix algebras and upper triangular matrix algebras over an arbitrary commutative ring. Section 2 contains the definition of generalized matrix algebra and some classical examples. In Section 3 (and Section 4), we provide sufficient conditions for each commuting trace (and centralizing trace) of arbitrary bilinear mappings on a generalized matrix algebra $\left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ with weakly loyal bimodule M to be proper. In Section 5 (and Section 6), we will use the main result of Section 4 (and Section 3) to characterize Lie triple isomorphisms (and Lie isomorphisms) on generalized matrix algebras with weakly loyal bimodules. As consequences, we can obtain a complete description of Lie triple isomorphisms (and Lie isomorphisms) on full matrix algebras and upper triangular matrix algebras over a 2-torsion free commutative ring.

2. Generalized matrix algebras and examples

Let us begin with the definition of generalized matrix algebras given by a Morita context. Let $\mathcal{R}$ be a commutative ring with identity. A *Morita context* consists of two unital $\mathcal{R}$-algebras A and B, two bimodules ${}_AM_B$ and ${}_BN_A$, and two bimodule homomorphisms called the pairings $\Phi_{MN}\colon\ M\otimes_B N\to A$ and $\Psi_{NM}\colon\ N\otimes_A M\to B$ satisfying the following commutative diagrams:

$$\begin{array}{ccc} M\otimes_B N\otimes_A M & \xrightarrow{\Phi_{MN}\otimes I_M} & A\otimes_A M \\ \Big\downarrow{\scriptstyle I_M\otimes\Psi_{NM}} & & \Big\downarrow{\scriptstyle\cong} \\ M\otimes_B B & \xrightarrow{\quad\cong\quad} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} N\otimes_A M\otimes_B N & \xrightarrow{\Psi_{NM}\otimes I_N} & B\otimes_B N \\ \Big\downarrow{\scriptstyle I_N\otimes\Phi_{MN}} & & \Big\downarrow{\scriptstyle\cong} \\ N\otimes_A A & \xrightarrow{\quad\cong\quad} & N \end{array}$$

Let us write this Morita context as $(A, B, M, N, \Phi_{MN}, \Psi_{NM})$. We refer the reader to [50] for the basic properties of Morita contexts. If $(A, B, M, N,\ \Phi_{MN}, \Psi_{NM})$ is a Morita context, then the set

$$\begin{bmatrix} A & M \\ N & B \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} :\ a\in A,\ m\in M,\ n\in N,\ b\in B \right\}$$

form an $\mathcal{R}$-algebra under matrix-like addition and matrix-like multiplication, where at least one of the two bimodules M and N is distinct from zero. Such an $\mathcal{R}$-algebra is usually called a *generalized matrix algebra* of order 2 and is denoted by

$$\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}.$$

In a similar way, one can define a generalized matrix algebra of order $n > 2$. It was shown that up to isomorphism, arbitrary generalized matrix algebra of order n ($n \geqslant 2$) is a generalized matrix algebra of order 2, see [40], Example 2.2. If one of the modules M and N is zero, then $\mathcal{G}$ exactly degenerates to an *upper triangular algebra* or a *lower triangular algebra*. In this case, we denote the resulted upper triangular algebra (and lower triangular algebra) by

$$\mathcal{T}^{\mathcal{U}} = \mathcal{T}(A, M, B) = \begin{bmatrix} A & M \\ O & B \end{bmatrix} \quad \left(\text{and } \mathcal{T}_{\mathcal{L}} = \mathcal{T}(A, N, B) = \begin{bmatrix} A & O \\ N & B \end{bmatrix}\right).$$

Let $\mathcal{M}_n(\mathcal{R})$ be the full matrix algebra consisting of all $n \times n$ matrices over $\mathcal{R}$. It is worth to point out that the notion of generalized matrix algebras efficiently unifies triangular algebras with full matrix algebras together. The feature of our systematic work is to deal with all questions related to (non-)linear mappings of triangular algebras and of full matrix algebras under a unified frame, which is the admired generalized matrix algebras frame, see [40], [41], [42], [61], [62], [63].

Let us list some classical examples of generalized matrix algebras which will be invoked in the sequel (Section 3–6). Since these examples have already been presented in many papers, we just state their titles without any details.

(i) Unital algebras with nontrivial idempotents,
(ii) full matrix algebras,
(iii) inflated algebras,
(iv) triangular algebras, such as upper or lower triangular matrix algebras, block upper (or lower) triangular matix algebras and nest algebras over a Hilbert space,
(v) factor von Neumann algebra acting on a Hilbert space,
(vi) von Neumann algebra with no central summand of type I_1,
(vii) algebra of all bounded linear operators over a Banach space X,
(viii) standard operator algebras over a Banach space.

These generalized matrix algebras frequently appear in the theory of associative algebras and noncommutative Noetherian algebras in the most diverse situations, which is due to its powerful persuasiveness and intuitive illustration effect. However, people pay less attention to the linear mappings of generalized matrix algebras. It was Krylov who initiated the study of linear mappings on generalized matrix algebras from the classifying point of view [34]. Since then many articles are devoted to this topic, and a number of interesting results are obtained (see [1], [3], [6], [7], [12], [19], [23], [24], [40], [41], [42], [60], [61], [62]). Nevertheless, it leaves so much to be desired. The representation theory, homological behavior, K-theory of generalized matrix algebras are intensively intestigated by Krylov and his coauthors in [34], [35], [36], [37], [38].

Throughout this article, we denote the generalized matrix algebra of order 2 originated from the Morita context $(A, B, {}_A M_B, {}_B N_A, \Phi_{MN}, \Psi_{NM})$ by

$$\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix},$$

where at least one of the two bimodules M and N is distinct from zero. We always assume that M is faithful as a left A-module and also as a right B-module, but without any constraint conditions on N. The center of $\mathcal{G}$ is

$$\mathcal{Z}(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : \; am = mb, \; na = bn \; \forall\, m \in M \; \forall\, n \in N \right\}.$$

Indeed, by [34], Lemma 1, we know that the center $\mathcal{Z}(\mathcal{G})$ consists of all diagonal matrices $\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$, where $a \in \mathcal{Z}(A)$, $b \in \mathcal{Z}(B)$ and $am = mb$, $na = bn$ for all $m \in M$, $n \in N$. However, in our situation when M is faithful as a left A-module and also as a right B-module, the conditions that $a \in \mathcal{Z}(A)$ and $b \in \mathcal{Z}(B)$ become redundant and can be deleted. Indeed, if $am = mb$ for all $m \in M$, then for any $a' \in A$ we get

$$(aa' - a'a)m = a(a'm) - a'(am) = (a'm)b - a'(mb) = 0.$$

The assumption that M is faithful as a left A-module leads to $aa' - a'a = 0$ and hence $a \in \mathcal{Z}(A)$. Likewise, we also have $b \in \mathcal{Z}(B)$.

Let us define two natural $\mathcal{R}$-linear projections $\pi_A \colon \mathcal{G} \to A$ and $\pi_B \colon \mathcal{G} \to B$ by

$$\pi_A \colon \begin{bmatrix} a & m \\ n & b \end{bmatrix} \mapsto a \quad \text{and} \quad \pi_B \colon \begin{bmatrix} a & m \\ n & b \end{bmatrix} \mapsto b.$$

By the above paragraph, it is not difficult to see that $\pi_A(\mathcal{Z}(\mathcal{G}))$ is a subalgebra of $\mathcal{Z}(A)$ and that $\pi_B(\mathcal{Z}(\mathcal{G}))$ is a subalgebra of $\mathcal{Z}(B)$. Given an element $a \in \pi_A(\mathcal{Z}(\mathcal{G}))$, if $\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right], \left[\begin{smallmatrix} a & 0 \\ 0 & b' \end{smallmatrix}\right] \in \mathcal{Z}(\mathcal{G})$, then we have $am = mb = mb'$ for all $m \in M$. Since M is faithful as a right B-module, $b = b'$. That implies there exists a unique $b \in \pi_B(\mathcal{Z}(\mathcal{G}))$, which is denoted by $\varphi(a)$, such that $\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right] \in \mathcal{Z}(\mathcal{G})$. It is easy to verify that the map $\varphi\colon\ \pi_A(\mathcal{Z}(\mathcal{G})) \to \pi_B(\mathcal{Z}(\mathcal{G}))$ is an algebraic isomorphism such that $am = m\varphi(a)$ and $na = \varphi(a)n$ for all $a \in \pi_A(\mathcal{Z}(\mathcal{G}))$, $m \in M$, $n \in N$.

Let A and B be algebras. Recall an (A, B)-bimodule M is *loyal* if $aMb = 0$ implies that $a = 0$ or $b = 0$ for all $a \in A$, $b \in B$. We say that an (A, B)-bimodule M is *weakly loyal* if

(1) for any $a \in A$ the condition $aM[B, B] = 0$ implies that $a = 0$ or $[B, B] = 0$,
(2) for any $b \in B$ the condition $[A, A]Mb = 0$ implies that $b = 0$ or $[A, A] = 0$.

Example 2.1. Let us see two prevailing generalized matrix algebras which are endowed with weakly loyal bimodules.

(1) Let $\mathcal{T}_n(\mathcal{R})$ be the algebra of all $n \times n$ upper triangular matrices over a unital commutative ring $\mathcal{R}$, where $n \geqslant 2$ is an integer. For every $1 \leqslant r \leqslant n-1$ let $A = \mathcal{T}_r(\mathcal{R})$, $M = \mathcal{M}_{r\times(n-r)}(\mathcal{R})$ and $B = \mathcal{T}_{n-r}(\mathcal{R})$. Then M is a weakly loyal (A, B)-bimodule. However, if $\mathcal{R}$ is not a domain, then M is not a loyal (A, B)-bimodule, see [58], Remark 2.1.

(2) Let $n \geqslant 2$ and $\mathcal{R}$ be a 2-torsion free unital commutative ring. $\mathcal{M}_n(\mathcal{R})$ be a full matrix algebra defined over $\mathcal{R}$. For every $1 \leqslant r \leqslant n-1$ let $A = \mathcal{M}_r(\mathcal{R})$, $M = \mathcal{M}_{r\times(n-r)}(\mathcal{R})$, $N = \mathcal{M}_{(n-r)\times r}(\mathcal{R})$, $B = \mathcal{M}_{n-r}(\mathcal{R})$. Then M is a weakly loyal (A, B)-bimodule. Indeed, let us write

$$\mathcal{M}_n(\mathcal{R}) = \begin{bmatrix} \mathcal{M}_r(\mathcal{R}) & \mathcal{M}_{r\times(n-r)}(\mathcal{R}) \\ \mathcal{M}_{(n-r)\times r}(\mathcal{R}) & \mathcal{M}_{n-r}(\mathcal{R}) \end{bmatrix}.$$

Note that $Z(\mathcal{M}_n(\mathcal{R})) = \mathcal{R} \cdot \mathrm{I}_{\mathcal{M}_n(\mathcal{R})}$. We claim that condition (1) holds true. If $n = 2$, there is nothing to prove, since A and B are commutative algebras. If $n > 2$, we may assume $[B, B] \neq 0$, then $n - r \geqslant 2$. Suppose that for all $a = [a_{i,j}]_{r\times r} \in A$, $aM[B, B] = 0$. Let us take $e_{i,r+1} \in M$ and $e_{r+1,r+1}, e_{r+1,n} \in B$ for all $1 \leqslant i \leqslant r$. Then we have

$$ae_{i,r+1}[e_{r+1,r+1}, e_{r+1,n}] = ae_{i,n} = 0.$$

We therefore get $a_{j,i} = 0$ for all $1 \leqslant j \leqslant r$. In view of the scope of i, we get $a_{j,i} = 0$ for all $1 \leqslant j$, $i \leqslant r$, i.e., $a = 0$.

3. Commuting traces of bilinear mappings on generalized matrix algebras

In this section we will establish sufficient conditions which enable each commuting trace of an arbitrary bilinear mapping on a generalized matrix algebra $\left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ to be proper (Theorem 3.13). Consequently, we are able to describe commuting traces of bilinear mappings on triangular algebras and those on full matrix algebras. Most important is that Theorem 3.13 can be used to characterize Lie isomorphisms from a generalized matrix algebra into another one in Section 6.

Simulating the proofs of [58], Lemmas 2.1–2.2, Lemmas 3.1–3.3 one can obtain similar results for generalized matrix algebras with weakly loyal bimodules.

Lemma 3.1. *Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ be a generalized matrix algebra with weakly loyal (A, B)-bimodule M.*

(1) *For any* $\lambda \in \pi_A(\mathcal{G})$ *the condition* $\lambda[A, A] = 0$ *implies that either* $\lambda = 0$ *or* $[A, A] = 0$.
(2) *For any* $\lambda \in \pi_B(\mathcal{G})$ *the condition* $\lambda[B, B] = 0$ *implies that either* $\lambda = 0$ *or* $[B, B] = 0$.

Lemma 3.2. *Let* M *be a weakly loyal* (A, B)*-bimodule.*
(1) *Let* $f, g\colon M \to A$ *be arbitrary mappings. Suppose that* $f(m)n + g(n)m = 0$ *holds true for all* $m, n \in M$. *If* B *is noncommutative, then* $f = g = 0$.
(2) *Let* $f, g\colon M \to B$ *be arbitrary mappings. Suppose that* $nf(m) + mg(n) = 0$ *holds true for all* $m, n \in M$. *If* A *is noncommutative, then* $f = g = 0$.

In 2012, Benkovic and Eremita (see [5]) introduced the following useful condition for an arbitrary $\mathcal{R}$-algebra $\mathcal{A}$:

$$(\heartsuit) \qquad [x, \mathcal{A}] \in \mathcal{Z}(\mathcal{A}) \Rightarrow x \in \mathcal{Z}(\mathcal{A}) \quad \forall\, x \in \mathcal{A}.$$

This amounts to saying that

$$[[x, \mathcal{A}], \mathcal{A}] = 0 \Rightarrow [x, \mathcal{A}] = 0 \in \mathcal{Z}(\mathcal{A}) \quad \forall\, x \in \mathcal{A}.$$

Note that $(\heartsuit)$ is equivalent to the condition that there do not exist nonzero central inner derivations on $\mathcal{A}$. The usual examples of algebras satisfying $(\heartsuit)$ are commutative algebras, prime algebras, and triangular algebras (see [5], Example 5.2 and Example 5.3). Except for these algebras, we next show that there do not exist nonzero central derivations on generalized matrix algebras.

Indeed, for an arbitrary central derivation d of a generalized matrix algebra $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$. Let us set $e = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $f = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Suppose that $d(e) = \left[\begin{smallmatrix} w & 0 \\ 0 & u \end{smallmatrix}\right]$. Since d is a central derivation of $\mathcal{G}$, we have that $d(e) = ed(e) + ed(e)$ and that $\left[\begin{smallmatrix} w & 0 \\ 0 & u \end{smallmatrix}\right] = 2\left[\begin{smallmatrix} w & 0 \\ 0 & 0 \end{smallmatrix}\right]$. And hence $w = 0$. Note that $u = \varphi(w) = 0$. Thus, we assert that $d(e) = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. In an analogous manner, one can show that $d(f) = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

For an arbitrary $\left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathcal{G}$ assume that $d\Big(\left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right]\Big) = \left[\begin{smallmatrix} w_a & 0 \\ 0 & u_a \end{smallmatrix}\right]$. Since d is a central derivation of $\mathcal{G}$, $d\Big(\left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right]\Big) = d\Big(e\left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right]\Big) = d(e)\left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right] + ed\Big(\left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right]\Big) = ed\Big(\left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right]\Big) = \left[\begin{smallmatrix} w_a & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Thus, we arrive at $u_a = 0$. Note that $w_a = \varphi^{-1}(u_a)$. So $w_a = 0$ and hence $d\Big(\left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right]\Big) = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Likewise, for an arbitrary $\left[\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right] \in \mathcal{G}$, we can show that $d\Big(\left[\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right]\Big) = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

Suppose that $d\Big(\left[\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right]\Big) = \left[\begin{smallmatrix} w_m & 0 \\ 0 & u_m \end{smallmatrix}\right]$ for all $\left[\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right] \in \mathcal{G}$. Since d is a central derivation of $\mathcal{G}$, $d\Big(\left[\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right]\Big) = d\Big(e\left[\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right]\Big) = d(e)\left[\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right] + ed\Big(\left[\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right]\Big) =$

$ed\left(\left[\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} w_m & 0 \\ 0 & 0 \end{smallmatrix}\right]$. We obtain that $u_m = 0$ and that $w_m = \varphi^{-1}(u_m) = 0$. Thus $d\left(\left[\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. In a similar way, we can prove that $d\left(\left[\begin{smallmatrix} 0 & 0 \\ n & 0 \end{smallmatrix}\right]\right) = 0$ for all $\left[\begin{smallmatrix} 0 & 0 \\ n & 0 \end{smallmatrix}\right] \in \mathcal{G}$. We therefore conclude that $d(x) = 0$ for all $x \in \mathcal{G}$.

Using [40], Theorem 3.4 and condition ($\heartsuit$), one easily obtains the following result for generalized matrix algebras. We omit its proof for conciseness.

Proposition 3.3. *Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ be a generalized matrix algebra over a commutative ring $\mathcal{R}$ with $\frac{1}{2} \in \mathcal{R}$. Then every centralizing mapping of $\mathcal{G}$ is proper if the following conditions are satisfied:*

(1) *$\pi_A(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(A)$ and $\pi_B(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(B)$,*
(2) *both A and B satisfy condition ($\heartsuit$),*
(3) *there exist $m_0 \in M$, $n_0 \in N$ such that*

$$\mathcal{Z}(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : \ am_0 = m_0 b, \ n_0 a = b n_0 \ \forall\, a \in \mathcal{Z}(A) \ \forall\, b \in \mathcal{Z}(B) \right\}.$$

Let us see several consequences of Proposotion 3.3.

Corollary 3.4 ([58], Proposition 2.1)**.** *Let $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ be a triangular algebra over a commutative ring $\mathcal{R}$ with $\frac{1}{2} \in \mathcal{R}$. Then every centralizing mapping of $\mathcal{T}$ is proper if the following conditions hold:*

(1) *$\pi_A(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(A)$ and $\pi_B(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(B)$,*
(2) *both A and B satisfy condition ($\heartsuit$),*
(3) *there exists $m_0 \in M$ such that*

$$\mathcal{Z}(\mathcal{T}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : \ am_0 = m_0 b \ \forall\, a \in \mathcal{Z}(A) \ \forall\, b \in \mathcal{Z}(B) \right\}.$$

Corollary 3.5. *Let n be a positive integer with $n \geqslant 2$ and $\mathcal{R}$ be a commutative ring with $\frac{1}{2} \in \mathcal{R}$. Then every centralizing linear mapping on $\mathcal{M}_n(\mathcal{R})$ is proper.*

Corollary 3.6 ([58], Corollary 2.1)**.** *Let n be a positive integer with $n \geqslant 2$ and $\mathcal{R}$ be a commutative ring with $\frac{1}{2} \in \mathcal{R}$. Then every centralizing linear mapping on $\mathcal{T}_n(\mathcal{R})$ is proper.*

Corollary 3.7 ([58], Corollary 2.2)**.** *Then every centralizing linear mapping on nest algebras is proper.*

In order to prove the main theorem of this section, we need much more elementary results.

