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ON THE NUMBER OF ISOMORPHISM CLASSES OF DERIVED SUBGROUPS

Leyli Jafari Taghvasani, Soran Marzang, Mohammad Zarrin, Sanandaj

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Abstract. We show that a finite nonabelian characteristically simple group G satisfies $n = |\pi(G)| + 2$ if and only if $G \cong A_5$, where n is the number of isomorphism classes of derived subgroups of G and $\pi(G)$ is the set of prime divisors of the group G. Also, we give a negative answer to a question raised in M. Zarrin (2014).

Keywords: derived subgroup; simple group

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1. Introduction and results

Following [3], we say that a group G has the property $\mathcal{GR}_n$ if it has a finite number n of derived subgroups. In 2005, de Giovanni and Robinson [3] and, independently, Herzog, Longobardi, Maj in [5] studied new finiteness conditions related to the derived subgroups of a group. They proved that every locally graded $\mathcal{GR}_n$-group is finite-by-abelian (that is, G' is finite). More recently the author in [11], improved this result, by proving that every locally graded $\mathcal{GR}_n$-group is nilpotent-by-abelian-by-(finite of order $\leqslant n!$)-by-abelian.

Subsequently, the authors in [8], [9] investigated the class of groups which have at most n isomorphism classes of derived subgroups (denoted by $\mathfrak{D}_n$) with $n \in \{2, 3\}$. Clearly a group is a $\mathfrak{D}_1$-group if and only if it is abelian. Also the authors, in [8], classified completely the locally finite $\mathfrak{D}_3$-groups. It seems interesting to study $\mathcal{GR}_n$-groups for a given value of n. In this paper, among other things, we first show that for every nonabelian characteristically simple $\mathfrak{D}_n$-group G, we have $n \geqslant |\pi(G)| + 2$. Moreover, we show that this inequality is proper except for the alternating group A_5. In fact, we have the following new characterization of A_5.

Theorem 1.1. *For every nonabelian characteristically simple $\mathfrak{D}_n$-group G we have $n = |\pi(G)| + 2$ if and only if $G \cong A_5$.*

Finally, we give a negative answer to the following question raised by the author in [11]: Let G be a group and H a finite simple group. Is it true that

$$G \cong H \Leftrightarrow G, H \in \mathcal{GR}_n \setminus \mathcal{GR}_{n-1} \text{ for some } n?$$

Or

$$G \cong H \Leftrightarrow G, H \in \mathfrak{D}_n \setminus \mathfrak{D}_{n-1} \text{ for some } n?$$

In this paper all groups will be finite and we use the usual notation, for example A_n, S_n, $\mathrm{PSL}(n, q)$, $\mathrm{PSU}(n, q)$ and $\mathrm{Sz}(q)$, respectively, denote the alternating group on n letters, the symmetric group on n letters, the projective special linear group of degree n over the finite field of size q, the projective special unitary group of degree n over the finite field of order q^2 and the Suzuki group over the field with q elements.

2. Proofs

Here, we first show that for every nonabelian characteristically simple $\mathfrak{D}_n$-group G, we have $n \geqslant |\pi(G)| + 2$. For this, we need the following lemmas.

Lemma 2.1 (Burnside). *Let P be a Sylow p-subgroup of a finite group G, p a prime. If $N_G(P) = C_G(P)$ then G is a p-nilpotent group.*

Lemma 2.2. *Let G be a finite group and suppose that G is not p_i-nilpotent, where p_i is a prime, $p_i \in \pi(G)$. Then there is a subgroup H_i of G such that H_i' is a nontrivial p_i-group. In particular if G is a $\mathfrak{D}_n$-group, then $n \geqslant |\pi(G)| + 1$.*

Proof. Let $p_i \in \pi(G)$, and $P_i \in \mathrm{Syl}_{p_i}(G)$. If $N_G(P_i) = C_G(P_i)$, then by Lemma 2.1, G is p_i-nilpotent, a contradiction. So $C_G(P_i) < N_G(P_i)$. Choose $x_i \in N_G(P_i) \setminus C_G(P_i)$, and let $H_i = \langle x_i, P_i\rangle$. Then $H_i' = P_i'[P_i, x]$, and H_i' is a nontrivial p_i-subgroup. $\square$

Lemma 2.3. *If G is a finite nonabelian simple $\mathfrak{D}_n$-group, then $n \geqslant |\pi(G)| + 2$.*

Proof. Since G is not p-nilpotent for every $p \in \pi(G)$ and $G' = G$, the assertion follows from Lemma 2.2. $\square$

Lemma 2.4. *Let H be a $\mathfrak{D}_{n_1}$-group, K a $\mathfrak{D}_{n_2}$-group and $G = H \times K$. Then the following statements are true:*

(1) *G is a D_t-group for some $t \geqslant n_1 n_2$.*
(2) *If $H \cong K$ and K is a simple group, then G is a D_t-group for some $t \geqslant n_1 n_2 + 1$.*
(3) *If $(|H|, |K|) = 1$, then G is a $D_{n_1 n_2}$-group.*

P r o o f. (1) Clearly.

