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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR GRADIENT SYSTEMS WITHOUT A COMPACTNESS EMBEDDING CONDITION

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Abstract. This paper is devoted to the existence and uniqueness of solutions for gradient systems of evolution which involve gradients taken with respect to time-variable inner products. The Gelfand triple (V, H, V') considered in the setting of this paper is such that the embedding $V \hookrightarrow H$ is only continuous.

Keywords: gradient system; existence and uniqueness of solution; Galerkin method; quadratic form; weakly lower semicontinuity; diffusion equation

MSC 2010: 35F20, 35F25, 35F30, 35K57, 47H05, 47J05

1. Introduction

Let V be a Banach space which is densely and continuously embedded into a Hilbert space H, let $E\colon V \to \mathbb{R}$ be a continuously differentiable and weakly coercive functional such that the derivative operator E' is monotone and bounded. Let further $g\colon [0, T] \to \operatorname{Inner}(H)$ be a function. Here, $\operatorname{Inner}(H)$ denotes the set of all inner products on H. Then we have $\langle u, v\rangle_{g(t)} = \langle Q(t)u, v\rangle_H$, where $Q(t) \in \mathcal{L}(H)$ is symmetric positive definite. We assume that the norms associated with the inner products $\langle \cdot, \cdot\rangle_{g(t)}$ are uniformly equivalent to a fixed one on H. We suppose further that the function $t \to Q(t)$ is continuously differentiable on $[0, T]$, $-Q'(t)$ is positive and $Q(t)$ is invertible for every $t \in [0, T]$. Let us consider the gradient system

$$\begin{cases} u'(t) + \nabla_{g(t)} E(u(t)) = f(t) & \text{for a.e. } t \in (0, T), \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $\nabla_{g(t)} E$ denotes the gradient of E in H with respect to the inner product $\langle \cdot, \cdot\rangle_{g(t)}$. Our main result, Theorem 2.1, says the following: under the preceding assumptions on V, H, E and g for every $f \in L^2(0, T; H)$ and every $u_0 \in V$

there exists a unique $u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$ such that $u(t) \in D(\nabla_{g(t)} E)$ for a.e. $t \in (0,T)$ which is a solution of problem (1.1). This theorem is a maximal regularity result in the sense that the two terms on the left-hand side of the above system have the same regularity as the right-hand side term. Our approach is based on the application of the Galerkin method combined with techniques and results of the theory of monotone operators and the theory of weakly lower semicontinuous quadratic forms on Hilbert spaces. The study of weakly lower semicontinuity for quadratic forms on Hilbert spaces is one of the important problems in the field of Calculus of Variations; a good reference about this subject is [11].

When the function g is constant, i.e. $g = \langle \cdot, \cdot \rangle_H$, problem (1.1) was considered in the literature; see for example [9], Theorem 6.1. When g depends further on the space variable, problem (1.1) was studied in [7], where the embedding $V \hookrightarrow H$ is supposed to be compact, the energy E is assumed to be H-elliptic (i.e., the functional $u \to E(u) + \frac{1}{2}\omega\|u\|_H^2$ is convex and weakly coercive for some constant $\omega \in \mathbb{R}$) instead of convex and weakly coercive and the metric g satisfied some continuity condition.

We apply our result in order to solve the diffusion equation governed by the p-Laplace operator

$$\frac{\partial u}{\partial t} - m(t,\cdot)\Delta_p u = f \quad \text{in } (0,T) \times \Omega,$$

complemented by the Dirichlet boundary condition

$$u = 0 \quad \text{on } (0,T) \times \partial\Omega.$$

Here $\Omega \subset \mathbb{R}^N$ is an open set of $\mathbb{R}^N$, $1 \leqslant p \leqslant 2$, and $m\colon [0,T] \times \Omega \to [\varepsilon, 1/\varepsilon]$ is a measurable function satisfying some suitable conditions.

In [1] the authors studied the existence of solutions of the evolution equation with nonmonotone perturbation

$$\text{(1.2)} \qquad \begin{cases} u'(t) + Au(t) + G(u(t)) = f(t) & \text{for a.e. } t \in (0,T), \\ u(0) = u_0 \end{cases}$$

under the following assumptions.