Lemma 3.8. *Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ be a generalized matrix algebra over a commutative ring $\mathcal{R}$ with $\frac{1}{2} \in \mathcal{R}$. Suppose that there exist $m_0 \in M$, $n_0 \in N$ such that*

$$\mathcal{Z}(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \colon \ am_0 = m_0 b, \ n_0 a = bn_0 \ \forall\, a \in \mathcal{Z}(A), b \in \mathcal{Z}(B) \right\}.$$

(1) *If $f\colon M \to \mathcal{Z}(A)$ and $g\colon M \to \mathcal{Z}(B)$ are $\mathcal{R}$-linear mappings such that*

$$f(m)m = mg(m), \quad nf(m) = g(m)n$$

for all $m \in M$, $n \in N$, then $f(m) \oplus g(m) \in \mathcal{Z}(\mathcal{G})$.

(2) *If $f\colon M \times M \to \mathcal{Z}(A)$ and $g\colon M \times M \to \mathcal{Z}(B)$ are $\mathcal{R}$-linear mappings such that*

$$f(m,m)m = mg(m,m), \quad g(m,m)n = nf(m,m) \tag{3.1}$$

for all $m \in M$, $n \in N$, then $f(m,m) \oplus g(m,m) \in \mathcal{Z}(\mathcal{G})$.

(3) *If $f\colon M \times N \to \mathcal{Z}(A)$ and $g\colon M \times N \to \mathcal{Z}(B)$ are $\mathcal{R}$-linear mappings such that*

$$f(m,n)m = mg(m,n), \quad g(m,n)n = nf(m,n) \tag{3.2}$$

for all $m \in M$, $n \in N$, then $f(m,n) \oplus g(m,n) \in \mathcal{Z}(\mathcal{G})$.

P r o o f. We only prove statements (2) and (3). Statement (1) can be proved in a similar way.

(2) Let us take $m = m_0 \in M$, $n = n_0 \in N$ in (3.1). Then we have

$$f(m_0,m_0)m_0 = m_0 g(m_0,m_0), \quad g(m_0,m_0)n_0 = n_0 f(m_0,m_0). \tag{3.3}$$

Furthermore, we know that $f(m_0,m_0) \oplus g(m_0,m_0) \in \mathcal{Z}(\mathcal{G})$,

$$f(m_0,m_0)m = mg(m_0,m_0) \tag{3.4}$$

for all $m \in M$. Replacing m by $m + m_0$ in (3.1) gives

$$f(m+m_0, m+m_0)(m+m_0) = (m+m_0)g(m+m_0, m+m_0)$$

for all $m \in M$. Expanding this identity and using both (3.3) and (3.4), we obtain

$$\begin{aligned} &(f(m_0,m)+f(m,m_0))m_0 + (f(m_0,m)+f(m,m_0))m + f(m,m)m_0 \\ &\quad = m_0(g(m_0,m)+g(m,m_0)) + m(g(m_0,m)+g(m,m_0)) + m_0 g(m,m) \end{aligned} \tag{3.5}$$

for all $m \in M$. Substituting $-m$ for m in (3.5) and comparing both identities we get

$$(f(m_0,m)+f(m,m_0))m_0 = m_0(g(m_0,m)+g(m,m_0)) \tag{3.6}$$

for all $m \in M$. Replacing m by $m+m_0$ and taking $n=n_0$ in (3.1) we arrive at the relation

$$g(m+m_0,m+m_0)n_0 = n_0 f(m+m_0,m+m_0)$$

for all $m \in M$. Combining this identity with (3.1) and (3.3), we conclude that

$$(g(m_0,m)+g(m,m_0))n_0 = n_0(f(m_0,m)+f(m,m_0)). \tag{3.7}$$

Comparing (3.6) with (3.7) yields

$$(f(m_0,m)+f(m,m_0)) \oplus (g(m_0,m)+g(m,m_0)) \in \mathcal{Z}(\mathcal{G}).$$

In view of the assumptions, we assert that

$$(f(m_0,m)+f(m,m_0))m = m(g(m_0,m)+g(m,m_0)) \tag{3.8}$$

for all $m \in M$. According to relations (3.5), (3.6) and (3.8), we get $f(m,m)m_0 = m_0 g(m,m)$ for all $m \in M$. Together with relation $g(m,m)n_0 = n_0 f(m,m)$, we obtain that $f(m,m) \oplus g(m,m) \in \mathcal{Z}(\mathcal{G})$.

(3) Taking $m=m_0$, $n=n_0$ in (3.2) leads to

$$f(m_0,n_0)m_0 = m_0 g(m_0,n_0), \quad g(m_0,n_0)n_0 = n_0 f(m_0,n_0) \tag{3.9}$$

and

$$f(m,n_0)m = m g(m,n_0), \quad g(m_0,n)n = n f(m_0,n). \tag{3.10}$$

It follows from (3.9) that

$$f(m_0,n_0)m = m g(m_0,n_0), \quad g(m_0,n_0)n = n f(m_0,n_0). \tag{3.11}$$

Substituting $m+m_0$ for m in (3.2) and comparing both identities, we observe

$$f(m,n)m_0 + f(m_0,n)m = m g(m_0,n) + m_0 g(m,n). \tag{3.12}$$

Substituting $n+n_0$ for n in (3.2) and comparing both identities, we know that

$$g(m,n)n_0 + g(m,n_0)n = n f(m,n_0) + n_0 f(m,n). \tag{3.13}$$

Setting $n = n_0$ in (3.12) and $m = m_0$ in (3.13) we see that

$$f(m, n_0)m_0 = m_0 g(m, n_0) \quad \text{and} \quad g(m_0, n)n_0 = n_0 f(m_0, n). \tag{3.14}$$

Substituting $m + m_0$ for m and $n + n_0$ for n in (3.2), we arrive at

$$\begin{aligned} f(m + m_0, n + n_0)(m + m_0) &= (m + m_0)g(m + m_0, n + n_0). \\ g(m + m_0, n + n_0)(n + n_0) &= (n + n_0)f(m + m_0, n + n_0). \end{aligned}$$

For the sake of relations (3.9)–(3.14), we obtain

$$f(m_0, n)m_0 = m_0 g(m_0, n) \quad \text{and} \quad g(m, n_0)n_0 = n_0 f(m, n_0). \tag{3.15}$$

Substituting $m + m_0$ for m and $n + n_0$ for n in (3.10) and using relation (3.10), we get

$$g(m_0, n)n_0 = n_0 f(m_0, n) \quad \text{and} \quad f(m, n_0)m_0 = m_0 g(m, n_0). \tag{3.16}$$

Combining (3.15) with (3.16) gives

$$\begin{bmatrix} f(m, n_0) & 0 \\ 0 & g(m, n_0) \end{bmatrix} \in \mathcal{Z}(\mathcal{G}) \quad \text{and} \quad \begin{bmatrix} f(m_0, n) & 0 \\ 0 & g(m_0, n) \end{bmatrix} \in \mathcal{Z}(\mathcal{G}). \tag{3.17}$$

It follows from (3.17) that

$$\begin{aligned} f(m_0, n)m &= m g(m_0, n) \quad \text{and} \quad g(m_0, n)n = n f(m_0, n), \\ f(m, n_0)m &= m g(m, n_0) \quad \text{and} \quad g(m, n_0)n = n f(m, n_0). \end{aligned} \tag{3.18}$$

Relations (3.12), (3.13) together with (3.18) imply that

$$f(m, n)m_0 = m_0 g(m, n) \quad \text{and} \quad g(m, n)n_0 = n_0 f(m, n).$$

That is

$$\begin{bmatrix} f(m, n) & 0 \\ 0 & g(m, n) \end{bmatrix} \in \mathcal{Z}(\mathcal{G}).$$

□

Lemma 3.9 ([58], Lemma 3.2). *Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ be a generalized matrix algebra over a commutative ring $\mathcal{R}$. Suppose that $[A, A] \neq 0$ and there exist $a_0 \in A$, $m_0 \in M$ such that $a_0 m_0$ and m_0 are independent over $\mathcal{Z}(A)$. If $f\colon M \to \mathcal{Z}(A)$ is an $\mathcal{R}$-linear mapping and $g\colon A \times M \to \mathcal{Z}(A)$ is an $\mathcal{R}$-bilinear mapping such that*

$$(f(m)a + g(a, m))m = 0$$

for all $a \in A$, $m \in M$, then $f(m) = 0 = g(a, m)$ for all $a \in A$, $m \in M$.

Lemma 3.10 ([58], Lemma 3.3). *Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ be a generalized matrix algebra over a commutative ring $\mathcal{R}$. Suppose that $[B, B] \neq 0$ and there exists $b_0 \in B$, $m_0 \in M$ such that $m_0 b_0$ and m_0 are independent over $\mathcal{Z}(B)$. If $f\colon M \to \mathcal{Z}(B)$ is an $\mathcal{R}$-linear mapping and $g\colon B \times M \to \mathcal{Z}(B)$ is an $\mathcal{R}$-bilinear mapping such that*

$$m(f(m)b + g(b, m)) = 0$$

for all $b \in B$, $m \in M$, then $f(m) = 0 = g(b, m)$ for all $b \in B$, $m \in M$.

To round off we need to give two more useful lemmas.

Lemma 3.11. *Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ be a generalized matrix algebra over a commutative ring $\mathcal{R}$ with $\frac{1}{2} \in \mathcal{R}$ such that $\pi_A(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(A)$ and $\pi_B(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(B)$. Then both A and B are commutative if and only if there exist an $\mathcal{R}$-liner mapping $\mu\colon \mathcal{G} \to \mathcal{Z}(\mathcal{G})$ and a trace $\nu\colon \mathcal{G} \to \mathcal{Z}(\mathcal{G})$ of some $\mathcal{R}$-bilinear mapping such that*

$$x^2 + \mu(x)x + \nu(x) = 0$$

for all $x \in G$.

P r o o f. We first assert that if both A and B are commutative, then $\left[\begin{smallmatrix} mn & 0 \\ 0 & nm \end{smallmatrix}\right] \in \mathcal{Z}(\mathcal{G})$ for all $m \in M$, $n \in N$.

Let us define two $\mathcal{R}$-bilinear mappings $f\colon M \times N \to \mathcal{Z}(A)$ and $g\colon N \times M \to \mathcal{Z}(B)$ by the relations $f(m, n) = mn$ and $g(m, n) = nm$, respectively. Then they satisfy the relations $f(m, n)m = mg(m, n)$ and $g(m, n)n = nf(m, n)$. In light of Lemma 3.8, we obtain the assertion.

Suppose that both A and B are commutative. We take

$$\mu\begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} -a - \varphi^{-1}(b) & 0 \\ 0 & -\varphi(a) - b \end{bmatrix}$$

and

$$\nu\begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} a\varphi^{-1}(b) - mn & 0 \\ 0 & \varphi(a)b - nm \end{bmatrix}$$

for all $a \in A$, $b \in B$, $n \in N$, $m \in M$. It is easy to verify that $x^2 + \mu(x)x + \nu(x) = 0$ holds true for all $x \in \mathcal{G}$.

Conversely, suppose that there exist an $\mathcal{R}$-linear mapping $\mu\colon \mathcal{G} \to \mathcal{Z}(\mathcal{G})$ and a trace $\nu\colon \mathcal{G} \to \mathcal{Z}(\mathcal{G})$ of some $\mathcal{R}$-bilinear mapping such that

$$x^2 + \mu(x)x + \nu(x) = 0$$

for all $x \in \mathcal{G}$. One can easily check that $[[x^2, y], [x, y]] = 0$ for all $x, y \in \mathcal{G}$. Applying [62], Lemma 4.1 yields that both A and B are commutative. $\square$

Adopting the same methods as [58], Lemma 2.4, one can prove the following lemma.

Lemma 3.12. *Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ be a generalized matrix algebra over a commutative ring $\mathcal{R}$ with $\frac{1}{2} \in \mathcal{R}$. Then both A and B are commutative if and only if*

$$[[[x^2, y], z], [x, y]] = 0$$

for all $x, y, z \in \mathcal{G}$.

We are ready to prove the main theorem of this section.

Theorem 3.13. *Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ be a generalized matrix algebra over a commutative ring $\mathcal{R}$ with $\frac{1}{2} \in \mathcal{R}$. Let $\mathfrak{q}\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ be an $\mathcal{R}$-bilinear mapping. Suppose that*

(1) *every commuting linear mapping on A or B is proper,*
(2) *$\pi_A(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(A)$ and $\pi_B(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(B)$,*
(3) *if $[A, A] \neq 0$ and $[B, B] = 0$, then there exist $a_0 \in A$, $m_0 \in M$ such that $a_0 m_0$ and m_0 are independent over $\mathcal{Z}(A)$,*
(4) *if $[A, A] = 0$ and $[B, B] \neq 0$, then there exist $b_0 \in B$, $m_0 \in M$ such that $m_0 b_0$ and m_0 are independent over $\mathcal{Z}(B)$,*
(5) *there exist $m_0 \in M$, $n_0 \in N$ such that*

$$\mathcal{Z}(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \colon\ am_0 = m_0 b,\ n_0 a = b n_0\ \forall\, a \in \mathcal{Z}(A),\, b \in \mathcal{Z}(B) \right\},$$

(6) *M is weakly loyal.*

If $\mathfrak{T}_{\mathfrak{q}}\colon \mathcal{G} \to \mathcal{G}$ is a commuting trace of the bilinear mapping $\mathfrak{q}$, then there exist $\lambda \in \mathcal{Z}(\mathcal{G})$, an $\mathcal{R}$-linear mapping $\mu\colon \mathcal{G} \to \mathcal{Z}(\mathcal{G})$ and a trace $\nu\colon \mathcal{G} \to \mathcal{Z}(\mathcal{G})$ of some $\mathcal{R}$-bilinear mapping such that

$$\mathfrak{T}_{\mathfrak{q}}(x) = \lambda x^2 + \mu(x) x + \nu(x)$$

for all $x \in \mathcal{G}$. If A and B are commutative, we may take $\lambda = 0$.

For convenience, let us write $A_1 = A$, $A_2 = M$, $A_3 = N$ and $A_4 = B$. Suppose that $\mathfrak{T}_{\mathfrak{q}}$ is an arbitrary trace of the $\mathcal{R}$-bilinear mapping $\mathfrak{q}$. Then there exist $\mathcal{R}$-bilinear mappings $f_{ij}\colon A_i \times A_j \to A_1$, $g_{ij}\colon A_i \times A_j \to A_2$, $h_{ij}\colon A_i \times A_j \to A_3$ and $k_{ij}\colon A_i \times A_j \to A_4$ $(1 \leqslant i \leqslant j \leqslant 4)$ such that

$$\begin{aligned} &\mathfrak{T}_{\mathfrak{q}}\colon \mathcal{G} \to \mathcal{G} \\ &\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mapsto \begin{bmatrix} F(a_1, a_2, a_3, a_4) & G(a_1, a_2, a_3, a_4) \\ H(a_1, a_2, a_3, a_4) & K(a_1, a_2, a_3, a_4) \end{bmatrix} \quad \text{for all } \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in \mathcal{G}, \end{aligned}$$

where

$$F(a_1,a_2,a_3,a_4)=\sum_{1\leqslant i\leqslant j\leqslant 4} f_{ij}(a_i,a_j),\quad G(a_1,a_2,a_3,a_4)=\sum_{1\leqslant i\leqslant j\leqslant 4} g_{ij}(a_i,a_j),$$
$$H(a_1,a_2,a_3,a_4)=\sum_{1\leqslant i\leqslant j\leqslant 4} h_{ij}(a_i,a_j),\quad K(a_1,a_2,a_3,a_4)=\sum_{1\leqslant i\leqslant j\leqslant 4} k_{ij}(a_i,a_j).$$

Since $\mathfrak{T}_{\mathfrak{q}}$ is commuting, we have

$$(\star)\qquad \begin{aligned} 0 &= \left[\begin{bmatrix} F & G \\ H & K \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right] \\ &= \begin{bmatrix} Fa_1+Ga_3-a_1F-a_2H & Fa_2+Ga_4-a_1G-a_2K \\ Ha_1+Ka_3-a_3F-a_4H & Ha_2+Ka_4-a_3G-a_4K \end{bmatrix} \end{aligned}$$

for all $\left[\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right] \in \mathcal{G}$.

Now we divide the proof of Theorem 3.13 into a series of lemmas for comfortable reading.

Lemma 3.14 ([62], Lemma 3.5). *With notations as above we have*

$$H(a_1,a_2,a_3,a_4)=h_{13}(a_1,a_3)+h_{23}(a_2,a_3)+h_{33}(a_3,a_3)+h_{34}(a_3,a_4).$$

Lemma 3.15 ([62], Lemma 3.6). *With notations as above we have*

$$G(a_1,a_2,a_3,a_4)=g_{12}(a_1,a_2)+g_{22}(a_2,a_2)+g_{23}(a_2,a_3)+g_{24}(a_2,a_4).$$

Lemma 3.16 ([62], Lemma 3.7). *With notations as above we have:*

(1) $a_1 \mapsto f_{11}(a_1,a_1)$ *is a commuting trace,*
(2) $a_1 \mapsto f_{12}(a_1,a_2)$, $a_1 \mapsto f_{13}(a_1,a_3)$, $a_1 \mapsto f_{14}(a_1,a_4)$ *are commuting linear mappings for each* $a_2 \in A_2$, $a_3 \in A_3$, $a_4 \in A_4$, *respectively,*
(3) f_{22}, f_{24}, f_{33}, f_{34}, f_{44} *map into* $\mathcal{Z}(A_1)$.

Lemma 3.17 ([62], Lemma 3.8). *With notations as above we have:*

(1) $a_4 \mapsto k_{44}(a_4,a_4)$ *is a commuting trace,*
(2) $a_4 \mapsto k_{14}(a_1,a_4)$, $a_4 \mapsto k_{24}(a_2,a_4)$, $a_4 \mapsto k_{34}(a_3,a_4)$ *are commuting mappings for each* $a_1 \in A_1$, $a_2 \in A_2$, $a_3 \in A_3$, *respectively,*
(3) k_{11}, k_{12}, k_{13}, k_{22}, k_{33} *map into* $\mathcal{Z}(A_4)$.