(2) For proof, we consider the diagonal subgroup of G which is of the form $T = \{(a,a)\colon a \in K\}$. Now as, by [7], every element of T is of the form $([a,b],[a,b])$, where $a, b \in K$, one can conclude that T is a perfect subgroup of G, that is $T' = T$. Hence the result follows from Lemma 2.3 and Lemma 2.2.

(3) Since $(|H|,|K|) = 1$, every subgroup T of G is of the form $T = T_1 \times T_2$ and so $T' = T_1' \times T_2'$, where T_1 and T_2 are subgroups of H and K, respectively. This completes the proof. $\square$

Theorem 2.5. *If G is a finite nonabelian characteristically simple $\mathfrak{D}_n$-group, then $n \geqslant |\pi(G)| + 2$.*

P r o o f. Let G be a characteristically simple $\mathfrak{D}_n$-group. Then $G \cong \prod\limits_{i=1}^{t} K_i$, where the K_i's are isomorphic to a simple $\mathfrak{D}_m$-group K. Hence, by Lemma 2.4 and Lemma 2.3, we have $n \geqslant m^t \geqslant (\pi(K)+2)^t = (\pi(G)+2)^t \geqslant (\pi(G)+2)$, since $\pi(G) = \pi(K)$, as wanted. $\square$

Corollary 2.6. *A_5 is the only nonabelian simple $\mathfrak{D}_5$-group.*

P r o o f. Let G be a nonabelian simple $\mathfrak{D}_5$-group, then by Theorem 2.5, $|\pi(G)| = 3$ and, by the results in [4], the nonabelian simple groups of order divisible by exactly three primes are the following eight groups: $\mathrm{PSL}(2,q)$, where $q \in \{5,7,8,9,17\}$, $\mathrm{PSL}(3,3)$, $U_3(3)$, $U_4(2)$. Now it is easy to see (by GAP [2] and also Lemmas 2.7 and 2.9, below) that A_5 is the only nonabelian simple $\mathfrak{D}_5$-group. $\square$

Now we can show that the inequality of Theorem 2.5, is proper except for the group A_5. In fact, in the sequel, we want to prove Theorem 1.1. For this purpose we need the following lemmas.

Lemma 2.7. *Let $G = \mathrm{PSL}(2,q)$ be a $\mathfrak{D}_n$-group such that $|\pi(G)| \geqslant 5$. Then $n > |\pi(G)| + 2$.*

P r o o f. By Lemma 2.2, it is enough to find a proper subgroup of G such that its derived subgroup is not a primary group. Suppose that $\{p,r,s,t,u\} \subseteq \pi(G)$, then since $|G| = q(q^2-1)/d$, where $d = (2, q-1)$, we can assume $\{r,s,t,u\} \subseteq \pi(q-1) \cup \pi(q+1)$. Thus one of the numbers $q-1$ or $q+1$ is of the form $2m$ where m is a number which is divided by at least two distinct odd prime numbers. Now by Dickson's Theorem [6], G has dihedral subgroups of the form D_{2z} where $z \mid (q \pm 1)/d$. The derived subgroup of D_{2z} has order divisible by at least two distinct primes, as desired. $\square$

Lemma 2.8. *Let $G = K \rtimes H$ be a Frobenius group, then $G' = KH'$.*

P r o o f. Obviously. $\square$

Lemma 2.9. *Let $G = Sz(q)$, $q = 2^{2m+1}$. Then $n > |\pi(G)| + 2$.*

P r o o f. Suppose that F is a Sylow 2-subgroup of G, then F is nonabelian of order q^2 and $N_G(F) = FH = T$ is a Frobenius group with cyclic complement H of order $q - 1$ and kernel F. Now since F is nonabelian, we have $1 < Z(F) < F$, on the other hand, $H \leqslant N_T(Z(F))$, so $S = Z(F)H$ is a Frobenius group and by Lemma 2.8, $|S'| = |Z(F)| = q$ and $|T'| = |F| = q^2$. So G has at least two nonisomorphic 2-subgroups. Hence $n > |\pi(G)| + 2$. $\square$

Remark 2.10. If G is a nonabelian simple group and $|\pi(G)| \in \{3, 4\}$, then we say that G is a K_n-group for $n = 3, 4$. Herzog in [4] proved that there are eight simple K_3-groups. Also Shi in [10] gave a characterization of all simple K_4-groups. By GAP software we can see that in these groups $n > |\pi(G)| + 2$, except for the group A_5. In the following theorem, we show that in fact $G = A_5$ is the only group among all simple groups whose number of nonisomorphic derived subgroups is equal to $|\pi(G)| + 2$.