(i) $A\colon V \to V'$ is monotone and hemicontinuous.

(ii) There exist positive constants c_1, c_2 such that for every $u \in V$

$$\|Au\|_{V'} \leqslant c_1(\|u\|_V^{p-1} + 1), \quad c_2\|u\|_V^p \leqslant c_3 + \langle Au, u \rangle_{V',V}.$$

(iii) $G\colon V \to V'$ is both continuous and weakly continuous.

(iv) There exist positive constants a, b and c such that for every $u \in V$

$$\langle G(u), u \rangle_{V',V} \geqslant -c, \quad \|Gu\|_{V'} \leqslant a\|u\|_V^{p-1} + b.$$

(v) If $u_n \rightharpoonup u$ in V, then we have

$$\langle G(u_n), u_n - u\rangle_{V',V} \to 0.$$

The authors proved that problem (1.2) can be rewritten as an algebraic equation in the Banach space $L^q(0,T;V')$ for which they applied the theory of pseudo-monotone operators. Further results concerning evolution equations involving nonmonotone perturbations can be found for example in [12], [18], [19].

In [8], the author considered the evolution problem of the form

$$\tag{1.3} \begin{cases} u'(t) + \partial\varphi(u(t)) \ni f(t) & \text{for a.e. } t \in (0,T), \\ u(0) = u_0, \end{cases}$$

where $\partial\varphi$ denotes the subdifferential of a functional φ defined on a Hilbert space H. Under the assumptions that φ is convex, lower semicontinuous and $\min\varphi = 0$, it was proved that if $f \in L^2(0,T;H)$ and $u_0 \in D(\varphi)$, problem (1.3) admits a unique strong solution $u \in W^{1,2}(0,T;H)$. The approach used in this study is based on the application of the theory of maximal monotone operators. Problem (1.3) was considered also in [9] where the authors used the so-called time-discretization method which consists in the discretization in time. Further existence and uniqueness results for multivalued evolution equations (namely in the nonautonomous case) can be found in [5] and the references therein.

Evolution problems of linear and nonlinear parabolic equations describe naturally several phenomena, e.g. in physical, chemical, biological etc. applications. We refer the reader to the monographs [2], [10], [13], [15], also to the books [4], [6], [20] and to the papers [3], [17] for classical results on linear and quasilinear second order parabolic equations. Let us point out that there is a large literature which deals with evolution problems involving monotone and pseudomonotone type operators with applications to problems involving the p-Laplacian with Dirichlet or Neumann boundary conditions both in bounded or unbounded domains. In this sense, we refer the reader to [16], [21], [22], [23].

2. Functional setting, main result and some preliminaries

Let V be a real reflexive and separable Banach space with norm $\|\cdot\|_V$, and let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle_H$ and induced norm $\|\cdot\|_H$ such that V is densely and continuously embedded into H. The duality bracket between the dual space V' and V is denoted by $\langle\cdot,\cdot\rangle_{V',V}$. Let $\mathrm{Inner}(H)$ be the set of all inner products on H. Let $g\colon [0,T] \to \mathrm{Inner}(H)$ be a function and denote by $\langle\cdot,\cdot\rangle_{g(t)}$

the inner product $g(t)$ at a time $t \in [0,T]$ and by $\|\cdot\|_{g(t)}$ the norm associated with this inner product. By the Riesz representation theorem, there exists a mapping $Q\colon [0,T] \to \mathcal{L}(H)$ such that for every $t \in [0,T]$, $Q(t)$ is symmetric, positive definite, and for every v, $w \in H$

$$\langle Q(t)v, w\rangle_H = \langle v, w\rangle_{g(t)}. \tag{2.1}$$

Let $E\colon V \to \mathbb{R}$ be a differentiable functional and denote by E' the Fréchet derivative of E.