Lemma 3.18.

$$\begin{bmatrix} f_{22}(a_2, a_2) & 0 \\ 0 & k_{22}(a_2, a_2) \end{bmatrix} \in \mathcal{Z}(\mathcal{G}) \quad \text{and} \quad \begin{bmatrix} f_{33}(a_3, a_3) & 0 \\ 0 & k_{33}(a_3, a_3) \end{bmatrix} \in \mathcal{Z}(\mathcal{G}).$$

P r o o f. By relation $(\star)$, we know that

$$Fa_2 + Ga_4 - a_1G - a_2K = 0 \tag{3.19}$$

for all $a_i \in A_i$, $i = 1, 2, 3, 4$. Assigning $a_1 = 0$, $a_4 = 0$ in (3.19), we obtain

$$(f_{22}(a_2, a_2) + f_{23}(a_2, a_3) + f_{33}(a_3, a_3))a_2 = a_2(k_{22}(a_2, a_2) + k_{23}(a_2, a_3) + k_{33}(a_3, a_3)) \tag{3.20}$$

for all $a_i \in A_i$ $(i = 2, 3)$. Furthermore, setting $a_3 = 0$ in (3.20), we get

$$f_{22}(a_2, a_2)a_2 = a_2k_{22}(a_2, a_2) \tag{3.21}$$

for all $a_2 \in A_2$. Taking into account (3.20) and (3.21), we conclude that

$$(f_{22}(a_2, a_3) + f_{33}(a_3, a_3))a_2 = a_2(k_{22}(a_2, a_3) + k_{33}(a_3, a_3)) \tag{3.22}$$

for all $a_i \in A_i$ $(i = 2, 3)$. Replacing a_3 by $-a_3$ in (3.22) and comparing both identities, we arrive at

$$f_{33}(a_3, a_3)a_2 = a_2k_{33}(a_3, a_3) \tag{3.23}$$

for all $a_i \in A_i$ $(i = 2, 3)$.

On the other hand, according to relations $(\star)$, we know that

$$Ha_1 + Ka_3 - a_3F - a_4H = 0 \tag{3.24}$$

for all $a_i \in A_i$ $(i = 1, 2, 3, 4)$. Adopting similar methods for (3.24), one can show that

$$a_3f_{22}(a_2, a_2) = k_{22}(a_2, a_2)a_3 \quad \text{and} \quad a_3f_{33}(a_3, a_3) = k_{33}(a_3, a_3)a_3 \tag{3.25}$$

for all $a_i \in A_i$ $(i = 2, 3)$. By relations (3.21), (3.23), (3.25) and Lemma 3.8, we assert that

$$\begin{bmatrix} f_{22}(a_2, a_2) & 0 \\ 0 & k_{22}(a_2, a_2) \end{bmatrix} \in \mathcal{Z}(\mathcal{G})$$

and

$$\begin{bmatrix} f_{33}(a_3, a_3) & 0 \\ 0 & k_{33}(a_3, a_3) \end{bmatrix} \in \mathcal{Z}(\mathcal{G}).$$

□

Lemma 3.19. *With notations as above, we have*

(1) $f_{12}(a_1, a_2) = \alpha(a_2)a_1 + \varphi^{-1}(k_{12}(a_1, a_2))$,

(2) $k_{24}(a_2, a_4) = \varphi(\alpha(a_2))a_4 + \varphi(f_{24}(a_2, a_4))$,

where $\alpha(a_2) = f_{12}(1, a_2) - \varphi^{-1}(k_{12}(1, a_2))$.

P r o o f. Let us take $a_1 = 0$ and $a_4 = 0$ in (3.19). Then (3.19) implies that

$$(f_{22}(a_2, a_2) + f_{23}(a_2, a_3) + f_{33}(a_3, a_3))a_2 = a_2(k_{22}(a_2, a_2) + k_{23}(a_2, a_3) + k_{33}(a_3, a_3)) \tag{3.26}$$

for all $a_2 \in A_2$, $a_3 \in A_3$. Taking $a_4 = 0$ in (3.19) and using (3.26), we see that

$$(f_{11}(a_1, a_1) + f_{12}(a_1, a_2) + f_{13}(a_1, a_3))a_2 - a_2(k_{11}(a_1, a_1) + k_{12}(a_1, a_2) + k_{13}(a_1, a_3)) - a_1(g_{12}(a_1, a_2) + g_{22}(a_2, a_2) + g_{23}(a_2, a_3)) = 0 \tag{3.27}$$

for all $a_2 \in A_2$, $a_3 \in A_3$. Substituting $-a_1$ for a_1 in (3.27), we get

$$(f_{11}(a_1, a_1) - f_{12}(a_1, a_2) - f_{13}(a_1, a_3))a_2 - a_2(k_{11}(a_1, a_1) - k_{12}(a_1, a_2) - k_{13}(a_1, a_3)) - a_1(g_{12}(a_1, a_2) - g_{22}(a_2, a_2) - g_{23}(a_2, a_3)) = 0 \tag{3.28}$$

for all $a_2 \in A_2$, $a_3 \in A_3$. Combining (3.27) with (3.28) gives

$$a_1 g_{12}(a_1, a_2) = f_{11}(a_1, a_1)a_2 - a_2 k_{11}(a_1, a_1), \tag{3.29}$$

$$a_1 g_{22}(a_2, a_2) = f_{12}(a_1, a_2)a_2 - a_2 k_{12}(a_1, a_2), \tag{3.30}$$

$$a_1 g_{23}(a_2, a_3) = f_{13}(a_1, a_3)a_2 - a_2 k_{13}(a_1, a_3) \tag{3.31}$$

for all $a_2 \in A_2$, $a_3 \in A_3$. In an analogous way, taking $a_1 = 0$ in (3.19) and using (3.26), we obtain

$$g_{24}(a_2, a_4)a_4 = a_2 k_{44}(a_4, a_4) - f_{44}(a_4, a_4)a_2, \tag{3.32}$$

$$g_{22}(a_2, a_2)a_4 = a_2 k_{24}(a_2, a_4) - f_{24}(a_2, a_4)a_2, \tag{3.33}$$

$$g_{23}(a_2, a_3)a_4 = a_2 k_{34}(a_3, a_4) - f_{34}(a_3, a_4)a_2 \tag{3.34}$$

for all $a_2 \in A_2$, $a_3 \in A_3$, $a_4 \in A_4$.

According to (3.24) and using similar methods, we obtain the relations

$$h_{23}(a_2, a_3)a_1 = a_3 f_{12}(a_1, a_2) - k_{12}(a_1, a_2)a_3, \tag{3.35}$$

$$h_{33}(a_3, a_3)a_1 = a_3 f_{13}(a_1, a_3) - k_{13}(a_1, a_3)a_3, \tag{3.36}$$

$$h_{13}(a_1, a_3)a_1 = a_3 f_{11}(a_1, a_1) - k_{11}(a_1, a_1)a_3, \tag{3.37}$$

$$a_3 f_{23}(a_2, a_3) = k_{23}(a_2, a_3)a_3, \tag{3.38}$$

$$a_4 h_{34}(a_3, a_4) = k_{44}(a_4, a_4)a_3 - a_3 f_{44}(a_4, a_4), \tag{3.39}$$

$$(3.40) \qquad a_4 h_{33}(a_3, a_3) = k_{34}(a_3, a_4)a_3 - a_3 f_{34}(a_3, a_4),$$

$$(3.41) \qquad a_4 h_{23}(a_2, a_3) = k_{24}(a_2, a_4)a_3 - a_3 f_{24}(a_2, a_4)$$

for all $a_i \in A_i$, $i = 1, 2, 3, 4$.

On the other hand, by (2) in Lemma 3.16, we know that $[f_{12}(a_1, a_2), a_1] = 0$ for all $a_1 \in A_1$, $a_2 \in A_2$. Substituting $a_1 + 1$ for a_1 in $[f_{12}(a_1, a_2), a_1] = 0$ leads to $f_{12}(1, a_2) \in \mathcal{Z}(A_1)$ for all $a_2 \in A_2$. By relation (3.30) it follows that $g_{22}(a_2, a_2) = \alpha(a_2)a_2$, where $\alpha(a_2) = f_{12}(1, a_2) - \varphi^{-1}(k_{12}(1, a_2)) \in \mathcal{Z}(A_1)$.

Let us set $E(a_1, a_2) = f_{12}(a_1, a_2) - \alpha(a_2)a_1 - \varphi^{-1}(k_{12}(a_1, a_2))$. Then (3.30) together with $g_{22}(a_2, a_2) = \alpha(a_2)a_2$ imply that

$$E(a_1, a_2)a_2 = 0$$

for all $a_1 \in A_1$ and $a_2 \in A_2$.

Using similar methods, setting $a_1 = 1$ in (3.35), one can show that

$$(3.42) \qquad h_{23}(a_2, a_3) = a_3 f_{12}(1, a_2) - k_{12}(1, a_2)a_3 = a_3\alpha(a_2).$$

Furthermore, by relation (3.35) and (3.42) it follows that

$$a_3 E(a_1, a_2) = 0.$$

Claim. $E(a_1, a_2) = 0$, i.e., $f_{12}(a_1, a_2) = \alpha(a_2)a_1 + \varphi^{-1}(k_{12}(a_1, a_2))$. The process can be divided into two cases.

Case 1: $[A_4, A_4] \neq 0$.

With the help of the definition of $E(a_1, a_2)$, one can define a mapping $E(a_1, \cdot)\colon A_2 \to A_1$. By the previous discussion, we know that this mapping satisfies the condition $E(a_1, a_2)a_2 = 0$ for all $a_1 \in A_1$. Replacing a_2 by $m_1 + m_2$ in $E(a_1, a_2)a_2 = 0$ for each $a_1 \in A_1$, we obtain

$$E(a_1, m_1)m_2 + E(a_1, m_2)m_1 = 0$$

for all $m_1, m_2 \in A_2$. Since A_4 is noncommutative, we conclude that $E(a_1, m_2) = 0$ by (1) of Lemma 3.2, hence $f_{12}(a_1, a_2) = \alpha(a_2)a_1 + \varphi^{-1}(k_{12}(a_1, a_2))$.

Case 2: $[A_4, A_4] = 0$.

In order to obtain this claim under this subcase, we will divide its proof into two subcases.

Case 2.1: $[A_1, A_1] = 0$.

In this subcase, A_1 is a commutative algebra. Thus, we get

$$E(a_1, a_2) = f_{12}(a_1, a_2) - \alpha(a_2)a_1 - \varphi^{-1}(k_{12}(a_1, a_2)) \in \mathcal{Z}(A).$$

On the other hand, we should remark that $E(a_1, a_2)a_2 = 0 = a_2 0$ and $a_3 E(a_1, a_2) = 0 = 0 a_3$ holds true for all $a_1 \in A_1$. By the conclusion (1) of Lemma 3.8, we can say that $E(a_1, a_2) \oplus 0 \in \mathcal{Z}(\mathcal{G})$ and hence $E(a_1, a_2) = 0$. That is,

$$f_{12}(a_1, a_2) = \alpha(a_2)a_1 + \varphi^{-1}(k_{12}(a_1, a_2)),$$

which is the desired form.

Case 2.2: $[A_1, A_1] \neq 0$.

Since $a_1 \mapsto f_{12}(a_1, a_2)$ is a commuting mapping of A_1 for each $a_2 \in A_2$, there exist mappings $\xi\colon A_4 \to \mathcal{Z}(A_1)$ and $\kappa\colon A_1 \times A_2 \to \mathcal{Z}(A_1)$ such that

$$f_{12}(a_1, a_2) = \xi(a_2)a_1 + \kappa(a_1, a_2), \tag{3.43}$$

where κ is $\mathcal{R}$-linear in the first argument. Let us show that ξ is $\mathcal{R}$-linear and that κ is $\mathcal{R}$-bilinear. It is not difficult to see that

$$f_{12}(a_1, a_2 + b_2) = \xi(a_2 + b_2)a_1 + \kappa(a_1, a_2 + b_2)$$

and

$$f_{12}(a_1, a_2) + f_{12}(a_1, b_2) = \xi(a_2)a_1 + \kappa(a_1, a_2) + \xi(b_2)a_1 + \kappa(a_1, b_2)$$

for all $a_1 \in A_1$ and $a_2, b_2 \in A_2$. We therefore have

$$(\xi(a_2 + b_2) - \xi(a_2) - \xi(b_2))a_1 + \kappa(a_1, a_2 + b_2) - \kappa(a_1, a_2) - \kappa(a_1, b_2) = 0$$

for all $a_1 \in A_1$ and $a_2, b_2 \in A_2$. Note that both ξ and κ map into $\mathcal{Z}(A_1)$. And hence $(\xi(a_2 + b_2) - \xi(a_2) - \xi(b_2))[a_1, b_1] = 0$ for all $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$. Applying Lemma 3.1 yields that ξ is $\mathcal{R}$-linear. Consequently, κ is $\mathcal{R}$-linear in the second argument as well.

Taking (3.43) in the equality $E(a_1, a_2)a_2 = 0$, we get

$$(\xi(a_2) - \alpha(a_2))a_1 a_2 + (\kappa(a_1, a_2) - \varphi^{-1}(k_{12}(a_1, a_2)))a_2 = 0$$

for all $a_1 \in A_1$ and $a_2 \in A_2$. It should be remarked that $\xi(a_2) - \alpha(a_2) \in \mathcal{Z}(A_1)$ and that $\kappa(a_1, a_2) - \varphi^{-1}(k_{12}(a_1, a_2)) \in \mathcal{Z}(A_1)$. By assumption (3) and Lemma 3.9 it follows that $\xi(a_2) = \alpha(a_2)$ and $\kappa(a_1, a_2) = \varphi^{-1}(k_{12}(a_1, a_2))$ for all $a_1 \in A_1$ and $a_2 \in A_2$. Hence f_{12} has the desired form.

In an analogous manner, one can use relations (3.33) and (3.41) to show that k_{24} is of the desired form as well. $\square$

Lemma 3.20. *With notations as above we have*

(1) $f_{13}(a_1, a_3) = \tau(a_3)a_1 + \varphi^{-1}(k_{13}(a_1, a_3))$,

(2) $k_{34}(a_3, a_4) = \varphi(\tau(a_3))a_4 + \varphi(f_{34}(a_3, a_4))$,

where $\tau(a_3) = f_{13}(1, a_3) - \varphi^{-1}(k_{13}(1, a_3))$.

Proof. In light of (2) of Lemma 3.16, we know that $[f_{13}(a_1, a_3), a_1] = 0$ for all $a_1 \in A_1$, $a_3 \in A_3$. Substituting $a_1 + 1$ for a_1 in $[f_{13}(a_1, a_3), a_1] = 0$ gives $f_{13}(1, a_3) \in \mathcal{Z}(A_1)$ for all $a_3 \in A_3$. Taking $a_1 = 1$ in (3.31), we see that

$$g_{23}(a_2, a_3) = f_{13}(1, a_3)a_2 - a_2k_{13}(1, a_3) = \tau(a_3)a_2,$$

where $\tau(a_3) = f_{13}(1, a_3) - \varphi^{-1}(k_{13}(1, a_3)) \in \mathcal{Z}(A_1)$.

Let us set $S(a_1, a_3) = f_{13}(a_1, a_3) - \tau(a_3)a_1 - \varphi^{-1}(k_{13}(a_1, a_3)) \in A_1$. It follows from (3.31) that $S(a_1, a_3)a_2 = 0$ for all $a_1 \in A_1$, $a_2 \in A_2$, $a_3 \in A_3$. Since $M = A_2$ is faithful as a left A-module, we get $S(a_1, a_3) = 0$. And hence $f_{13}(a_1, a_3) = \tau(a_3)a_1 + \varphi^{-1}(k_{13}(a_1, a_3))$. Likewise, using (3.34), one can prove that k_{34} is of the desired form as well. □

Lemma 3.21. *With notations as above we have:*

(1) *There exist linear mapping* $\gamma\colon A_4 \to \mathcal{Z}(A_1)$ *and bilinear mapping* $\delta\colon A_1 \times A_4 \to \mathcal{Z}(A_1)$ *such that* $f_{14}(a_1, a_4) = \gamma(a_4)a_1 + \delta(a_1, a_4)$.

(2) $k_{14}(a_1, a_4) = \gamma'(a_1)a_4 + \varphi(\delta(a_1, a_4))$, *where* $\gamma'(a_1) = k_{14}(a_1, 1) - \varphi(\delta(a_1, 1))$.

Proof. Notice the fact $[f_{11}(a_1, a_1), a_1] = 0$, which is due to (1) of Lemma 3.16. Substituting $a_1 + 1$ for a_1 in $[f_{11}(a_1, a_1), a_1] = 0$ yields

$$[f_{11}(a_1, 1) + f_{11}(1, a_1) + f_{11}(1, 1), a_1] = 0.$$

Replacing a_1 by $-a_1$ in this equality and comparing both identities, we obtain $[f_{11}(1, 1), a_1] = 0$. That is, $f_{11}(1, 1) \in \mathcal{Z}(A_1)$. Taking $a_1 = 1$ in (3.29), we arrive at

$$g_{12}(1, a_2) = f_{11}(1, 1)a_2 - a_2k_{11}(1, 1) = a_2\zeta, \tag{3.44}$$

where $\zeta = \varphi(f_{11}(1, 1)) - k_{11}(1, 1)$ for all $a_2 \in A_2$.

On the other hand, relations (3.29)–(3.34) together with (3.19) imply that

$$f_{14}(a_1, a_4)a_2 + g_{12}(a_1, a_2)a_4 = a_1g_{24}(a_2, a_4) + a_2k_{14}(a_1, a_4) \tag{3.45}$$

for all $a_1 \in A_1$, $a_2 \in A_2$, $a_4 \in A_4$. Let us take $a_1 = 1$ in (3.45). Then we have

$$g_{24}(a_2, a_4) = a_2(\zeta a_4 + \varphi(f_{14}(1, a_4)) - k_{14}(1, a_4)) \tag{3.46}$$

for all $a_2 \in A_2$, $a_4 \in A_4$, where $\zeta = \varphi(f_{11}(1,1)) - k_{11}(1,1)$. In a similar discussion, considering relations (3.32), (3.35) together with equality $[k_{44}(a_4,a_4),a_4] = 0$, we arrive at

$$g_{12}(a_1,a_2) = (\theta a_1 + \varphi^{-1}(k_{14}(a_1,1)) - f_{14}(a_1,1))a_2 \tag{3.47}$$

for all $a_1 \in A_1$, $a_2 \in A_2$, where $\theta = \varphi^{-1}(k_{44}(1,1)) - f_{44}(1,1)$.

In order to obtain the conclusion of this lemma, we divide its proof into two different cases.

Case 1: $[A_1,A_1] \neq 0$.

Since $a_1 \mapsto f_{14}(a_1,a_4)$ is a commuting mapping of A_1 for all $a_4 \in A_4$, there exist mappings $\gamma\colon A_4 \to \mathcal{Z}(A_1)$ and $\delta\colon A_1 \times A_4 \to \mathcal{Z}(A_1)$ such that

$$f_{14}(a_1,a_4) = \gamma(a_4)a_1 + \delta(a_1,a_4), \tag{3.48}$$

where δ is $\mathcal{R}$-linear in the first argument. Let us show that γ is $\mathcal{R}$-linear and that δ is $\mathcal{R}$-bilinear. It is easy to observe that

$$f_{14}(a_1,a_4+b_4) = \gamma(a_4+b_4)a_1 + \delta(a_1,a_4+b_4)$$

and

$$f_{14}(a_1,a_4) + f_{14}(a_1,b_4) = \gamma(a_4)a_1 + \delta(a_1,a_4) + \gamma(b_4)a_1 + \delta(a_1,b_4)$$

for all $a_1 \in A_1$ and $a_4,b_4 \in A_4$. We therefore assert that

$$\big(\gamma(a_4+b_4) - \gamma(a_4) - \gamma(b_4)\big)a_1 + \delta(a_1,a_4+b_4) - \delta(a_1,a_4) - \delta(a_1,b_4) = 0$$

for all $a_1 \in A_1$ and $a_4,b_4 \in A_4$. Note that both γ and δ map into $\mathcal{Z}(A_1)$. Thus, $(\gamma(a_4+b_4) - \gamma(a_4) - \gamma(b_4))[a_1,b_1] = 0$ for all $a_1,b_1 \in A_1$ and $a_4,b_4 \in A_4$. Applying Lemma 3.1 yields that γ is $\mathcal{R}$-linear. Consequently, δ is $\mathcal{R}$-linear in the second argument.