Lemma 2.11. *Let G be a nonabelian simple $\mathfrak{D}_n$-group. Then $n = |\pi(G)| + 2$ if and only if $G \cong A_5$.*

P r o o f. Let G be a nonabelian simple $\mathfrak{D}_n$-group, other than A_5. By Lemma 2.1, it is enough to find $p \in \pi(G)$ and two subgroups H_1 and H_2 of G such that H_1' and H_2' are nonisomorphic p-groups, or to find a subgroup H whose derived subgroup is not a p-group. It follows that $n > |\pi(G)| + 2$. It is well-known that every nonabelian simple group contains a minimal simple group (see [1]). So if G is not a minimal simple group, let $H < G$ be a proper minimal simple subgroup. Thus $|H'| = |H|$ is not a p-group, so $n > |\pi(G)| + 2$. Therefore it is enough to consider the following minimal simple groups:

(1) $\mathrm{PSL}(2, 2^p)$ where p is a prime number.
(2) $\mathrm{PSL}(2, 3^p)$ where p is an odd prime.
(3) $\mathrm{PSL}(2, p)$ where $p > 3$ and $5 \mid p^2 + 1$.
(4) $\mathrm{SZ}(2^p)$ where p is an odd prime.
(5) $\mathrm{PSL}(3, 3)$.

Now by Lemmas 2.7, 2.9 and Remark 2.10, the proof is complete. $\square$

Now we are ready to prove the main result.

P r o o f of Theorem 1.1. Let G be a characteristically simple $\mathfrak{D}_n$-group. Then $G \cong \prod_{i=1}^{i=t} K_i$, where the K_i's are isomorphic to a simple $\mathfrak{D}_m$-group K. Now, by Lemma 2.11, we get $n \geqslant m^t \geqslant (\pi(K)+2)^t \geqslant (\pi(G)+2)^t$, since $\pi(G) = \pi(K)$. Therefore $t = 1$ and the result follows. □

Example 2.12. Consider the nonsolvable symmetric group $G = S_n$, for $n \geqslant 5$. Since for every $m \leqslant n$, $S_m \leqslant S_n$, we have that $\mathcal{D} = \{A_n, A_{n-1}, \ldots, A_4, V_4, 1\}$ is a set of nonisomorphic derived subgroups of G. Now $|\mathcal{D}| = n-1$, therefore, if $G \in \mathfrak{D}_t$, we have $n \leqslant t+1$, thus $|\pi(G)| \leqslant t+1$.

Note that generally, the relation in Lemma 2.3, is not true for all nonsolvable groups. For example, see the following.

Example 2.13. Let H be an arbitrary (in particular an insolvable group) $\mathfrak{D}_n$-group, with $\pi(H) = \{p_1, p_2, \ldots, p_t\}$. If $n \geqslant t+1$, then consider the group $G = H \times Z_{p_{t+1}} \times Z_{p_{t+2}} \times \ldots \times Z_{p_n}$, where $p_1 < p_2 < \ldots < p_t < p_{t+1} < \ldots < p_n$ are prime numbers. By Lemma 2.4, G is a $\mathfrak{D}_n$-group with $|\pi(G)| = n$.

Note that generally, two groups with the same number of derived subgroups (or even with the same number of isomorphism classes of derived subgroups) need not be necessarily isomorphic. In fact we give a negative answer to a question raised in [11].

Proposition 2.14. *Let $G = D_{2^n} = \langle r, s \colon r^{2^{n-1}} = s^2 = 1, r^s = r^{-1}\rangle$ be the dihedral group of order 2^n. Then $G \in \mathfrak{D}_{n-1} \cap \mathcal{RG}_{n-1}$.*

P r o o f. G' is cyclic of order 2^{n-2} and the derived subgroup of every subgroup of G is one of the $n-1$ subgroups of G'. On the other hand, each of these subgroups of G' is the derived subgroup of some subgroup of G. □

Example 2.15. Let $G = D_{2^6}$, $S = A_5$ and $H = D_{2^{24}}$, then G, S are $\mathfrak{D}_5$-groups and H, S are $\mathcal{RG}_{23}$-groups.

Finally, in view of the above results, we raise the following conjecture.

Conjecture 2.16. *Let G be a group and S a finite simple group such that $|G| = |S|$. Is it true that*

$$G \cong S \Leftrightarrow G, S \in \mathcal{GR}_n \setminus \mathcal{GR}_{n-1}, \text{ for some } n?$$

Or

$$G \cong S \Leftrightarrow G, S \in \mathfrak{D}_n \setminus \mathfrak{D}_{n-1}, \text{ for some } n?$$

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Authors' address: Leyli Jafari Taghvasani, Soran Marzang, Mohammad Zarrin (corresponding author), Department of Mathematics, University of Kurdistan, P.O. Box: 416, Sanandaj, Iran, e-mail: `L.jafari@sci.uok.ac.ir`, `S.marzang@sci.uok.ac.ir`, `M.zarrin@sci.uok.ac.ir`.