Definition 2.1. We define the gradient of E in H with respect to the inner product $\langle\cdot,\cdot\rangle_H$ by

$$\begin{aligned} &D(\nabla E) = \{u \in V\colon \exists\, w \in H\ \forall\, v \in V,\ E'(u)v = \langle w, v\rangle_H\},\\ &\nabla E(u) = w. \end{aligned}$$

Definition 2.2. For every $t \in [0,T]$, we define the gradient of E in H with respect to the inner product $\langle\cdot,\cdot\rangle_{g(t)}$ by

$$\begin{aligned} &D(\nabla_{g(t)} E) = \{u \in V\colon \exists\, w \in H\ \forall\, v \in V,\ E'(u)v = \langle w, v\rangle_{g(t)}\},\\ &\nabla_{g(t)} E(u) = w. \end{aligned}$$

Remark 2.1. We note that in the finite-dimensional setting, the gradients $\nabla E(u)$ and $\nabla_{g(t)} E(u)$ exist and are unique for every $t \in [0,T]$ and every $u \in V$ by the Riesz representation theorem.

Let us consider the evolution equation

$$\begin{cases} u'(t) + \nabla_{g(t)} E(u(t)) = f(t) & \text{for a.e. } t \in (0,T),\\ u(0) = u_0, \end{cases} \tag{2.2}$$

where f is a given function from $[0,T]$ into H and u_0 is a given initial data in V. We call this evolution equation an *abstract gradient system*. We are concerned with solutions of (2.2) given in the following sense.

Definition 2.3. A function $u\colon [0,T] \to V$ is called a solution of problem (2.2) if:

$$u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V), \quad u(t) \in D(\nabla_{g(t)} E) \quad \text{for a.e. } t \in [0,T]$$

and u satisfies the gradient system (2.2).

Remark 2.2. Note that, by the Sobolev embedding

$$W^{1,2}(0,T;H) \hookrightarrow C([0,T];H),$$

it makes sense to evaluate an element $u \in W^{1,2}(0,T;H)$ pointwise, and if u is a solution of problem (2.2), the initial condition $u(0) = u_0$ has a well defined meaning.

Remark 2.3. The reader shall be referred to [9] for a comprehensive discussion on the notions of gradients and gradient systems in finite- and infinite-dimensional spaces.

Now, we state the assumptions on E and g which will be needed in the sequel.

(H1) E is continuously differentiable.

(H2) E is a weakly coercive functional, i.e.,

$$\lim_{\|u\|_V \to \infty} E(u) = \infty.$$

(H3) $E' \colon V \to V'$ is a monotone operator, i.e., for every $u, v \in V$

$$\langle E'(u) - E'(v), u - v \rangle_{V',V} \geqslant 0.$$

(H4) E' is a bounded operator, i.e., for every $R > 0$ there exists $C_R > 0$ such that for every $u \in V$ the implication

$$\|u\|_V \leqslant R \Rightarrow \|E'(u)\|_{V'} \leqslant C_R$$

holds true.

(H5) There exist two constants c_1, $c_2 > 0$ such that for every $v \in H$ and for every $t \in [0,T]$

$$c_1\|v\|_H \leqslant \|v\|_{g(t)} \leqslant c_2\|v\|_H.$$

(H6) The function $Q \colon [0,T] \to \mathcal{L}(H)$ is continuously differentiable on $[0,T]$.

(H7) The negative derivative $-Q'$ is positive, i.e., for every $t \in [0,T]$ and every $u \in H$

$$\langle -Q'(t)u, u \rangle_H \geqslant 0.$$

(H8) For every $t \in [0,T]$, $Q(t)$ is invertible.

Let us provide some remarks on assumption (H5).

Remark 2.4. Assume that assumption (H5) is satisfied. Then we have for every $t \in [0,T]$ and $v, w \in H$

$$|\langle Q(t)v, w \rangle_H| = |\langle v, w \rangle_{g(t)}| \leqslant \|v\|_{g(t)}\|w\|_{g(t)} \leqslant c_2^2\|v\|_H\|w\|_H.$$

Therefore

$$\|Q(t)\|_{\mathcal{L}(H)} \leqslant c_2^2.$$

Remark 2.5. Assume that assumption (H5) is satisfied. Then we claim that we have the following assertion:

$$\left.\begin{array}{l} u_n \rightharpoonup u \quad \text{in } L^2(0,T;H),\\ v \in L^2(0,T;H) \end{array}\right\} \Rightarrow \int_0^T \langle u_n, v\rangle_{g(t)}\,\mathrm{d}t \to \int_0^T \langle u, v\rangle_{g(t)}\,\mathrm{d}t.$$

Indeed, if $v \in L^2(0,T;H)$, then we deduce from the preceding remark that $Q(\cdot)v \in L^2(0,T;H)$, and the desired result follows from the definition of the weak convergence in $L^2(0,T;H)$.