Now equalities (3.45)–(3.48) jointly show that

$$\begin{aligned}(\gamma(a_4)a_1 + \delta(a_1,a_4))a_2 + (\theta a_1 + \varphi^{-1}(k_{14}(a_1,1)) - f_{14}(a_1,1))a_2a_4 \\ = a_2k_{14}(a_1,a_4) + a_1a_2(\zeta a_4 + \varphi(f_{14}(1,a_4)) - k_{14}(1,a_4))\end{aligned}$$

for all $a_1 \in A_1$, $a_2 \in A_2$, $a_4 \in A_4$. That is,

$$\begin{aligned}a_1a_2(\zeta + \varphi(\gamma(1) - \theta)a_4 + \varphi(\delta(1,a_4)) - k_{14}(1,a_4)) \\ = a_2(\gamma'(a_1)a_4 + \varphi(\delta(a_1,a_4)) - k_{14}(a_1,a_4)),\end{aligned} \tag{3.49}$$

where $\gamma'(a_1) = k_{14}(a_1, 1) - \varphi(\delta(a_1, 1))$ for all $a_1 \in A_1$, $a_2 \in A_2$, $a_4 \in A_4$. Replacing a_2 by $b_1 a_2$ in (3.49) and subtracting the left multiplication of (3.49) by b_1 gives

$$[a_1, b_1]a_2(\zeta + \varphi(\gamma(1) - \theta)a_4 + \varphi(\delta(1, a_4)) - k_{14}(1, a_4)) = 0$$

for all $a_1, b_1 \in A_1$, $a_2 \in A_2$, $a_4 \in A_4$. Note that $M = A_2$ is weakly loyal and $A = A_1$ is noncommutative. It follows that

$$k_{14}(1, a_4) = (\zeta + \varphi(\gamma(1) - \theta)a_4 + \varphi(\delta(1, a_4))$$

for all $a_4 \in A_4$. Consequently, relation (3.49) implies that

$$A_2(\gamma'(a_1)a_4 + \varphi(\delta(a_1, a_4)) - k_{14}(a_1, a_4)) = 0$$

for all $a_1, a_4 \in A_4$. Since $A_2 = M$ is weakly loyal as a right B-module, k_{14} is of the desired form.

Case 2: $[A_1, A_1] = 0$.

It follows from relations (3.45)–(3.47) that

$$\begin{aligned} f_{14}(a_1, a_4)a_2 + (\theta a_1 + \varphi^{-1}(k_{14}(a_1, 1)) - f_{14}(a_1, 1))a_2 a_4 \\ = a_1 a_2(\zeta a_4 + \varphi(f_{14}(1, a_4)) - k_{14}(1, a_4)) + a_2 k_{14}(a_1, a_4) \end{aligned} \tag{3.50}$$

for all $a_1 \in A_1$, $a_2 \in A_2$, $a_4 \in A_4$. This implies that

$$\begin{aligned} a_2\varphi(f_{14}(a_1, a_4)) + (\theta a_1 + \varphi^{-1}(k_{14}(a_1, 1)) - f_{14}(a_1, 1))a_2 a_4 \\ = a_2\varphi(a_1)(\zeta a_4 + \varphi(f_{14}(1, a_4)) - k_{14}(1, a_4)) + a_2 k_{14}(a_1, a_4) \end{aligned} \tag{3.51}$$

for all $a_1 \in A_1$, $a_2 \in A_2$, $a_4 \in A_4$. Since A_2 is faithful as a right A_4-module, from (3.51) we get

$$\begin{aligned} \varphi(f_{14}(a_1, a_4)) + (\varphi(\theta)\varphi(a_1) + k_{14}(a_1, 1) - \varphi(f_{14}(a_1, 1)))a_4 \\ = \varphi(a_1)(\zeta a_4 + \varphi(f_{14}(1, a_4)) - k_{14}(1, a_4)) + k_{14}(a_1, a_4) \end{aligned} \tag{3.52}$$

for all $a_1 \in A_1$, $a_2 \in A_2$, $a_4 \in A_4$. We further get the following equality based on the deformation of the above equality (3.52):

$$\begin{aligned} \varphi(f_{14}(a_1, a_4) - a_1(f_{14}(1, a_4) - \varphi^{-1}(k_{14}(1, a_4)))) \\ = k_{14}(a_1, a_4) + (\varphi(a_1)\zeta - (\varphi(\theta)\varphi(a_1) + k_{14}(a_1, 1)) + \varphi(f_{14}(a_1, 1)))a_4. \end{aligned} \tag{3.53}$$

Let us set

$$
\begin{aligned}
(3.54) \qquad \gamma(a_4) &= f_{14}(1,a_4) - \varphi^{-1}(k_{14}(1,a_4)) \\
\delta(a_1,a_4) &= -\gamma(a_4)a_1 + f_{14}(a_1,a_4) \\
\gamma^{'}(a_1) &= -\varphi(a_1)\varsigma + \varphi(\theta)\varphi(a_1) + k_{14}(a_1,1) - \varphi(f_{14}(a_1,1))
\end{aligned}
$$

for all $a_1 \in A_1$, $a_4 \in A_4$.

We claim that $\gamma'(a_1) = k_{14}(a_1,1) - \varphi(\delta(a_1,1))$. Indeed, setting $a_1 = 1$, $a_4 = 1$ in (3.45), we obtain

$$
(3.55) \qquad f_{14}(1,1)a_2 + g_{12}(1,a_2) = g_{24}(a_2,1) + a_2k_{14}(1,1).
$$

Taking $a_1 = 1$, $a_4 = 1$ in (3.47) and (3.46), respectively, we get

$$
\begin{aligned}
(3.56) \qquad g_{24}(a_2,1) &= a_2(\zeta + \varphi(f_{14}(1,1)) - k_{14}(1,1)), \\
g_{12}(1,a_2) &= (\theta + \varphi^{-1}(k_{14}(1,1)) - f_{14}(1,1))a_2,
\end{aligned}
$$

where $\zeta = \varphi(f_{11}(1,1)) - k_{11}(1,1)$, $\theta = \varphi^{-1}(k_{44}(1,1)) - f_{44}(1,1)$ for all $a_2 \in A_2$, $a_4 \in A_4$. Combining (3.55) with (3.56), we see that $\varphi(\theta) - \zeta = \varphi(f_{14}(1,1)) - k_{14}(1,1)$. Furthermore, we arrive at

$$
(3.57) \qquad \varphi(\theta)\varphi(a_1) - \zeta\varphi(a_1) = \varphi(f_{14}(1,1))\varphi(a_1) - k_{14}(1,1)\varphi(a_1)
$$

for all $a_1 \in A_1$. On the other hand,

$$
\begin{aligned}
(3.58) \qquad & k_{14}(a_1,1) - \varphi(\delta(a_1,1)) \\
&= k_{14}(a_1,1) - \varphi((f_{14}(1,1) - \varphi^{-1}(k_{14}(1,1)))a_1 - f_{14}(a_1,1)) \\
&= k_{14}(a_1,1) - \varphi(f_{14}(a_1,1)) - k_{14}(1,1)\varphi(a_1) + \varphi((f_{14}(1,1))\varphi(a_1).
\end{aligned}
$$

Considering relations (3.54), (3.57), (3.58) together with the definition of $\gamma'(a_1)$, this claim holds true.

Thus, equality (3.52) can be rewritten as

$$
k_{14}(a_1,a_4) - \gamma'(a_1)a_4 = \varphi(f_{14}(a_1,a_4)) - a_1\delta(a_1,a_4)) = \varphi(\delta(a_1,a_4)).
$$

That is, $k_{14}(a_1,a_4) = \gamma'(a_1)a_4 + \varphi(\delta(a_1,a_4))$. So $f_{14}(a_1,a_4)$ and $k_{14}(a_1,a_4)$ have the desired forms. $\square$

P r o o f of Theorem 3.13. Let us write $\varepsilon = \theta - \gamma(1)$ and $\varepsilon' = \zeta - \gamma'(1)$. Using equalities (3.46)–(3.47) and considering the form of f_{14}, k_{14}, we conclude that

$$g_{12}(a_1, a_2) = \varepsilon a_1 a_2 + \varphi^{-1}(\gamma'(a_1))a_2, \quad g_{24}(a_2, a_4) = a_2(\varepsilon' a_4 + \varphi(\gamma(a_4))) \tag{3.59}$$

for all $a_1 \in A_1$, $a_2 \in A_2$, $a_4 \in A_4$. Taking into accounts equalities (3.24) and (3.35)–(3.41) and adopting similar computational procedures, we obatin

$$h_{13}(a_1, a_3) = a_3 \varepsilon a_1 + \gamma'(a_1)a_3, \quad h_{34}(a_3, a_4) = \varepsilon' a_4 a_3 + \varphi(\gamma(a_4))a_3 \tag{3.60}$$

for all $a_1 \in A_1$, $a_3 \in A_3$, $a_4 \in A_4$. Taking $a_1 = 1$ and $a_4 = 1$ in (2.45) and combining Lemma 3.20 with (3.59), we arrive at $\varepsilon a_2 = a_2 \varepsilon'$ for all $a_2 \in A_2$. Note that $\varepsilon \in \mathcal{Z}(A_1) = \pi_A(\mathcal{Z}(\mathcal{G}))$ and $\varepsilon' \in \mathcal{Z}(A_4) = \pi_B(\mathcal{Z}(\mathcal{G}))$. We get $\left[\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon' \end{smallmatrix}\right] \in \mathcal{Z}(\mathcal{G})$.

It follows from (3.29) and (3.59) that

$$(f_{11}(a_1, a_1) - \varepsilon a_1^2 - \varphi^{-1}(\gamma'(a_1))a_1 - \varphi^{-1}(k_{11}(a_1, a_1)))a_2 = 0$$

for all $a_1 \in A_1$, $a_2 \in A_2$. Since $A_2 = M$ is faithful as a left A-module,

$$f_{11}(a_1, a_1) = \varepsilon a_1^2 + \varphi^{-1}(\gamma'(a_1))a_1 + \varphi^{-1}(k_{11}(a_1, a_1))$$

for all $a_1 \in A_1$. Likewise, in light of relations (3.39) and (3.60), we assert that

$$k_{44}(a_4, a_4) = \varepsilon' a_4^2 + \varphi(\gamma(a_4))a_4 + \varphi(f_{44}(a_4, a_4))$$

for all $a_4 \in A_4$.

Finally, let us set $z = \left[\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon' \end{smallmatrix}\right]$ and define the mapping $\mu\colon \mathcal{G} \to \mathcal{Z}(\mathcal{G})$ by

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mapsto \begin{bmatrix} \varphi^{-1}(\gamma'(a_1)) + \gamma(a_4) + \alpha(a_2) + \tau(a_3) & 0 \\ 0 & \gamma'(a_1) + \varphi(\gamma(a_4) + \alpha(a_2) + \tau(a_3)) \end{bmatrix}.$$

In view of all obtained results as above, we see that

$$\begin{aligned} \nu(x) &:= \mathfrak{T}_{\mathfrak{q}}(x) - zx^2 - \mu(x)x \\ &\equiv \begin{bmatrix} f_{23}(a_2, a_3) - \varepsilon a_2 a_3 & 0 \\ 0 & k_{23}(a_2, a_3) - \varepsilon' a_3 a_2 \end{bmatrix} \bmod \mathcal{Z}(\mathcal{G}), \end{aligned}$$

where $x = \left[\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right]$. Thus, we can write

$$\mathfrak{T}_{\mathfrak{q}}(x) = zx^2 + \mu(x)x + \begin{bmatrix} f_{23}(a_2, a_3) - \varepsilon a_2 a_3 & 0 \\ 0 & k_{23}(a_2, a_3) - \varepsilon' a_3 a_2 \end{bmatrix} + c$$

for some $c \in \mathcal{Z}(\mathcal{G})$. Since $\mathfrak{q}$ is a commuting mapping, we have

$$\left[\begin{bmatrix} f_{23}(a_2, a_3) - \varepsilon a_2 a_3 & 0 \\ 0 & k_{23}(a_2, a_3) - \varepsilon' a_3 a_2 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right] = 0.$$

This implies that $f_{23}(a_2, a_3) - \varepsilon a_2 a_3 \in \mathcal{Z}(A_1) = \pi_A(\mathcal{Z}(\mathcal{G}))$ and $k_{23}(a_2, a_3) - \varepsilon' a_3 a_2 \in \mathcal{Z}(A_4) = \pi_B(\mathcal{Z}(\mathcal{G}))$. Moreover, it shows that

$$(f_{23}(a_2, a_3) - \varepsilon a_2 a_3) a_2 = a_2 (k_{23}(a_2, a_3) - \varepsilon' a_3 a_2)$$

and

$$a_3 (f_{23}(a_2, a_3) - \varepsilon a_2 a_3) = (k_{23}(a_2, a_3) - \varepsilon' a_3 a_2) a_3$$

for all $a_2 \in A_2, a_3 \in A_3$. For convenience, let us write $\mathfrak{f}(a_2, a_3) = f_{23}(a_2, a_3) - \varepsilon a_2 a_3$ and $\mathfrak{k}(a_2, a_3) = k_{23}(a_2, a_3) - \varepsilon' a_3 a_2$, where $\mathfrak{f}\colon A_2 \times A_3 \to \mathcal{Z}(A_1)$ is an $\mathcal{R}$-bilinear mapping and $\mathfrak{k} : A_2 \times A_3 \to \mathcal{Z}(A_4)$ is also an $\mathcal{R}$-bilinear mapping. They satisfy the following relations:

$$\mathfrak{f}(a_2, a_3) a_2 = a_2 \mathfrak{k}(a_2, a_3) \quad \text{and} \quad a_3 \mathfrak{f}(a_2, a_3) = \mathfrak{k}(a_2, a_3) a_3.$$

We therefore get $\left[\begin{smallmatrix} \mathfrak{f}(a_2,a_3) & 0 \\ 0 & \mathfrak{k}(a_2,a_3) \end{smallmatrix}\right] \in \mathcal{Z}(\mathcal{G})$, which is due to (3) of Lemma 3.8. That is, $\left[\begin{smallmatrix} f_{23}(a_2,a_3) - \varepsilon a_2 a_3 & 0 \\ 0 & k_{23}(a_2,a_3) - \varepsilon' a_3 a_2 \end{smallmatrix}\right] \in \mathcal{Z}(\mathcal{G})$. Hence, ν maps $\mathcal{G}$ into $\mathcal{Z}(\mathcal{G})$.

In the case when A and B are commutative and considering Lemma 3.11, we may take $\lambda = 0$. $\square$

In particular, we have:

Corollary 3.22. *Let $\mathcal{R}$ be a 2-torsionfree commutative ring and $\mathcal{M}_n(\mathcal{R})$ $(n \geqslant 2)$ be the full matrix algebra over $\mathcal{R}$. Suppose that $\mathfrak{q}\colon \mathcal{M}_n(\mathcal{R}) \times \mathcal{M}_n(\mathcal{R}) \to \mathcal{M}_n(\mathcal{R})$ is an $\mathcal{R}$-bilinear mapping. Then every commuting trace $\mathfrak{T}_{\mathfrak{q}}\colon \mathcal{M}_n(\mathcal{R}) \to \mathcal{M}_n(\mathcal{R})$ of $\mathfrak{q}$ is proper.*

Corollary 3.23. *Let $\mathcal{R}$ be a 2-torsionfree commutative ring and V be an $\mathcal{R}$-linear space and $B(\mathcal{R}, V, \gamma)$ be the inflated algebra of $\mathcal{R}$ along V. Suppose that $\mathfrak{q}\colon B(\mathcal{R}, V, \gamma) \times B(\mathcal{R}, V, \gamma) \to B(\mathcal{R}, V, \gamma)$ is an $\mathcal{R}$-bilinear mapping. Then every commuting trace $\mathfrak{T}_{\mathfrak{q}}\colon B(\mathcal{R}, V, \gamma) \to B(\mathcal{R}, V, \gamma)$ of $\mathfrak{q}$ is proper.*

It should be remarked that Corollary 3.22 or Corollary 3.23 removes the assumption that $\mathcal{R}$ is a domain in [62], Corollary 3.18 or Corollary 3.19.

Corollary 3.24 ([58], Theorem 4.1)**.** *Let $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ be a 2-torsionfree triangular algebra over a commutative ring $\mathcal{R}$ and $\mathfrak{q}\colon \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ be an $\mathcal{R}$-bilinear mapping. Suppose that*

(1) *every commuting linear mapping on A or B is proper,*
(2) $\pi_A(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(A)$ *and* $\pi_B(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(B)$,
(3) *if $[A, A] \neq 0$ and $[B, B] = 0$, then there exist $a_0 \in A$, $m_0 \in M$ such that a_0m_0 and m_0 are independent over $\mathcal{Z}(A)$,*
(4) *if $[A, A] = 0$ and $[B, B] \neq 0$, then there exist $b_0 \in B$, $m_0 \in M$ such that m_0b_0 and m_0 are independent over $\mathcal{Z}(B)$,*
(5) *there exist $m_0 \in M$, $n_0 \in N$ such that*

$$\mathcal{Z}(\mathcal{T}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : \ am_0 = m_0b \ \forall\, a \in \mathcal{Z}(A), \ b \in \mathcal{Z}(B) \right\},$$

(6) *M is weakly loyal.*

If $\mathfrak{q}\colon \mathcal{T} \to \mathcal{T}$ is a commuting trace of the bilinear mapping $\mathfrak{q}$, then there exist $\lambda \in \mathcal{Z}(\mathcal{T})$, an $\mathcal{R}$-linear mapping $\mu\colon \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ and a trace $\nu\colon \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ of some $\mathcal{R}$-bilinear mapping such that

$$\mathfrak{T}_{\mathfrak{q}}(x) = \lambda x^2 + \mu(x)x + \nu(x)$$

for all $x \in \mathcal{T}$. If A and B are commutative, we may take $\lambda = 0$.

4. Centralizing traces of bilinear mappings on generalized matrix algebras

This section will be devoted to building the sufficient condition under which each centralizing trace of an arbitrary bilinear mapping on a generalized matrix algebra $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ is proper (Theorem 4.2). Accordingly, we can characterize centralizing traces of bilinear mappings on triangular algebras and those on full matrix algebras. In addition, the main theorem will be applied to describe Lie triple isomorphisms from a generalized matrix algebra into another one in Section 5.

We begin with the following technical result whose proof is basically similar to that of [58], Lemma 4.1.

Lemma 4.1. *Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ be a generalized matrix algebra over a commutative ring $\mathcal{R}$ with $\frac{1}{2} \in \mathcal{R}$.*
(1) *If centralizing linear mapping on A is proper, then A satisfies $(\heartsuit)$.*
(2) *If centralizing linear mapping on B is proper, then B satisfies $(\heartsuit)$.*

P r o o f. We only provide the proof of statement (2). Statement (1) can be proved in an analogous manner. Suppose that for any $b \in B$, $[b, B] \subseteq \mathcal{Z}(B)$. We may assume that $[B, B] \neq 0$. Then

$$[b, y^2] = [b, y]y + y[b, y] = 2[b, y]y = 2[by, y] \in \mathcal{Z}(B)$$

for all $y \in B$. This implies that $[by, y] \in \mathcal{Z}(B)$. Let us define a mapping $f\colon B \to B$ satisfying $f(y) = by$ for some $b \in B$. Then f is a linear mapping satisfying the condition $[by, y] \in \mathcal{Z}(B)$. Since every centralizing linear mapping on B is proper, we assert that there exist $\alpha \in \mathcal{Z}(B)$ and some central mapping $\beta\colon B \to \mathcal{Z}(B)$ satisfying the relation

$$f(y) = \lambda y + \beta(y).$$

Furthermore, we know that $by - \lambda y \in \mathcal{Z}(B)$ for all $y \in B$. Setting $y = 1$ in $by - \lambda y \in \mathcal{Z}(B)$, we obtain $b - \lambda \in \mathcal{Z}(B)$, i.e., $b \in \mathcal{Z}(B)$. $\square$

Applying Lemma 4.1 and Theorem 3.13, we can prove the following theorem.

Theorem 4.2. *Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ be a generalized matrix algebra over a commutative ring $\mathcal{R}$ with $\frac{1}{2} \in \mathcal{R}$. Let $\mathfrak{q}\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ be an $\mathcal{R}$-bilinear mapping. Suppose that*

(1) *every centralizing linear mapping on A or B is proper,*
(2) *$\pi_A(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(A)$ and $\pi_B(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(B)$,*
(3) *if $[A, A] \neq 0$ and $[B, B] = 0$, then there exist $a_0 \in A$, $m_0 \in M$ such that $a_0 m_0$ and m_0 are independent over $\mathcal{Z}(A)$,*
(4) *if $[A, A] = 0$ and $[B, B] \neq 0$, then there exist $b_0 \in B$, $m_0 \in M$ such that $m_0 b_0$ and m_0 are independent over $\mathcal{Z}(B)$,*
(5) *there exist $m_0 \in M$, $n_0 \in N$ such that*

$$\mathcal{Z}(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \colon\ am_0 = m_0 b,\ n_0 a = b n_0\ \forall\, a \in \mathcal{Z}(A),\ b \in \mathcal{Z}(B) \right\},$$

(6) *M is weakly loyal.*

If $\mathfrak{T}_{\mathfrak{q}}\colon \mathcal{G} \to \mathcal{G}$ is a centralizing trace of the bilinear mapping $\mathfrak{q}$, then there exist $\lambda \in \mathcal{Z}(\mathcal{G})$, an $\mathcal{R}$-linear mapping $\mu\colon \mathcal{G} \to \mathcal{Z}(\mathcal{G})$ and a trace $\nu\colon \mathcal{G} \to \mathcal{Z}(\mathcal{G})$ of some $\mathcal{R}$-bilinear mapping such that

$$\mathfrak{T}_{\mathfrak{q}}(x) = \lambda x^2 + \mu(x)x + \nu(x)$$

for all $x \in \mathcal{G}$.

P r o o f. Now following the same notations as in Theorem 3.13, we get

$$(\diamondsuit)\qquad \begin{bmatrix} Fa_1+Ga_3-a_1F-a_2H & Fa_2+Ga_4-a_1G-a_2K \\ Ha_1+Ka_3-a_3F-a_4H & Ha_2+Ka_4-a_3G-a_4K \end{bmatrix} \in \mathcal{Z}(\mathcal{G}).$$

By relation $(\diamondsuit)$ we know that

$$[F,a_1]+Ga_3-a_2H=\varphi^{-1}([K,a_4]+Ha_2-a_3G) \tag{4.1}$$

for all $a_i\in A_i$, $i=1,2,3,4$. Taking $a_2=0$, $a_3=0$ in (4.1) we obtain

$$\begin{aligned}&[f_{11}(a_1,a_1)+f_{14}(a_1,a_4)+f_{44}(a_4,a_4),a_1]\\ &\qquad =\varphi^{-1}([k_{11}(a_1,a_1)+k_{14}(a_1,a_4)+k_{44}(a_4,a_4),a_4])\end{aligned} \tag{4.2}$$

for all $a_1\in A_1$, $a_4\in A_4$. Let us choose $a_4=0$. Thus $[f_{11}(a_1,a_1),a_1]=0$. Adopting similar computational techniques, we arrive at $[k_{44}(a_4,a_4),a_4]=0$. Replacing a_4 by $-a_4$ in (4.2) yields

$$\varphi([f_{14}(a_1,a_4),a_1])=[k_{11}(a_1,a_1),a_4]\in\mathcal{Z}_{A_4}(\mathcal{G}) \tag{4.3}$$

and

$$[f_{44}(a_4,a_4),a_1]=\varphi^{-1}([k_{14}(a_1,a_4),a_4])\in\mathcal{Z}_{A_1}(\mathcal{G}) \tag{4.4}$$

for all $a_1\in A_1$, $a_4\in A_4$. For each $a_4\in A_4$ we know that $f_{14}(\cdot,a_4)\colon A_1\times A_4\to A_1$ is a centralizing $\mathcal{R}$-linear mapping. By assumption (1) and Lemma 4.1, we assert that $f_{14}(a_1,a_4)\in\mathcal{Z}(A_1)$. Thus $[k_{11}(a_1,a_1),a_1]=0$. In an analogous manner, we can show that $[k_{14}(a_1,a_4),a_4]=0$ and hence $f_{44}(a_4,a_4)\in\mathcal{Z}(A)$.

Let us take $a_1=0$, $a_4=0$ in (4.1) and together with Lemma 3.14 and Lemma 3.15 we conclude that

$$\begin{aligned}&(g_{22}(a_2,a_2)+g_{23}(a_2,a_3))a_3-a_2(h_{23}(a_2,a_3)+h_{33}(a_3,a_3))\\ &\qquad =\varphi^{-1}((h_{23}(a_2,a_3)+h_{33}(a_3,a_3))a_2-a_3(g_{22}(a_2,a_2)+g_{23}(a_2,a_3)))\end{aligned} \tag{4.5}$$

for all $a_1\in A_1$, $a_2\in A_2$, $a_3\in A_3$. Replacing a_2 by $-a_2$ in (4.5) and comparing both identities we get that

$$g_{22}(a_2,a_2)a_3-a_2h_{23}(a_2,a_3)=\varphi^{-1}(h_{23}(a_2,a_3)a_2-a_3g_{22}(a_2,a_2)) \tag{4.6}$$

and

$$g_{23}(a_2,a_3)a_3-a_2h_{33}(a_3,a_3)=\varphi^{-1}(h_{33}(a_3,a_3)a_2-a_3g_{23}(a_2,a_3)) \tag{4.7}$$

for all $a_2\in A_2$, $a_3\in A_3$, $a_4\in A_4$.

Combining (4.1) with (4.6) and (4.7) we obtain

$$
\begin{aligned}
(4.8)\qquad & [f_{12}(a_1,a_2)+f_{13}(a_1,a_3)+f_{22}(a_2,a_2)+f_{23}(a_2,a_3)\\
&\quad +f_{24}(a_2,a_4)+f_{33}(a_3,a_3)+f_{34}(a_3,a_4),a_1]\\
&\quad +(g_{12}(a_1,a_2)+g_{24}(a_2,a_4))a_3-a_2(h_{13}(a_1,a_3)+h_{34}(a_3,a_4))\\
&=\varphi^{-1}([k_{12}(a_1,a_2)+k_{13}(a_1,a_3)+k_{22}(a_2,a_2)\\
&\quad +k_{23}(a_2,a_3)+k_{24}(a_2,a_4)+k_{33}(a_3,a_3)+k_{34}(a_3,a_4),a_4]\\
&\quad -a_3(g_{12}(a_1,a_2)+g_{24}(a_2,a_4))+(h_{13}(a_1,a_3)+h_{34}(a_3,a_4))a_2)
\end{aligned}
$$

for all $a_1\in A_1$, $a_2\in A_2$, $a_3\in A_3$, $a_4\in A_4$. Substituting $-a_1$ for a_1 in (4.8) and comparing both identities we get

$$
\begin{aligned}
(4.9)\quad & [f_{12}(a_1,a_2)+f_{13}(a_1,a_3),a_1]+g_{24}(a_2,a_4)a_3-a_2h_{34}(a_3,a_4)\\
&=\varphi^{-1}([k_{22}(a_2,a_2)+k_{23}(a_2,a_3)+k_{24}(a_2,a_4)+k_{33}(a_3,a_3)+k_{34}(a_3,a_4),a_4]\\
&\quad -a_3g_{24}(a_2,a_4)+h_{34}(a_3,a_4)a_2)
\end{aligned}
$$

and

$$
\begin{aligned}
(4.10)\qquad & [f_{22}(a_2,a_2)+f_{23}(a_2,a_3)+f_{24}(a_2,a_4)+f_{33}(a_3,a_3)+f_{34}(a_3,a_4),a_1]\\
&\quad +g_{12}(a_1,a_2)a_3-a_2h_{13}(a_1,a_3)\\
&=\varphi^{-1}([k_{12}(a_1,a_2)+k_{13}(a_1,a_3),a_4]-a_3g_{12}(a_1,a_2)+h_{13}(a_1,a_3)a_2)
\end{aligned}
$$

for all $a_1\in A_1$, $a_2\in A_2$, $a_3\in A_3$, $a_4\in A_4$. Let us take $a_3=0$, $a_4=0$; $a_1=0$, $a_3=0$; $a_2=0$, $a_4=0$; $a_1=0$, $a_2=0$ in (4.9) and (4.10), respectively. Thus we have

$$
\begin{aligned}
(4.11)\qquad & [f_{12}(a_1,a_2),a_1]=0, \quad [f_{22}(a_2,a_2),a_1]=0,\\
& [k_{22}(a_2,a_2),a_4]=0, \quad [k_{24}(a_2,a_4),a_4]=0,\\
& [f_{13}(a_1,a_2),a_3]=0, \quad [f_{33}(a_3,a_3),a_1]=0,\\
& [k_{34}(a_3,a_4),a_4]=0, \quad [k_{33}(a_3,a_3),a_4]=0
\end{aligned}
$$

for all $a_1\in A_1$, $a_2\in A_2$, $a_3\in A_3$, $a_4\in A_4$. Let us choose $a_2=0$ and $a_3=0$ in (4.10). We, respectively, arrive at

$$
(4.12)\qquad [f_{34}(a_3,a_4),a_1]=\varphi^{-1}([k_{13}(a_1,a_3),a_4])\in\pi_{A_1}(\mathcal{Z}(\mathcal{G}))
$$

and

$$
(4.13)\qquad \varphi([f_{24}(a_2,a_4),a_1])=[k_{12}(a_1,a_2),a_4]\in\pi_{A_4}(\mathcal{Z}(\mathcal{G}))
$$

for all $a_1 \in A_1$, $a_2 \in A_2$, $a_3 \in A_3$, $a_4 \in A_4$. It follows from Lemma 4.1 that $k_{12}(a_1, a_2) \in \mathcal{Z}(A_4)$ and $[f_{12}(a_1, a_2), a_1] = 0$, and that $f_{34}(a_4, a_4) \in \mathcal{Z}(A_1)$ and $[k_{13}(a_1, a_3), a_4] = 0$.

Setting $a_1 = 0$ in (4.9) and considering (4.11) gives

$$\begin{aligned}(4.14)\quad & g_{24}(a_2, a_4)a_3 - a_2h_{34}(a_3, a_4) \\ &\qquad = \varphi^{-1}([k_{23}(a_2, a_3), a_4] - a_3g_{24}(a_2, a_4) + h_{34}(a_3, a_4)a_2)\end{aligned}$$

for all $a_2 \in A_2$, $a_3 \in A_3$, $a_4 \in A_4$. Putting $a_4 = 0$ in (4.10) and using (4.11) yields

$$\begin{aligned}(4.15)\quad & [f_{23}(a_2, a_3), a_1] + g_{12}(a_1, a_2)a_3 - a_2h_{13}(a_1, a_3) \\ &\qquad = \varphi^{-1}(h_{13}(a_1, a_3)a_2 - a_3g_{12}(a_1, a_2))\end{aligned}$$

for all $a_1 \in A_1$, $a_2 \in A_2$, $a_3 \in A_3$.

Claim 1. With notations as above, we have

$$g_{22}(a_2, a_2)a_3 = a_2h_{23}(a_2, a_3), \quad g_{23}(a_2, a_3)a_3 = a_2h_{33}(a_3, a_3)$$

for $a_i \in A_i$ $(i = 2, 3)$. Let us take $a_1 = 1$ in (3.35). We get

$$(4.16)\quad h_{23}(a_2, a_3) = a_3f_{12}(1, a_2) - k_{12}(1, a_2)a_3 = (\varphi(f_{12}(1, a_2)) - k_{12}(1, a_2))a_3$$

for all $a_2 \in A_2$, $a_3 \in A_3$. In a similar way, setting $a_1 = 1$ in (3.30), we obtain

$$(4.17)\quad g_{22}(a_2, a_2) = f_{12}(1, a_2)a_2 - a_2k_{12}(1, a_2) = a_2(\varphi(f_{12}(1, a_2)) - k_{12}(1, a_2))$$

for all $a_2 \in A_2$. Relations (4.6), (4.7), (4.16) and (4.17) jointly imply that

$$g_{22}(a_2, a_2)a_3 = a_2h_{23}(a_2, a_3)$$

for all $a_2 \in A_2$, $a_3 \in A_3$. In other words, we can say that equation (4.6) is actually zero.

Similarly, taking $a_4 = 1$ in (3.34) and taking $a_4 = 1$ in (3.40) we arrive at $g_{23}(a_2, a_3)a_3 = a_2h_{33}(a_3, a_3)$. That is, both sides of equation (4.7) are zero.

Claim 2. With notations as above, we have

$$h_{13}(a_1, a_3)a_2 = a_3g_{12}(a_1, a_2), \quad g_{24}(a_2, a_4)a_3 = a_2h_{34}(a_3, a_4)$$

for all $a_i \in A_i$ $(i = 1, 2, 3, 4)$. Setting $a_1 = 1$ in (3.37) and (3.29) we see that

$$h_{13}(1, a_3) = a_3\varpi \quad \text{and} \quad g_{12}(1, a_3) = \varpi a_2,$$

where $\varpi = f_{11}(1,1) - \tau(k_{11}(1,1)) \in \mathcal{Z}(A_1)$. Replacing a_1 by $a_1 + 1$ in (3.37) and (3.29) we get

$$h_{13}(a_1,a_3) = a_3(f_{11}(a_1,1) + f_{11}(1,a_1)) - (k_{11}(a_1,1) + k_{11}(1,a_1))a_3 + a_3\varpi a_1$$
$$g_{12}(a_1,a_3) = (f_{11}(a_1,1) + f_{11}(1,a_1))a_2 - a_2(k_{11}(a_1,1) + k_{11}(1,a_1)) + a_1\varpi a_3.$$

Note that $(k_{11}(a_1,1) + k_{11}(1,a_1)) \in \mathcal{Z}(A_4)$. We therefore have

$$h_{13}(a_1,a_3)a_2 = a_3 g_{12}(a_1,a_2).$$

Accordingly, we see that equality (4.15) is zero.

Considering relations (3.32) and (3.39) and using similar arguments, one can show that $g_{24}(a_2,a_4)a_3 = a_2 h_{34}(a_3,a_4)$. In view of Claim 1 and Claim 2, we assert that

$$Fa_1 + Ga_3 - a_1F - a_2H = 0, \quad Ha_2 + Ka_4 - a_3G - a_4K = 0.$$

Hence, the $\mathcal{R}$-bilinear mapping $\mathfrak{q}$ is actually commuting. This theorem follows from Theorem 3.13. $\square$

In particular, we have:

Corollary 4.3 ([58], Theorem 4.1)**.** *Let $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ be a 2-torsionfree triangular algebra over a commutative ring $\mathcal{R}$. Let $\mathfrak{q}\colon \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ be an $\mathcal{R}$-bilinear mapping. Suppose that*

(1) *every centralizing linear mapping on A or B is proper,*
(2) *$\pi_A(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(A)$ and $\pi_B(\mathcal{Z}(\mathcal{T})) = \mathcal{Z}(B)$,*
(3) *if $[A,A] \neq 0$ and $[B,B] = 0$, then there exist $a_0 \in A$, $m_0 \in M$ such that a_0m_0 and m_0 are independent over $\mathcal{Z}(A)$,*
(4) *if $[A,A] = 0$ and $[B,B] \neq 0$, then there exist $b_0 \in B$, $m_0 \in M$ such that m_0b_0 and m_0 are independent over $\mathcal{Z}(B)$,*
(5) *there exist $m_0 \in M, n_0 \in N$ such that*

$$\mathcal{Z}(\mathcal{T}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \colon am_0 = m_0b,\ \forall\, a \in \mathcal{Z}(A),\ b \in \mathcal{Z}(B) \right\},$$

(6) *M is weakly loyal.*

If $\mathfrak{T}_{\mathfrak{q}}\colon \mathcal{T} \to \mathcal{T}$ is a centralizing trace of bilinear mapping $\mathfrak{q}$, then there exist $\lambda \in \mathcal{Z}(\mathcal{T})$, an $\mathcal{R}$-linear mapping $\mu\colon \mathcal{G} \to \mathcal{Z}(\mathcal{T})$ and a trace $\nu\colon \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ of some $\mathcal{R}$-bilinear mapping such that

$$\mathfrak{T}_{\mathfrak{q}}(x) = \lambda x^2 + \mu(x)x + \nu(x)$$

for all $x \in \mathcal{T}$.