The purpose of this paper is to prove the following result.

Theorem 2.1. *Suppose that assumptions* (H1)–(H8) *hold. Then for every* $u_0 \in V$ *and every* $f \in L^2(0,T;H)$*, there exists a unique function* u *which is a solution of problem* (2.2).

The following result is crucial for the proof of uniqueness of solutions for problem (2.2).

Lemma 2.1. *Assume that assumption* (H8) *is satisfied. Then we have*

$$\begin{aligned} &D(\nabla_{g(t)}E) = D(\nabla E),\\ &\nabla_{g(t)}E(u) = Q(t)^{-1}\nabla E(u) \quad \forall\, t \in [0,T]\ \forall\, u \in D(\nabla E). \end{aligned}$$

P r o o f. Let $u \in D(\nabla E)$. Then we have from identity (2.1)

$$E'(u)v = \langle \nabla E(u), v\rangle_H = \langle Q(t)Q(t)^{-1}\nabla E(u), v\rangle_H = \langle Q(t)^{-1}\nabla E(u), v\rangle_{g(t)}$$

for all $v \in V$ and all $t \in [0,T]$. This shows that $u \in D(\nabla_{g(t)}E)$ and $\nabla_{g(t)}E(u) = Q(t)^{-1}\nabla E(u)$. In a similar way we prove that $D(\nabla_{g(t)}E) \subset D(\nabla E)$. □

3. Proof of Theorem 2.1

Uniqueness. Let $u_1, u_2 \in W^{1,2}(0,T;V) \cap L^\infty(0,T;V)$ be two solutions of problem (2.2). Let us introduce the function $\psi\colon [0,T] \to \mathbb{R}$ defined for every $t \in [0,T]$ by

$$\psi(t) = \frac{1}{2}\|u_1(t) - u_2(t)\|^2_{g(t)}.$$

By assumption (H6), the function ψ is almost everywhere differentiable on $[0,T]$, and by using the fact that u_1 and u_2 are solutions of problem (2.2), assumption (H8) and Lemma 2.1, we have for almost every $t \in [0,T]$

$$\begin{aligned}\psi'(t) &= \frac{1}{2}\langle Q'(t)(u_1-u_2), u_1-u_2\rangle_H + \langle Q(t)(u_1'-u_2'), u_1-u_2\rangle_H \\ &= \frac{1}{2}\langle Q'(t)(u_1-u_2), u_1-u_2\rangle_H - \langle \nabla E(u_1)-\nabla E(u_2), u_1-u_2\rangle_H \\ &= \frac{1}{2}\langle Q'(t)(u_1-u_2), u_1-u_2\rangle_H - (E'(u_1)-E'(u_2))(u_1-u_2).\end{aligned}$$

Then, exploiting assumptions (H3) and (H7), we deduce for almost every $t \in [0,T]$ that

$$\psi'(t) \leqslant 0.$$

The integration of both sides over $(0,T)$ and the initial conditions $u_1(0) = u_2(0) = u_0$ yield for every $t \in [0,T]$

$$\psi(t) = 0.$$

Therefore

$$u_1 = u_2,$$

which proves the uniqueness of solutions for problem (2.2).

Existence. The proof of existence of solutions for problem (2.2) is based on the application of the Galerkin method. We start by constructing approximate finite-dimensional problems.