As direct consequences of Theorem 4.2, we have:

Corollary 4.4. *Let $\mathcal{R}$ be a 2-torsionfree commutative ring and $\mathcal{M}_n(\mathcal{R})$ $(n \geqslant 2)$ be the full matrix algebra over $\mathcal{R}$. Suppose that $\mathfrak{q}\colon \mathcal{M}_n(\mathcal{R}) \times \mathcal{M}_n(\mathcal{R}) \to \mathcal{M}_n(\mathcal{R})$ is an $\mathcal{R}$-bilinear mapping. Then every centralizing trace $\mathfrak{T}_{\mathfrak{q}}\colon \mathcal{M}_n(\mathcal{R}) \to \mathcal{M}_n(\mathcal{R})$ of $\mathfrak{q}$ is proper.*

Corollary 4.5. *Let $\mathcal{R}$ be a 2-torsionfree commutative ring, V be an $\mathcal{R}$-linear space and $B(\mathcal{R}, V, \gamma)$ be the inflated algebra of $\mathcal{R}$ along V. Suppose that $\mathfrak{q}\colon B(\mathcal{R}, V, \gamma) \times B(\mathcal{R}, V, \gamma) \to B(\mathcal{R}, V, \gamma)$ is an $\mathcal{R}$-bilinear mapping. Then every centralizing trace $\mathfrak{T}_{\mathfrak{q}}\colon B(\mathcal{R}, V, \gamma) \to B(\mathcal{R}, V, \gamma)$ of $\mathfrak{q}$ is proper.*

It should be remarked that Corollary 4.4 and Corollary 4.5 removes the assumption that $\mathcal{R}$ is a domain in [43], Corollary 3.22 and Corollary 3.23.

Corollary 4.6 ([58], Corollary 4.1)**.** *Let $\mathcal{R}$ be a 2-torsion-free commutative ring and $\mathcal{T}_n(\mathcal{R})$ $(n \geqslant 2)$ be the upper triangular matrix algebra over $\mathcal{R}$. Suppose that $\mathfrak{q}\colon \mathcal{T}_n(\mathcal{R}) \times \mathcal{T}_n(\mathcal{R}) \to \mathcal{T}_n(\mathcal{R})$ is an $\mathcal{R}$-bilinear mapping. Then every centralizing trace $\mathfrak{T}_{\mathfrak{q}}\colon \mathcal{T}_n(\mathcal{R}) \to \mathcal{T}_n(\mathcal{R})$ of $\mathfrak{q}$ is proper.*

5. Lie triple isomorphisms on generalized matrix algebras

In this section we shall use Theorem 4.2 to describe the form of Lie triple isomorphisms of generalized matrix algebras. As applications of Theorem 4.2, we characterize Lie triple isomorphisms of a class of generalized matrix algebras. The involved algebras include upper triangular matrix algebras over a commutative ring $\mathcal{R}$ and full matrix algebras over a commutative ring $\mathcal{R}$.

Throughout this section, we denote the generalized matrix algebra of order 2 originated from the Morita context $(A, B, {}_A M_B, {}_B N_A, \Phi_{MN}, \Psi_{NM})$ by

$$\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix},$$

where at least one of the two bimodules M and N is distinct from zero. We always assume that M is weak loyal as a left A-module and also as a right B-module, but without any constraint conditions on N.

Lemma 5.1 ([62], Lemma 4.1)**.** *Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ be a 2-torsionfree generalized matrix algebra over a commutative ring $\mathcal{R}$. Then $\mathcal{G}$ satisfies the polynomial identity $[[x^2, y], [x, y]]$ if and only if both A and B are commutative.*

Lemma 5.2. *Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ be a 2-torsionfree generalized matrix algebra over a commutative ring $\mathcal{R}$. Suppose that M is a weakly loyal (A, B)-bimodule.*
(1) *For all $a \in A$ the condition $[a, A] \subseteq \pi_A(\mathcal{Z}(\mathcal{G}))$ implies that $a \in \mathcal{Z}(A)$.*
(2) *For all $b \in B$ the condition $[b, B] \subseteq \pi_B(\mathcal{Z}(\mathcal{G}))$ implies that $b \in \mathcal{Z}(B)$.*

P r o o f. We only provide the proof of statement (1). Statement (2) can be proved in an analogous manner. Suppose that $[a, A] \subseteq \pi_A(\mathcal{Z}(\mathcal{G})) \subseteq \mathcal{Z}(A)$ for all $a \in A$. Then

$$[a, a_1^2] = [a, a_1]a_1 + a_1[a, a_1] = 2[a, a_1]a_1 \in \pi_A(\mathcal{Z}(\mathcal{G}))$$

for all $a_1 \in A$. Thus, we assert that $2[a, a_1]a_1 \in \pi_A(\mathcal{Z}(\mathcal{G}))$ and so $[a, a_1]a_1 \in \pi_A(\mathcal{Z}(\mathcal{G})) \subseteq \mathcal{Z}(A)$ for all $a_1 \in A$. We therefore have $[[a, a_1]a_1, a] = 0$. It can be rewritten as

$$0 = [[a, a_1]a_1, a] = -[a, a_1]^2.$$

Hence

$$[a, a_1]^2 m = \{0\} = [a, a_1]m\varphi([a, a_1])$$

holds true for all $m \in M$. Since M is weakly loyal as a left A-module and also as a right B-module, we obtain $[a, a_1] = 0$ for all $a_1 \in A$. This shows that $a \in \mathcal{Z}(A)$. $\square$

Lemma 5.3. *Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ and $\mathcal{G}' = \left[\begin{smallmatrix} A' & M' \\ N' & B' \end{smallmatrix}\right]$ be 2-torsionfree generalized matrix algebra over a commutative ring $\mathcal{R}$. Let $\theta\colon \mathcal{G} \to \mathcal{G}'$ be a Lie triple isomorphism.*
(1) *If $\pi_{A'}(\mathcal{Z}(\mathcal{G}')) = \mathcal{Z}(A')$ and $\pi_{B'}(\mathcal{Z}(\mathcal{G}')) = \mathcal{Z}(B')$, then $\theta(\mathcal{Z}(\mathcal{G})) \subseteq \mathcal{Z}(\mathcal{G}')$.*
(2) *If $\pi_{A'}(\mathcal{Z}(\mathcal{G}')) = \mathcal{Z}(A')$, $\pi_{B'}(\mathcal{Z}(\mathcal{G}')) = \mathcal{Z}(B')$ and $\theta(w) \in \mathcal{Z}(\mathcal{G}')$, then $w \in \mathcal{Z}(\mathcal{G})$.*

P r o o f. (1) According to the assumption, we know that

$$\theta[[x, y], z] = [[\theta(x), \theta(y)], \theta(z)] = 0$$

for all $x \in \mathcal{Z}(\mathcal{G})$, $y, z \in \mathcal{G}$. Then we obtain

$$[\theta(x), \mathcal{G}'] \subseteq \mathcal{Z}(\mathcal{G}'),$$

since $\theta\colon \mathcal{G} \to \mathcal{G}'$ is a Lie triple isomorphism for all $y, z \in \mathcal{G}$.

Furthermore, we claim that $\theta(x) \in \mathcal{Z}(\mathcal{G}')$. Indeed, let us define $\theta(x) = \left[\begin{smallmatrix} a & m \\ n & b \end{smallmatrix}\right]$. For $y = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathcal{G}'$ we have

$$[\theta(x), y] = \begin{bmatrix} a & m \\ n & b \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & m \\ n & b \end{bmatrix} = \begin{bmatrix} 0 & -m \\ n & 0 \end{bmatrix} \in \mathcal{Z}(\mathcal{G}').$$

Thus $m = 0 = n$.

Let us take $y = \left[\begin{smallmatrix} a_1 & 0 \\ 0 & b_1 \end{smallmatrix}\right] \in \mathcal{G}'$ (for all $a_1 \in A_1, b_1 \in A_4$). Then we have

$$[\theta(x), y] = \left[\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}\right] = \begin{bmatrix} [a, a_1] & 0 \\ 0 & [b, b_1] \end{bmatrix} \in \mathcal{Z}(\mathcal{G}').$$

Furthermore, we arrive at $[a, a_1] \in \pi_{A'}(\mathcal{Z}(\mathcal{G}'))$ and $[b, b_1] \in \pi_{B'}(\mathcal{Z}(\mathcal{G}'))$. By Lemma 5.2 it follows that $a \in \mathcal{Z}(A')$ and that $b \in \mathcal{Z}(B')$. Moreover, we have $\theta(x) \in \mathcal{Z}(\mathcal{G}')$, which is due to the assumption.

(2) It can be shown by an analogous manner. $\square$

The following theorem is a much more common generalization of [58], Theorem 5.1.

Theorem 5.4. *Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ and $\mathcal{G}' = \left[\begin{smallmatrix} A' & M' \\ N' & B' \end{smallmatrix}\right]$ be generalized matrix algebras over a commutative ring $\mathcal{R}$ with $\frac{1}{2} \in \mathcal{R}$. Let $\mathfrak{l}\colon \mathcal{G} \to \mathcal{G}'$ be a Lie triple isomorphism. If*

(1) *every centralizing trace of an arbitrary $\mathcal{R}$-bilinear mapping on $\mathcal{G}'$ is proper,*
(2) *every centralizing $\mathcal{R}$-linear mapping on $\mathcal{G}$ is proper,*
(3) *at least one of A, B and at least one of A', B' are noncommutative,*
(4) *M and M' are weakly loyal,*

then $\mathfrak{l} = \lambda\mathfrak{m} + \mathfrak{n}$ with $\lambda^2 = 1_{\mathcal{G}'}$, where $\mathfrak{m}\colon \mathcal{G} \to \mathcal{G}'$ is Jordan homomorphism, $\mathfrak{m}$ is injective, and $\mathfrak{n}\colon \mathcal{G} \to \mathcal{Z}(\mathcal{G}')$ is a linear mapping vanishing on each second commutator. Moreover, if $\mathcal{G}'$ is central over $\mathcal{R}$, then $\mathfrak{m}$ is surjective.

Proof. For arbitrary elements $x, z \in \mathcal{G}$ it is easy to check that $\mathfrak{l}$ satisfies the relation $[[\mathfrak{l}(x^2), \mathfrak{l}(x)], \mathfrak{l}(z)] = \mathfrak{l}([[x^2, x], z]) = 0$. Since $\mathfrak{l}$ is onto, $[\mathfrak{l}(x^2), \mathfrak{l}(x)] \in \mathcal{Z}(\mathcal{G}')$ for all $x \in \mathcal{G}$. Replacing x by $\mathfrak{l}^{-1}(y)$, we get $[\mathfrak{l}(\mathfrak{l}^{-1}(y)^2), y] \in \mathcal{Z}(\mathcal{G}')$ for all $y \in \mathcal{G}'$. This means that the mapping $\mathfrak{T}_{\mathfrak{q}}(y) = \mathfrak{l}(\mathfrak{l}^{-1}(y)^2)$ is a centralizing trace of the bilinear mapping $\mathfrak{q}\colon \mathcal{G}' \times \mathcal{G}' \to \mathcal{G}'$ defined by $\mathfrak{q}(y, z) = \mathfrak{l}(\mathfrak{l}^{-1}(y)\mathfrak{l}^{-1}(z))$. By hypothesis (1), there exist $\lambda \in \mathcal{Z}(\mathcal{G}')$, an $\mathcal{R}$-linear mapping $\mu_1\colon \mathcal{G}' \to \mathcal{Z}(\mathcal{G}')$ and a trace $\nu_1\colon \mathcal{G}' \to \mathcal{Z}(\mathcal{G}')$ of some $\mathcal{R}$-bilinear mapping such that

$$\mathfrak{l}(\mathfrak{l}^{-1}(y)^2) = \lambda y^2 + \mu_1(y)y + \nu_1(y) \tag{5.1}$$

for all $y \in \mathcal{G}'$. Let us set $\mu = \mu_1\mathfrak{l}$ and $\nu = \nu_1\mathfrak{l}$. Then μ and ν are mappings of $\mathcal{G}$ into $\mathcal{Z}(\mathcal{G}')$ and μ is $\mathcal{R}$-linear. Hence (5.1) can be rewritten as

$$\mathfrak{l}(x^2) = \lambda\mathfrak{l}(x)^2 + \mu(x)\mathfrak{l}(x) + \nu(x) \tag{5.2}$$

for all $x \in \mathcal{G}$. We conclude that $\lambda \neq 0$. Otherwise, we have $\mathfrak{l}(x^2) - \mu(x)\mathfrak{l}(x) \in \mathcal{Z}(\mathcal{G}')$ by (5.2) and hence

$$\begin{aligned} \mathfrak{l}([[x^2, y], [x, y]]) &= [[\mathfrak{l}(x^2), \mathfrak{l}(y)], \mathfrak{l}([x, y])] = [[\mu(x)\mathfrak{l}(x), \mathfrak{l}(y)], \mathfrak{l}([x, y])] \\ &= \mu(x)[[\mathfrak{l}(x), \mathfrak{l}(y)], \mathfrak{l}([x, y])] = \mu(x)\mathfrak{l}([[x, y], [x, y]]) = 0 \end{aligned}$$

for all $x, y \in \mathcal{G}$. Consequently, $[[x^2, y], [x, y]] = 0$ for all $x, y \in \mathcal{G}$. According to our assumption this contradicts with Lemma 5.1. Thus $\lambda \neq 0$.

Now we define a linear mapping $\mathfrak{m}\colon \mathcal{G} \to \mathcal{G}^{'}$ by

$$\mathfrak{m}(x) = \lambda\mathfrak{l}(x) + \tfrac{1}{2}\mu(x) \tag{5.3}$$

for $x \in \mathcal{G}$. Of course, $\mathfrak{m}$ is an $\mathcal{R}$-linear mapping. Our goal is to show that $\mathfrak{m}$ is a Jordan homomorphism. In view of (5.2) and (5.3), we have

$$\mathfrak{m}(x^2) = \lambda\mathfrak{l}(x^2) + \tfrac{1}{2}\mu(x) = \lambda^2\mathfrak{l}(x)^2 + \lambda\mu(x)\mathfrak{l}(x) + \lambda\nu(x) + \tfrac{1}{2}\mu(x^2),$$

while

$$\mathfrak{m}(x)^2 = (\lambda\mathfrak{l}(x) + \tfrac{1}{2}\mu(x))^2 = \lambda^2\mathfrak{l}(x)^2 + \lambda\mu(x)\mathfrak{l}(x) + \tfrac{1}{4}\mu(x)^2.$$

Comparing the above two identities we get

$$\mathfrak{m}(x^2) - \mathfrak{m}(x)^2 \in \mathcal{Z}(\mathcal{G}') \tag{5.4}$$

for all $x \in \mathcal{G}$. Linearizing (5.4) gives

$$\mathfrak{m}(x \circ y) - \mathfrak{m}(x) \circ \mathfrak{m}(y) \in \mathcal{Z}(\mathcal{G}')$$

for all $x, y \in \mathcal{G}$. Define a mapping $\varepsilon\colon \mathcal{G} \times \mathcal{G} \to \mathcal{Z}(\mathcal{G}')$ by

$$\varepsilon(x, y) = \mathfrak{m}(x \circ y) - \mathfrak{m}(x) \circ \mathfrak{m}(y). \tag{5.5}$$

Clearly, ε is a symmetric bilinear mapping. It is clear that $[[x, y], z] = x \circ (y \circ z) - y \circ (x \circ z)$ for all $x, y, z \in \mathcal{G}$. Thus, we obtain

$$\mathfrak{m}([[x, y], z]) = \lambda\mathfrak{l}([[x, y], z]) + \tfrac{1}{2}\mu([[x, y], z]) = \lambda\mathfrak{l}([[\mathfrak{l}(x), \mathfrak{l}(y)], \mathfrak{l}(z)]) + \tfrac{1}{2}\mu([[x, y], z])$$

for all $x, y, z \in \mathcal{G}$. On the other hand, by invoking (5.5) we get

$$\begin{aligned}
&\mathfrak{m}(x \circ (y \circ z)) - \mathfrak{m}(y \circ (x \circ z)) \\
&\quad = \mathfrak{m}(x) \circ \mathfrak{m}(y \circ z) - \mathfrak{m}(y) \circ \mathfrak{m}(x \circ z) + \varepsilon(x, y \circ z) - \varepsilon(y, x \circ z) \\
&\quad = \mathfrak{m}(x) \circ (\mathfrak{m}(y) \circ \mathfrak{m}(z)) + 2\varepsilon(y, z)\mathfrak{m}(x) - \mathfrak{m}(y) \circ (\mathfrak{m}(x) \circ \mathfrak{m}(z)) \\
&\qquad - 2\varepsilon(x, z)\mathfrak{m}(y) + \varepsilon(x, y \circ z) - \varepsilon(y, x \circ z) \\
&\quad = [[\mathfrak{m}(x), \mathfrak{m}(y)], \mathfrak{m}(z)] + 2\varepsilon(y, z)\mathfrak{m}(x) - 2\varepsilon(x, z)\mathfrak{m}(y) + \varepsilon(x, y \circ z) - \varepsilon(y, x \circ z) \\
&\quad = \lambda^3([[\mathfrak{l}(x), \mathfrak{l}(y)], \mathfrak{l}(z)]) + 2\lambda\varepsilon(y, z)\mathfrak{l}(x) + \varepsilon(y, z)\mu(x) \\
&\qquad - 2\lambda\varepsilon(x, z)\mathfrak{l}(y) - \varepsilon(x, z)\mu(y) + \varepsilon(x, y \circ z) - \varepsilon(y, x \circ z)
\end{aligned}$$

for all $x, y, z \in \mathcal{G}$. $\square$

According to the above identities, we see that

$$(5.6) \qquad (\lambda^3-\lambda)([[\mathfrak{l}(x),\mathfrak{l}(y)],\mathfrak{l}(z)]) + 2\lambda\varepsilon(y,z)\mathfrak{l}(x) - 2\lambda\varepsilon(x,z)\mathfrak{l}(y) \in \mathcal{Z}(\mathcal{G}')$$

for all $x, y, z \in \mathcal{G}$.

Claim. $\lambda^3 = \lambda$ and $\lambda\varepsilon = 0$.

Indeed, we suppose that A' is noncommutative. Pick any $a_1, a_2 \in A'$ and $m \in M'$. There exist $x_0, y_0, z_0 \in \mathcal{G}$ such that

$$\mathfrak{l}(x_0) = \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathfrak{l}(y_0) = \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathfrak{l}(z_0) = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}.$$

Replacing x, y, z by x_0, y_0, z_0 in (5.6), respectively, we get

$$\pi_{A'}(\lambda^3-\lambda)[a_1,a_2]m = 0$$

for some $a_1, a_2 \in A'$ and an arbitrary $m \in M'$. Because M' is faithful as a left A-module, we arrive at $\pi_{A'}(\lambda^3-\lambda)[A',A'] = 0$, hence $\pi_{A'}(\lambda^3-\lambda) = 0$ by Lemma 3.1. This shows that $\lambda^3 = \lambda$.

Thus, relation (5.6) can be changed into

$$\lambda\varepsilon(y,z)\mathfrak{l}(x) - \lambda\varepsilon(x,z)\mathfrak{l}(y) \in \mathcal{Z}(\mathcal{G}')$$

for all $x, y, z \in \mathcal{G}$. This implies that

$$(5.7) \qquad \lambda\varepsilon(y,z)[\mathfrak{l}(x),\mathfrak{l}(y)] = 0$$

for all $x, y, z \in \mathcal{G}$. For any $m \in M'$, we assume that $x_0, y_0 \in \mathcal{G}$ such that

$$\mathfrak{l}(x_0) = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathfrak{l}(y_0) = \begin{bmatrix} 0 & 0 \\ 0 & 1_{B'} \end{bmatrix}$$

for some $x_0, y_0 \in \mathcal{G}$. Replacing x, y by x_0, y_0 in (5.7), respectively, we assert that $\pi_{A'}(\lambda\varepsilon(y_0,z))m = 0$ for all $z \in \mathcal{G}$ and $m \in M'$. Then we obatin $\lambda\varepsilon(y_0,z) = 0$ for all $z \in \mathcal{G}$. Replacing x, y by x_0, y_0+y in (5.7), we arrive at $\pi_{A'}(\lambda\varepsilon(y,z))m = 0$ for all $y, z \in \mathcal{G}$ and $m \in M'$. So we know that $\lambda\varepsilon(y,z) = 0$ for all $y, z \in \mathcal{G}$.

We further claim that $\lambda^2 = 1_{\mathcal{G}'}$ and $\varepsilon = 0$. Set $\beta = \lambda^2 - 1_{\mathcal{G}'}$. Then $\beta\lambda = 0$. It follows from (5.2) that

$$(5.8) \qquad \beta\mathfrak{l}(x^2) = \beta\mu(x)\mathfrak{l}(x) + \beta\nu(x)$$

for all $x \in \mathcal{G}$. Note that $\mathfrak{l}^{-1}$ is also a Lie triple isomorphism of $\mathcal{G}'$ onto $\mathcal{G}$. Applying the identity $[[\beta\mathfrak{l}(x), \mathfrak{l}(x)], \mathfrak{l}(y)] = 0$ for all $x, y \in \mathcal{G}$ yields

$$[\mathfrak{l}^{-1}(\beta\mathfrak{l}(x)), x] \in \mathcal{Z}(\mathcal{G})$$

for all $x \in \mathcal{G}$. That is, $x \mapsto \mathfrak{l}^{-1}(\beta\mathfrak{l}(x))$ is a centralizing $\mathcal{R}$-linear mapping on $\mathcal{G}$. Since every centralizing mapping on $\mathcal{G}$ is proper, we conclude that there exist $\gamma \in \mathcal{Z}(\mathcal{G})$ and an additive mapping $\omega\colon \mathcal{G} \to \mathcal{Z}(\mathcal{G})$ such that

$$\mathfrak{l}^{-1}(\beta\mathfrak{l}(x)) = \gamma x + \omega(x)$$

for all $x \in \mathcal{G}$ and hence

$$\beta\mathfrak{l}(x) = \mathfrak{l}(\gamma x) + \mathfrak{l}(\omega(x)). \tag{5.9}$$

In view of Lemma 5.3, we see that $\mathfrak{l}(\omega(x)) \in \mathcal{Z}(\mathcal{G}')$ for all $x \in \mathcal{G}$. Combining (5.8) with (5.9) gives

$$\begin{aligned}
\mathfrak{l}([\gamma z_1, [z_2, [[x^2, y], [x, y]]]]) &= [\mathfrak{l}(\gamma z_1), [\mathfrak{l}(z_2), \mathfrak{l}([[x^2, y], [x, y]])]] \\
&= [\beta\mathfrak{l}(z_1), [\mathfrak{l}(z_2), \mathfrak{l}([[\mathfrak{l}(x^2), \mathfrak{l}(y)], \mathfrak{l}([x, y])])]] \\
&= [\mathfrak{l}(z_1), [\mathfrak{l}(z_2), [[\beta\mathfrak{l}(x^2), \mathfrak{l}(y)], \mathfrak{l}([x, y])]]] \\
&= [\mathfrak{l}(z_1), [\mathfrak{l}(z_2), [[\beta\mu(x)\mathfrak{l}(x), \mathfrak{l}(y)], \mathfrak{l}([x, y])]]] \\
&= \beta\mu(x)[\mathfrak{l}(z_1), [\mathfrak{l}(z_2), [[\mathfrak{l}(x), \mathfrak{l}(y)], \mathfrak{l}([x, y])]]] \\
&= \beta\mu(x)[\mathfrak{l}(z_1), [\mathfrak{l}(z_2), \mathfrak{l}([[x, y], [x, y]])]] = 0
\end{aligned}$$

for all $x, y, z_1, z_2 \in \mathcal{G}$. Since $\mathfrak{l}$ is one-to-one, we obtain

$$\gamma[z_1, [z_2, [[x^2, y], [x, y]]]] = 0$$

for all $x, y, z_1, z_2 \in \mathcal{G}$. We may assume that $[A, A] \neq 0$. Let us set

$$z_1 = z_2 = \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}, \quad x = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} a' & m \\ 0 & 0 \end{bmatrix},$$

where $a, a' \in A$, $m \in M$. Then we obtain $\pi_A(\gamma)a[a, a']am = 0$ for all $a, a' \in A$, $m \in M$. So $\pi_A(\gamma)a[a, a']a = 0$, which is due to the fact that M is weakly loyal for all $a, a' \in A$. Replacing a by $a \pm 1$ and comparing both identities, we have $\pi_A(\gamma)[a, a'] = 0$ for all $a, a' \in A$. By Lemma 3.1, we assert that $\pi_A(\gamma) = 0$. Hence $\gamma = 0$. Consequently, applying (5.9) yields that $\beta\mathcal{G}' \subseteq \mathcal{Z}(\mathcal{G}')$. Thus, $\beta\mathcal{G}'$ is a central ideal of $\mathcal{G}'$. By [62], Lemma 3.3, we known that every generalized matrix algebra does not contain nonzero central ideals. We get $\beta = 0$. So $\lambda^2 = 1_{\mathcal{G}'}$ and $\varepsilon = 0$. This shows that $\mathfrak{m}$ is a Jordan homomorphism. Let us set $\mathfrak{n} = -\frac{1}{2}\lambda\mu$. Then we see that $\mathfrak{l} = \lambda\mathfrak{m} + \mathfrak{n}$.

We next prove that $\mathfrak{m}$ is one-to-one. Suppose that $\mathfrak{m}(w) = 0$ for some $w \in \mathcal{G}$. Then $\mathfrak{l}(w) \in \mathcal{Z}(\mathcal{G}')$ and hence $w \in \mathcal{Z}(\mathcal{G})$ by (2) of Lemma 5.3. This implies that $\mathfrak{m}^{-1}(0) \subseteq \mathcal{Z}(\mathcal{G})$. That is, $\mathfrak{m}^{-1}(0)$ is a Jordan ideal of $\mathcal{Z}(\mathcal{G})$. However, by [43], Lemma 4.1 it follows that $\mathfrak{m}^{-1}(0) = 0$.

It remains to prove that $\mathfrak{m}$ is onto in the case when $\mathcal{G}'$ is central over $\mathcal{R}$. Let us first show that $\mathfrak{m}(1_{\mathcal{G}}) = 1_{\mathcal{G}'}$. Since $\mathfrak{l}$ is a Lie triple isomorphism, we have $\mathfrak{l}(1_{\mathcal{G}}) \in \mathcal{Z}(\mathcal{G}')$ and $\mathfrak{m}(1_{\mathcal{G}}) \in \mathcal{Z}(\mathcal{G}')$. Note that every $\mathfrak{m}$ is a Jordan homomorphism. We see that

$$2\mathfrak{m}(x) = \mathfrak{m}(x \circ 1_{\mathcal{G}}) = 2\mathfrak{m}(x)\mathfrak{m}(1_{\mathcal{G}}).$$

Since $\frac{1}{2} \in \mathcal{R}$, $(\mathfrak{m}(1_{\mathcal{G}}) - 1_{\mathcal{G}'})\mathfrak{m}(x) = 0$. which can be rewritten as $(\mathfrak{m}(1_{\mathcal{G}}) - 1_{\mathcal{G}'})\mathfrak{m}(\mathcal{G}) \subseteq \mathcal{Z}(\mathcal{G}')$. Note that every generalized matrix algebra does not contain nonzero central ideal by [62], Lemma 3.3. We get that $\mathfrak{m}(1_{\mathcal{G}}) = 1_{\mathcal{G}'}$. Obviously, we may write $\mathfrak{n}(x) = f(x)1_{\mathcal{G}'}$ for some linear mapping $f\colon \mathcal{G} \to \mathcal{R}$. Since $\mathfrak{m}$ is $\mathcal{R}$-linear, we know that $\mathfrak{m}(x) = \lambda\mathfrak{m}(x) + f(x)1_{\mathcal{G}'} = \mathfrak{m}(\lambda x + f(x)1_{\mathcal{G}})$ for all $x \in \mathcal{G}$. Consequently, $\mathfrak{m}$ is onto, which is due to the fact that $\mathfrak{l}$ is bijective. $\square$

In particular, we have:

Corollary 5.5 ([58], Theorem 5.1)**.** *Let* $\mathcal{T} = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$ *and* $\mathcal{T}' = \begin{bmatrix} A' & M' \\ O & B' \end{bmatrix}$ *be 2-torsionfree triangular algebras over a commutative ring* $\mathcal{R}$. *Let* $\mathfrak{l}\colon \mathcal{T} \to \mathcal{T}'$ *be a Lie triple isomorphism. If*

(1) *every centralizing trace of an arbitrary bilinear mapping on* $\mathcal{T}'$ *is proper,*
(2) *every centralizing linear mapping on* $\mathcal{T}$ *is proper,*
(3) *at least one of* A, B, *and at least one of* A', B' *are noncommutative,*
(4) M *and* M' *are weakly loyal,*

then $\mathfrak{l} = \lambda\varrho + \tau$ *with* $\lambda^2 = 1_{\mathcal{G}'}$, *where* $\varrho\colon \mathcal{T} \to \mathcal{T}'$ *is a Jordan homomorphism,* ϱ *is one-to-one and* $\tau\colon \mathcal{T} \to \mathcal{Z}(\mathcal{T}')$ *is a linear mapping vanishing on each second commutator. Moreover, if* $\mathcal{T}'$ *is central over* $\mathcal{R}$, *then* ϱ *is onto.*

Theorem 5.6. *Let* $\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ *and* $\mathcal{G}' = \begin{bmatrix} A' & M' \\ N' & B' \end{bmatrix}$ *be 2-torsionfree generalized matrix algebras over a commutative ring* $\mathcal{R}$ *with* $\frac{1}{2} \in \mathcal{R}$. *Let* $\mathfrak{l}\colon \mathcal{G} \to \mathcal{G}'$ *be a Lie triple isomorphism. If*

(1) $\pi_{A'}(\mathcal{Z}(\mathcal{G}')) = \mathcal{Z}(A')$ *and* $\pi_{B'}(\mathcal{Z}(\mathcal{G}')) = \mathcal{Z}(B')$,
(2) *both* A' *and* B' *are commutative,*

(3) *there exist* $m_0 \in M'$, $n_0 \in N'$ *such that*

$$\mathcal{Z}(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \colon \ am_0 = m_0 b,\ n_0 a = bn_0 \ \forall\, a \in \mathcal{Z}(A'),\ b \in \mathcal{Z}(B') \right\},$$

then $\mathfrak{l} = \mathfrak{m} + \mathfrak{n}$, *where* $\mathfrak{m}\colon \mathcal{G} \to \mathcal{G}'$ *is a Jordan homomorphism,* $\mathfrak{m}$ *is injective and* $\mathfrak{n}\colon \mathcal{G} \to \mathcal{Z}(\mathcal{G}')$ *is a linear mapping vanishing on each second commutator. Moreover, both* A *and* B *are commutative. If* $\mathcal{G}'$ *is central over* $\mathcal{R}$, *then* $\mathfrak{m}$ *is surjective.*

Proof. In view of Theorem 4.2, we know that each centralizing trace of a bilinear map on $\mathcal{G}'$ is proper. Using the same arguments as in the proof of Theorem 5.4 we get that there exists an $\mathcal{R}$-linear mapping $\varrho\colon \mathcal{G} \to \mathcal{Z}(\mathcal{G}')$ such that

$$\mathfrak{l}(x^2) - \varrho(x)\mathfrak{l}(x) \in \mathcal{Z}(\mathcal{G}')$$

for all $x \in \mathcal{G}$. Since both A' or B' are commutative, it follows from Lemma 3.11 that there exists a linear mapping $\eta\colon \mathcal{G} \to Z(\mathcal{G}')$ such that

$$\mathfrak{l}(x)^2 - \eta(x)\mathfrak{l}(x) \in \mathcal{Z}(\mathcal{G}')$$

for all $x \in \mathcal{G}$. Let us now define $\mathfrak{m}\colon \mathcal{G} \to \mathcal{G}'$ by

$$\mathfrak{m} = \mathfrak{l} - \tfrac{1}{2}(\eta - \varrho).$$

It is easy to verify that $\mathfrak{m}(x^2) - \mathfrak{m}(x)^2 \in \mathcal{Z}(\mathcal{G}')$ for all $x \in \mathcal{G}$. This gives

$$\mathfrak{m}(x \circ y) - \mathfrak{m}(x) \circ \mathfrak{m}(x) \in \mathcal{Z}(\mathcal{G}')$$

for all $x \in \mathcal{G}$. Let us define

$$\varepsilon(x, y) = \mathfrak{m}(x \circ y) - \mathfrak{m}(x) \circ \mathfrak{m}(y)$$

for all $x, y \in \mathcal{G}$. Following the same arguments as in Theorem 5.4, we assert that

$$\varepsilon(y, z)[\mathfrak{l}(x), \mathfrak{l}(y)] = 0 \tag{5.10}$$

for all $x, y, z \in \mathcal{G}$. Pick any $m \in M'$, there exist $x_0, y_0 \in \mathcal{G}$ such that

$$\mathfrak{l}(x_0) = \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathfrak{l}(y_0) = \begin{bmatrix} 1_{A'} & 0 \\ 0 & 0 \end{bmatrix}$$

for some $y_0 \in \mathcal{G}$. Then from (5.10) we get that $\pi_{A'}(\varepsilon(y_0, z))m = 0$ for all $m \in M'$. Since M' is faithful as a left A'-module, we have $\pi_{A'}(\varepsilon(y_0, z)) = 0$ and $\varepsilon(y_0, z) = 0$ for all $z \in \mathcal{G}$. Replacing y by $y_0 + y$ in (5.10), we arrive at

$$\varepsilon(y, \mathcal{G})[\mathfrak{l}(y_0), \mathcal{G}'] = 0$$

for all $y \in \mathcal{G}$. In a similar manner, we obtain $\varepsilon = 0$. Therefore $\mathfrak{m}$ is a Jordan homomorphism. Let us set $\mathfrak{n} = \frac{1}{2}(\eta - \varrho)$. Thus $\mathfrak{l} = \mathfrak{m} + \mathfrak{n}$.

In view of Lemma 3.12, it follows that

$$\begin{aligned}\mathfrak{l}([[[x^2, y], z], [x, y]]) &= [\mathfrak{l}([x^2, y], z]), [\mathfrak{l}(x), \mathfrak{l}(y)]] = [[[\mathfrak{l}(x^2), \mathfrak{l}(y)], \mathfrak{l}(z)], [\mathfrak{l}(x), \mathfrak{l}(y)]] \\ &= [[[\mathfrak{m}(x^2), \mathfrak{m}(y)], \mathfrak{m}(z)], [\mathfrak{m}(x), \mathfrak{m}(y)]] \\ &= [[[\mathfrak{m}^2(x), \mathfrak{m}(y)], \mathfrak{m}(z)], [\mathfrak{m}(x), \mathfrak{m}(y)]] = 0\end{aligned}$$

for all $x, y, z \in \mathcal{G}$. Hence, $[[[x^2, y], z], [x, y]] = 0$ for all $x, y, z \in \mathcal{G}$. Applying Lemma 3.12 yields that both A and B are commutative. □

In particular, we have:

Corollary 5.7 ([58], Theorem 5.2). *Let $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ and $\mathcal{T}' = \left[\begin{smallmatrix} A' & M' \\ O & B' \end{smallmatrix}\right]$ be 2-torsionfree triangular algebras over a commutative ring $\mathcal{R}$. Let $\mathfrak{l}\colon \mathcal{T} \to \mathcal{T}'$ be a Lie triple isomorphism. If*

(1) *$\pi_{A'}(\mathcal{Z}(\mathcal{T}')) = \mathcal{Z}(A')$ and $\pi_{B'}(\mathcal{Z}(\mathcal{T}')) = \mathcal{Z}(B')$,*
(2) *both $A^{'}$ and $B^{'}$ are commutative,*
(3) *there exits $m_0 \in M^{'}$ such that*

$$\mathcal{Z}(\mathcal{T}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} :\ am_0 = m_0 b \ \forall\, a \in \mathcal{Z}(A'),\ b \in \mathcal{Z}(B') \right\},$$

then $\mathfrak{l} = \mathfrak{m} + \mathfrak{n}$, where $\mathfrak{m}\colon \mathcal{T} \to \mathcal{T}'$ is a Jordan homomorphism, $\mathfrak{m}$ is injective and $\mathfrak{n}\colon \mathcal{T} \to \mathcal{Z}(\mathcal{T}')$ is a linear mapping vanishing on each second commutator. Moreover, both A and B are commutative. If $\mathcal{T}'$ is central over $\mathcal{R}$, then $\mathfrak{m}$ is surjective.

As direct consequences of Theorem 5.6 we have:

Corollary 5.8 ([58], Corollary 5.1). *Let n, n' $(n, n' \geqslant 2)$ be integers and $\mathcal{R}$ be a commutative ring with $\frac{1}{2} \in \mathcal{R}$. If $\mathfrak{l}\colon \mathcal{T}_n(\mathcal{R}) \to \mathcal{T}_{n'}(\mathcal{R})$ is a Lie triple isomorphism, then $\mathfrak{l} = \mathfrak{m}+\mathfrak{n}$, where $\mathfrak{m}\colon \mathcal{T}_n(\mathcal{R}) \to \mathcal{T}_{n'}(\mathcal{R})$ is a Jordan isomorphism and $\mathfrak{n}\colon \mathcal{T}_n(\mathcal{R}) \to \mathcal{R}1_{\mathcal{T}_{n'}(\mathcal{R})}$ is a linear mapping vanishing on each second commutator. Moreover, $n = 2$ if and only if $n' = 2$.*

For the Lie triple isomorphisms on full matrix algebras, we have similar characterizations.

Corollary 5.9. *Let n, n' $(n, n' \geqslant 2)$ be integers and $\mathcal{R}$ be a commutative ring with $\frac{1}{2} \in \mathcal{R}$. If $\mathfrak{l}\colon \mathcal{M}_n(\mathcal{R}) \to \mathcal{M}_{n'}(\mathcal{R})$ is a Lie triple isomorphism, then $\mathfrak{l} = \lambda\mathfrak{m} + \mathfrak{n}$, where $\lambda \in \mathcal{R}$ with $\lambda^2 = 1$, $\mathfrak{m}\colon \mathcal{M}_n(\mathcal{R}) \to \mathcal{M}_{n'}(\mathcal{R})$ is a Jordan isomorphism and $\mathfrak{n}\colon \mathcal{M}_n(\mathcal{R}) \to \mathcal{R}1_{\mathcal{M}_{n'}(\mathcal{R})}$ is a linear mapping vanishing on each second commutator. Moreover, $n = 2$ if and only if $n' = 2$.*

P r o o f. Suppose that $n, n' > 2$. In view of Example 2.1, Corollary 3.5 and Theorem 4.2, we see that all assumptions of Theorem 5.4 are satisfied. The result follows from Theorem 5.4.