3.1. Approximate problems and existence of approximate solutions. Let $u_0 \in V$ and $f \in L^2(0,T;H)$. Since V is a separable space, there exists a countable set of linearly independent elements $\{w_n \colon n \geqslant 1\}$ such that their finite linear combinations are dense in V. For every $m \in \mathbb{N}$, put

$$V_m = \operatorname{span}\{w_n,\ 1 \leqslant n \leqslant m\}.$$

As $\bigcup_m V_m$ is dense in V and (V_m) is increasing, we can choose $u_0^m \in V_m$ such that

$$u_0 = \lim_{m\to\infty} u_0^m \quad \text{in } V. \tag{3.1}$$

Let E_m be the restriction of E to V_m, $g_m \colon [0,T] \to \operatorname{Inner}(V_m)$ be the function defined for every $t \in [0,T]$ and for every $v, w \in V_m$ by

$$\langle v, w\rangle_{g_m(t)} = \langle v, w\rangle_{g(t)}.$$

Let further $P_m(t)\colon H \to V_m$ be the orthogonal projection from H onto V_m with respect to the inner product $\langle \cdot, \cdot \rangle_{g(t)}$. Let us consider the approximate problems in V_m

$$\begin{cases} u_m'(t) + \nabla_{g_m(t)} E_m(u_m(t)) = P_m(t) f(t), & \text{a.e. } t \in (0, T), \\ u_m(0) = u_0^m. \end{cases} \tag{3.2}$$

Here $\nabla_{g_m(t)} E_m$ denotes the gradient of E_m in V_m with respect to the inner product $\langle \cdot, \cdot \rangle_{g_m(t)}$.

Lemma 3.1. *Problem* (3.2) *admits a maximal solution* $u_m \in W^{1,2}_{\mathrm{loc}}([0, T_m); V_m)$.

Proof. See [7], Proof of Theorem 4, Part 1. □

Lemma 3.2. *We assume that* (H1) *is satisfied. Then we have the chain rule*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(u_m(t)) = \langle \nabla_{g_m(t)} E_m(u_m(t)), u_m'(t) \rangle_{g_m(t)} \textit{ for almost every } t \in (0, T).$$

Proof. We have for almost every $t \in (0, T)$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E(u_m(t)) &= \frac{\mathrm{d}}{\mathrm{d}t} E_m(u_m(t)) \text{ (by the definition of } E_m) \\ &= E_m'(u_m(t))(u_m'(t)) \text{ (by [9], Lemma 8.4 (a))} \\ &= \langle \nabla_{g_m(t)} E_m(u_m(t)), u_m'(t) \rangle_{g_m(t)} \text{ (by the definition of } \nabla_{g_m(t)} E_m). \end{aligned}$$

□

3.2. Estimates. At first, we establish a priori estimates for the solutions u_m of problem (3.2).

Lemma 3.3. *There exist constants C and C' independent on m such that*

$$\sup_{t \in [0, T_m)} \|u_m(t)\|_V \leqslant C$$

and

$$\sup_{m \in \mathbb{N}} \int_0^{T_m} \|u_m'\|_H^2 \, \mathrm{d}t \leqslant C'.$$

Proof. Multiply (3.2) by u_m' with respect to the inner product $\langle \cdot, \cdot \rangle_{g_m(t)}$, then integrate over the interval $(0, t)$, $t \in (0, T_m)$, and obtain

$$\int_0^t \|u_m'(s)\|_{g(s)}^2 \, \mathrm{d}s + E(u_m(t)) - E(u_0^m) = \int_0^t \langle f(s), u_m'(s) \rangle_{g(s)} \, \mathrm{d}s$$

using Lemma 3.2. Assumption (H5), the continuity of E, and the fact that (u_0^m) is a bounded sequence in V yield

$$c_1^2 \int_0^t \|u_m'(s)\|_H^2 \,\mathrm{d}s + E(u_m(t)) \leqslant C_1 + c_2^2 \int_0^t \|f(s)\|_H \|u_m'(s)\|_H \,\mathrm{d}s,$$

where $C_1 > 0$ is independent of m. Then, by virtue of the Young inequality, this implies that there exists a constant c_3 which is independent of m and t such that

$$\frac{c_1^2}{2} \int_0^t \|u_m'(s)\|_H^2 \,\mathrm{d}s + E(u_m(t)) \leqslant c_3. \tag{3.3}$$