Suppose that $n' = 2$. The result follows from Theorem 5.6. If $n = 2$, applying Theorem 5.6 to $\mathfrak{l}^{-1}$, we get that $n' = 2$. □

Similarly, we can prove that the following corollary.

Corollary 5.10. *Let $\mathcal{R}$ be a 2-torsionfree commutative ring, V_i be an $\mathcal{R}$-linear space and $B(\mathcal{R}, V_i, \gamma_i)$ be the inflated algebra of $\mathcal{R}$ along V_i $(i = 1, 2)$. Suppose that $\mathfrak{l}\colon B(\mathcal{R}, V_1, \gamma_1) \to B(\mathcal{R}, V_2, \gamma_2)$ is a Lie triple isomorphism. Then $\mathfrak{l} = \lambda\mathfrak{m} + \mathfrak{n}$, where $\lambda \in \mathcal{R}$ with $\lambda^2 = 1$, $\mathfrak{m}\colon B(\mathcal{R}, V_1, \gamma_1) \to B(\mathcal{R}, V_2, \gamma_2)$ is a Jordan isomorphism and $\mathfrak{n}\colon B(\mathcal{R}, V_1, \gamma_1) \to B(\mathcal{R}, V_2, \gamma_2)$ is a linear mapping vanishing on each second commutator. Moreover,* $\dim(V_1) = 2$ *if and only if* $\dim(V_2) = 2$.

6. Lie isomorphisms on generalized matrix algebras

It is clear that each Lie isomorphism is a Lie triple isomorphism. Applying Theorem 5.4, we can give a detailed description of Lie isomorphism on generalized matrix algebras.

The following theorem is a much more common generalization of [58], Theorem 6.1. For completeness and for reading convenience, we here give its proof which is rather similar to the proof of [58], Theorem 6.1.

Theorem 6.1. *Let $\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ and $\mathcal{G}' = \begin{bmatrix} A' & M' \\ N' & B' \end{bmatrix}$ be 2-torsionfree generalized matrix algebras over a commutative ring $\mathcal{R}$. Let $\mathfrak{l}\colon \mathcal{G} \to \mathcal{G}'$ be a Lie isomorphism. If*

(1) *every commuting trace of an arbitrary bilinear mapping on $\mathcal{G}'$ is proper,*
(2) *every commuting linear mapping on $\mathcal{G}$ is proper,*
(3) *at least one of A, B, and at least one of A', B' are noncommutative,*
(4) *both M and M' are weakly loyal,*

then $\mathfrak{l} = \lambda\mathfrak{m} + \mathfrak{n}$, *where* $\lambda \in \mathcal{Z}(\mathcal{G}')$ *with* $\lambda^2 = 1_{\mathcal{G}'}$, $\mathfrak{m}\colon \mathcal{G} \to \mathcal{G}'$ *is a sum of homomorphism and the negative of an anti-homomorphism,* $\mathfrak{m}$ *is injective and* $\mathfrak{n}\colon \mathcal{G} \to \mathcal{Z}(\mathcal{G}')$ *is a linear mapping vanishing on each commutator. Moreover, if* $\mathcal{G}'$ *is central over* $\mathcal{R}$, *then* $\mathfrak{m}$ *is onto.*

P r o o f. Let us set $1 = 1_{\mathcal{G}'}$. By Theorem 5.4 it follows that

$$\mathfrak{l} = \lambda\mathfrak{m} + \mathfrak{n},$$

where $\lambda \in \mathcal{Z}(\mathcal{G}')$ with $\lambda^2 = 1$, $\mathfrak{m}\colon \mathcal{G} \to \mathcal{G}'$ is a Jordan homomorphism, $\mathfrak{m}$ is one-to-one and $\mathfrak{n}\colon \mathcal{G} \to \mathcal{G}'$ is a linear mapping vanishing on each commutator. Moreover, if $\mathcal{G}^{'}$ is central over R, then $\mathfrak{m}$ is onto. Since $\mathfrak{l}$ is a Lie isomorphism, one can easily check that

$$\lambda\mathfrak{m}([x,y]) - [\mathfrak{m}(x), \mathfrak{m}(y)] = -\mathfrak{n}([x,y]) \in \mathcal{Z}(\mathcal{G}') \tag{6.1}$$

for all $x, y \in \mathcal{G}$. Since $\mathfrak{m}$ is a Jordan homomorphism, from (6.1) we get

$$\lambda\mathfrak{m}(xy) - \tfrac{1}{2}(\lambda+1)\mathfrak{m}(x)\mathfrak{m}(y) - \tfrac{1}{2}(\lambda-1)\mathfrak{m}(y)\mathfrak{m}(x) \in \mathcal{Z}(\mathcal{G}')$$

for all $x, y \in \mathcal{G}$. Consequently, the mapping

$$\varepsilon\colon \mathcal{G} \times \mathcal{G} \to \mathcal{Z}(\mathcal{G}')$$

satisfies the relation

$$\varepsilon(x,y) = \lambda\mathfrak{m}(xy) - \tfrac{1}{2}(\lambda+1)\mathfrak{m}(x)\mathfrak{m}(y) - \tfrac{1}{2}(\lambda-1)\mathfrak{m}(y)\mathfrak{m}(x).$$

Let us define $\alpha = \frac{1}{2}(\lambda + 1)$. Thus $\alpha^2 = \alpha$. We therefore have

$$\lambda\mathfrak{m}(xy) = \alpha\mathfrak{m}(x)\mathfrak{m}(y) + (\alpha-1)\mathfrak{m}(y)\mathfrak{m}(x) + \varepsilon(x,y) \tag{6.2}$$

for all $x, y \in \mathcal{G}$. In light of (6.2), we conclude

$$\begin{aligned}
\mathfrak{m}(xyz) = \mathfrak{m}(x(yz)) &= \alpha\mathfrak{m}(x)\mathfrak{m}(yz) + (\alpha-1)\mathfrak{m}(yz)\mathfrak{m}(x) + \varepsilon(x,yz) \\
&= \alpha\mathfrak{m}(x)(\alpha\mathfrak{m}(y)\mathfrak{m}(z) + (\alpha-1)(\mathfrak{m}(z)\mathfrak{m}(y)) + \varepsilon(y,z)) \\
&\quad + (\alpha-1)(\alpha\mathfrak{m}(y)\mathfrak{m}(z) + (\alpha-1)(\mathfrak{m}(z)\mathfrak{m}(y)) + \varepsilon(y,z))\mathfrak{m}(x) + \varepsilon(x,yz) \\
&= \alpha^2\mathfrak{m}(x)\mathfrak{m}(y)\mathfrak{m}(z) + (\alpha-1)^2\mathfrak{m}(z)\mathfrak{m}(y)\mathfrak{m}(x) + \varepsilon(x,yz) + \varepsilon(y,z)\mathfrak{m}(x).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathfrak{m}(xyz) &= \mathfrak{m}((xy)z) = \alpha\mathfrak{m}(xy)\mathfrak{m}(z) + (\alpha-1)\mathfrak{m}(z)\mathfrak{m}(xy) + \varepsilon(xy,z)\\ &= \alpha(\alpha\mathfrak{m}(x)\mathfrak{m}(y) + (\alpha-1)(\mathfrak{m}(y)\mathfrak{m}(x)) + \varepsilon(x,y))\mathfrak{m}(z)\\ &\quad + (\alpha-1)\mathfrak{m}(z)(\alpha\mathfrak{m}(x)\mathfrak{m}(y) + (\alpha-1)\mathfrak{m}(y)\mathfrak{m}(x)) + \varepsilon(x,y)) + \varepsilon(xy,z)\\ &= \alpha^2\mathfrak{m}(x)\mathfrak{m}(y)\mathfrak{m}(z) + (\alpha-1)^2\mathfrak{m}(z)\mathfrak{m}(y)\mathfrak{m}(x) + \varepsilon(xy,z) + \varepsilon(x,y)\mathfrak{m}(z).\end{aligned}$$

Comparing the two identities above gives

$$\varepsilon(y,z)\mathfrak{m}(x) - \varepsilon(x,y)\mathfrak{m}(z) \in \mathcal{Z}(\mathcal{G}')$$

for all $x, y, z \in \mathcal{G}$. Hence

$$\varepsilon(y,z)\mathfrak{l}(x) - \varepsilon(x,y)\mathfrak{l}(z) \in \mathcal{Z}(\mathcal{G}') \tag{6.3}$$

for all $x, y, z \in \mathcal{G}$. Following the same argument as in Theorem 5.4, we assert that $\varepsilon = 0$. This implies that $\alpha\mathfrak{m}$ is a homomorphism and $(1-\alpha)\mathfrak{m}$ is the negative of an antihomomorphism. Hence, $\mathfrak{m}$ is a sum of a homomorphism and the negative of an antihomomorphism. □

We immediately get the following corollary:

Corollary 6.2 ([58], Theorem 6.1). *Let $\mathcal{T} = \begin{bmatrix} A & M \\ O & B \end{bmatrix}$ and $\mathcal{T}' = \begin{bmatrix} A' & M' \\ O & B' \end{bmatrix}$ be 2-torsionfree triangular algebras over a commutative ring $\mathcal{R}$. Let $\mathfrak{l}\colon \mathcal{T} \to \mathcal{T}'$ be a Lie isomorphism. If*

(1) *every commuting trace of an arbitrary bilinear mapping on $\mathcal{T}'$ is proper,*
(2) *every commuting linear mapping on $\mathcal{T}$ is proper,*
(3) *at least one of A, B, and at least one of A', B' are noncommutative,*
(4) *both M and M' are weakly loyal,*

then $\mathfrak{l} = \lambda\mathfrak{m} + \mathfrak{n}$, where $\lambda \in \mathcal{Z}(\mathcal{T}')$ with $\lambda^2 = 1_{\mathcal{T}'}$, $\mathfrak{m}\colon \mathcal{T} \to \mathcal{T}'$ is a sum of a homomorphism and the negative of an anti-homomorphism, $\mathfrak{m}$ is one-to-one and $\mathfrak{n}\colon \mathcal{T} \to \mathcal{Z}(\mathcal{T}')$ is a linear mapping vanishing on each commutator. Moreover, if $\mathcal{T}'$ is central over $\mathcal{R}$, then $\mathfrak{m}$ is onto.

Using Theorem 5.6 we can prove the following result.

Theorem 6.3. *Let $\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}$ and $\mathcal{G}' = \begin{bmatrix} A' & M' \\ N' & B' \end{bmatrix}$ be 2-torsionfree generalized matrix algebras over a commutative ring $\mathcal{R}$. Let $\mathfrak{l}\colon \mathcal{G} \to \mathcal{G}'$ be a Lie isomorphism. If*

(1) *$\pi_{A'}(\mathcal{Z}(\mathcal{G}')) = \mathcal{Z}(A')$ and $\pi_{B'}(\mathcal{Z}(\mathcal{G}')) = \mathcal{Z}(B')$,*
(2) *both A' and B' are commutative,*

(3) *there exist* $m_0 \in M'$, $n_0 \in N'$ *such that*

$$\mathcal{Z}(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \colon\ am_0 = m_0 b,\ n_0 a = b n_0\ \forall\, a \in \mathcal{Z}(A'),\ b \in \mathcal{Z}(B') \right\},$$

then $\mathfrak{l} = \mathfrak{m} + \mathfrak{n}$, *where* $\mathfrak{m}\colon \mathcal{G} \to \mathcal{G}'$ *is a homomorphism,* $\mathfrak{m}$ *is one-to-one and* $\mathfrak{n}\colon \mathcal{G} \to \mathcal{Z}(\mathcal{G}')$ *is a linear mapping vanishing on each commutator. Moreover, both* A *and* B *are commutative. If* $\mathcal{G}'$ *is central over* $\mathcal{R}$, *then* $\mathfrak{m}$ *is onto.*

P r o o f. In view of Theorem 5.6, we know that $\mathfrak{l} = \mathfrak{m} + \mathfrak{n}$, where $\mathfrak{m}\colon \mathcal{G} \to \mathcal{G}'$ is a Jordan homomorphism, $\mathfrak{m}$ is one-to-one and $\mathfrak{n}\colon \mathcal{G} \to \mathcal{Z}(\mathcal{G}')$ is a linear mapping vanishing on each commutator. Since $\mathfrak{l}$ is a Lie isomorphism, we get

$$\mathfrak{m}([x, y]) - [\mathfrak{m}(x), \mathfrak{m}(y)] \in \mathcal{Z}(\mathcal{G}')$$

for all $x, y \in \mathcal{G}$. Thus, we obtain

$$\mathfrak{m}(xy) - \mathfrak{m}(x)\mathfrak{m}(y) \in \mathcal{Z}(\mathcal{G}')$$

for all $x, y \in \mathcal{G}$. Let us set

$$\varepsilon(x, y) = \mathfrak{m}(xy) - \mathfrak{m}(x)\mathfrak{m}(y)$$

for all $x, y \in \mathcal{G}$. Following the same argument as in Theorem 6.1, we assert that $\varepsilon(y, z)[\mathfrak{l}(x), \mathfrak{l}(y)] = 0$ for all $x, y, z \in \mathcal{G}$ and that $\mathfrak{m}$ is a homomorphism. $\square$

As a consequence of both Theorem 6.1 and Theorem 6.3 we have:

Corollary 6.4 ([58], Theorem 6.2)**.** *Let* $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ *and* $\mathcal{T}' = \left[\begin{smallmatrix} A' & M' \\ O & B' \end{smallmatrix}\right]$ *be 2-torsionfree triangular algebras over a commutative ring* $\mathcal{R}$. *Let* $\mathfrak{l}\colon \mathcal{T} \to \mathcal{T}'$ *be a Lie isomorphism. If*

(1) $\pi_{A'}(\mathcal{Z}(\mathcal{G}')) = \mathcal{Z}(A')$ *and* $\pi_{B'}(\mathcal{Z}(\mathcal{G}')) = \mathcal{Z}(B')$,

(2) *both* A' *and* B' *are commutative,*

(3) *there exits* $m_0 \in M'$ *such that*

$$\mathcal{Z}(\mathcal{T}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \colon\ am_0 = m_0 b\ \forall\, a \in \mathcal{Z}(A'),\ b \in \mathcal{Z}(B') \right\},$$

then $\mathfrak{l} = \mathfrak{m} + \mathfrak{n}$, *where* $\mathfrak{m}\colon \mathcal{T} \to \mathcal{T}'$ *is a homomorphism,* $\mathfrak{m}$ *is one-to-one and* $\mathfrak{n}\colon \mathcal{T} \to \mathcal{Z}(\mathcal{T}')$ *is a linear mapping vanishing on each commutator. Moreover, both* A *and* B *are commutative. If* $\mathcal{T}'$ *is central over* $\mathcal{R}$, *then* $\mathfrak{m}$ *is onto.*

In addition, we also get:

Corollary 6.5. *Let $\mathcal{R}$ be a 2-torsionfree commutative ring. If $\mathfrak{l}\colon \mathcal{M}_n(\mathcal{R}) \to \mathcal{M}_{n'}(\mathcal{R})$ $(n, n' \geqslant 2)$ is a Lie isomorphism, then $\mathfrak{l} = \lambda\mathfrak{m} + \mathfrak{n}$, where $\lambda \in \mathcal{R}$ with $\lambda^2 = 1$, $\mathfrak{m}\colon \mathcal{M}_n(\mathcal{R}) \to \mathcal{M}_{n'}(\mathcal{R})$ is a sum of an isomorphism and the negative of an anti-isomorphism and $\mathfrak{n}\colon \mathcal{M}_n(\mathcal{R}) \to \mathcal{R}1_{\mathcal{M}_{n'}(\mathcal{R})}$ is a linear mapping vanishing on each commutator. Moreover, $n = 2$ if and only if $n' = 2$.*

Corollary 6.6. *Let $\mathcal{R}$ be a 2-torsionfree commutative ring, V_i be an $\mathcal{R}$-linear space and $B(\mathcal{R}, V_i, \gamma_i)$ be the inflated algebra of $\mathcal{R}$ along V_i $(i = 1, 2)$. If $\mathfrak{l}\colon B(\mathcal{R}, V_1, \gamma_1) \to B(\mathcal{R}, V_2, \gamma_2)$ is a Lie isomorphism, then $\mathfrak{l} = \lambda\mathfrak{m}+\mathfrak{n}$, where $\lambda \in \mathcal{R}$ with $\lambda^2 = 1$, $\mathfrak{m}\colon B(\mathcal{R}, V_1, \gamma_1) \to B(\mathcal{R}, V_2, \gamma_2)$ is a sum of an isomorphism and the negative of an anti-isomorphism and $\mathfrak{n}\colon B(\mathcal{R}, V_1, \gamma_1) \to B(\mathcal{R}, V_2, \gamma_2)$ is a linear mapping vanishing on each commutator. Moreover, $\dim(V_1) = 2$ if and only if $\dim(V_2) = 2$.*

Corollary 6.7 ([58], Corollary 6.1)**.** *Let $\mathcal{R}$ be a 2-torsionfree commutative ring. If $\mathfrak{l}\colon \mathcal{T}_n(\mathcal{R}) \to \mathcal{T}_{n'}(\mathcal{R})$ $(n, n' \geqslant 2)$ is a Lie isomorphism, then $\mathfrak{l} = \lambda\mathfrak{m}+\mathfrak{n}$, where $\lambda \in \mathcal{R}$ with $\lambda^2 = 1$, $\mathfrak{m}\colon \mathcal{T}_n(\mathcal{R}) \to \mathcal{T}_{n'}(\mathcal{R})$ is a sum of an isomorphism and the negative of an anti-isomorphism and $\mathfrak{n}\colon \mathcal{T}_n(\mathcal{R}) \to \mathcal{R}1_{\mathcal{T}_{n'}(\mathcal{R})}$ is a linear mapping vanishing on each commutator. Moreover, $n = 2$ if and only if $n' = 2$. In particular, if $n = 2$, then φ is an isomorphism.*

It should be remarked that Corollary 6.5 or Corollary 6.6 removes the assumption that $\mathcal{R}$ is a domain in [62], Corollary 4.7 or Corollary 4.8.

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Authors' addresses: Xinfeng Liang, School of Mathematics and Big Data, Anhui University of Science & Technology, Huainan, 232001, P. R. China. e-mail: `xfliang@aust.edu.cn`; Feng Wei (corresponding author), School of Mathematics and Statistics, Beijing Institute of Technology, 5 South Zhongguancun Street, Haidian District, Beijing, 100081, P. R. China, e-mail: `daoshuo@hotmail.com, daoshuowei@gmail.com`; Ajda Fošner, Faculty of Management, University of Primorska, Cankarjeva 5, SI-6104 Koper, Slovenia, e-mail: `ajda.fosner@fm-kp.si`.