Thus, we obtain the estimate

$$\sup_{t \in [0,T_m)} E(u_m(t)) \leqslant c_3.$$

Since, by assumption (H2), E is weakly coercive, the last estimate yields that there exists a constant C independent of m such that

$$\sup_{t \in [0,T_m)} \|u_m(t)\|_V \leqslant C.$$

The functional E is of class C^1, and by assumption (H3), E' is monotone, so that by [16], Proposition 1.1, page 158, E is convex, which implies, since E is coercive on the reflexive Banach space V, that E is bounded from below. Thus, we deduce from estimate (3.3) that there exists a constant C' which is independent of m such that

$$\sup_{m \in \mathbb{N}} \int_0^{T_m} \|u_m'\|_H^2 \,\mathrm{d}t \leqslant C'.$$

□

3.3. Limiting procedure. Next, we derive the convergence of u_m to a solution u of (2.2) as $m \to \infty$. From the above a priori estimates we can deduce that the solutions u_m are global, that is $T_m = T$, and therefore the sequence (u_m) is bounded in $W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$, and by virtue of assumption (H4) we get that the sequence $(E'(u_m))$ is bounded in $L^\infty(0,T;V')$. From this boundedness, by extracting a sequence, which will be also denoted by (u_m), we can derive convergences of (u_m).

Lemma 3.4. *There exist $u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$ and $\chi \in L^\infty(0,T;V')$ such that*

$$u_m \rightharpoonup u \quad \text{in } W^{1,2}(0,T;H), \tag{3.4}$$

$$u_m \overset{\mathrm{w}^*}{\to} u \quad \text{in } L^\infty(0,T;V), \tag{3.5}$$

$$E'(u_m) \overset{\mathrm{w}^*}{\to} \chi \quad \text{in } L^\infty(0,T;V'). \tag{3.6}$$

Now, we justify that the function u satisfies the initial condition of problem (2.2).

Lemma 3.5. *The function u satisfies the condition $u(0) = u_0$.*

Proof. Since, by (3.4), $(u_m(0))$ converges weakly to $u(0)$ in H, and since $u_m(0) = u_0^m$ and (u_0^m) converges strongly to u_0 in V, we claim by uniqueness of the limit that $u(0) = u_0$. $\square$

Next, we prove that u satisfies the evolution equation of problem (2.2).

Lemma 3.6. *It holds that $u(t) \in D(\nabla_{g(t)} E)$ for a.e. $t \in (0,T)$ and u satisfies the evolution equation of problem* (2.2).

Proof. Let $m \in \mathbb{N}$, $w \in V_m$ and $\varphi \in L^2(0,T)$ be a scalar function. Then for every $n \geqslant m$ we have from (3.2)

$$\int_0^T \langle u_n', \varphi(t)w\rangle_{g(t)} \,\mathrm{d}t + \int_0^T \langle E'(u_n), \varphi(t)w\rangle_{V',V} \,\mathrm{d}t = \int_0^T \langle f(t), \varphi(t)w\rangle_{g(t)} \,\mathrm{d}t.$$

Recalling (3.4), (3.6) and Remark 2.5, we obtain upon passing to weak limits in this last identity

$$\int_0^T \langle u', \varphi(t)w\rangle_{g(t)} \,\mathrm{d}t + \int_0^T \langle \chi, \varphi(t)w\rangle_{V',V} \,\mathrm{d}t = \int_0^T \langle f(t), \varphi(t)w\rangle_{g(t)} \,\mathrm{d}t.$$

By [21], Lemma 8.28, $\bigcup\limits_{k\in\mathbb{N}} L^2(0,T;V_k)$ is dense in $L^2(0,T;V)$, so that this last identity holds true if we replace $\varphi(\cdot)w$ by any $v \in L^2(0,T;V)$, which implies in particular

$$\int_0^T \langle u', u\rangle_{g(t)} \,\mathrm{d}t + \int_0^T \langle \chi, u\rangle_{V',V} \,\mathrm{d}t = \int_0^T \langle f(t), u\rangle_{g(t)} \,\mathrm{d}t. \tag{3.7}$$

Returning once more to (3.2), integrating by parts, and using assumption (H6) and the fact that $Q(t)$ is a symmetric operator, we obtain

$$\begin{aligned}\int_0^T E'(u_n)u_n \,\mathrm{d}t &= \int_0^T \langle f(t), u_n\rangle_{g(t)} \,\mathrm{d}t - \int_0^T \langle u_n', u_n\rangle_{g(t)} \,\mathrm{d}t \\ &= \int_0^T \langle f(t), u_n\rangle_{g(t)} \,\mathrm{d}t - \int_0^T \langle Q(t)u_n', u_n\rangle_H \,\mathrm{d}t \\ &= \int_0^T \langle f(t), u_n\rangle_{g(t)} \,\mathrm{d}t + \frac{1}{2}\int_0^T \langle Q'(t)u_n, u_n\rangle_H \,\mathrm{d}t \\ &\quad + \frac{1}{2}\langle Q(0)u_0^n, u_0^n\rangle_H - \frac{1}{2}\langle Q(T)u_n(T), u_n(T)\rangle_H.\end{aligned} \tag{3.8}$$

By assumption (H7), $u \mapsto -\int_0^T \langle Q'(t)u, u\rangle_H \, dt$ defines a positive quadratic form on $L^2(0,T;H)$. Since this quadratic form is continuous on $L^2(0,T;H)$, it is weakly lower semicontinuous on $L^2(0,T;H)$. Using the convergence (3.4), we have

$$-\int_0^T \langle Q'(t)u, u\rangle_H \, \mathrm{d}t \leqslant -\liminf_{n\to\infty} \int_0^T \langle Q'(t)u_n, u_n\rangle_H \, \mathrm{d}t. \tag{3.9}$$

Similarly, using the fact that $Q(T)$ is a positive operator, we deduce that $u \mapsto \langle Q(T)u_n(T), u_n(T)\rangle_H$ defines a weakly lower semicontinuous quadratic form on H. The convergence (3.4) yields that $u_n(T) \rightharpoonup u(T)$ in H, and consequently

$$\langle Q(T)u(T), u(T)\rangle_H \leqslant \liminf_{n\to\infty} \langle Q(T)u_n(T), u_n(T)\rangle_H. \tag{3.10}$$

The convergence (3.1) yields

$$\langle Q(0)u_0^n, u_0^n\rangle_H \to \langle Q(0)u(0), u(0)\rangle_H. \tag{3.11}$$

Employing Remark 2.5 and (3.4), we get

$$\int_0^T \langle f(t), u_n\rangle_{g(t)} \, \mathrm{d}t \to \int_0^T \langle f(t), u\rangle_{g(t)} \, \mathrm{d}t. \tag{3.12}$$

By combining (3.12), (3.11), (3.10), (3.9) and (3.8) we get

$$\begin{aligned}\limsup_{n\to\infty} \int_0^T E'(u_n)u_n \, \mathrm{d}t \leqslant \int_0^T \langle f(t), u\rangle_{g(t)} \, \mathrm{d}t + \frac{1}{2}\int_0^T \langle Q'(t)u, u\rangle_H \, \mathrm{d}t \\ + \frac{1}{2}\langle Q(0)u(0), u(0)\rangle_H - \frac{1}{2}\langle Q(T)u(T), u(T)\rangle_H.\end{aligned}$$

Integrate by parts again to get the identity

$$\int_0^T \langle u', u\rangle_{g(t)} = -\frac{1}{2}\int_0^T \langle Q'(t)u, u\rangle_H \, \mathrm{d}t - \frac{1}{2}\langle Q(0)u(0), u(0)\rangle_H + \frac{1}{2}\langle Q(T)u(T), u(T)\rangle_H.$$

From (3.7) we deduce that

$$\begin{aligned}\limsup_{n\to\infty} \int_0^T E'(u_n)u_n \, \mathrm{d}t &\leqslant \int_0^T \langle f(t), u\rangle_{g(t)} \, \mathrm{d}t - \int_0^T \langle u', u\rangle_{g(t)} \\ &= \int_0^T \langle \chi, u\rangle_{V',V} \, \mathrm{d}t.\end{aligned} \tag{3.13}$$

By assumption (H3) we have

$$\int_0^T \langle E'(u_n), u_n - u\rangle_{V',V} \, \mathrm{d}t \geqslant \int_0^T \langle E'(u), u_n - u\rangle_{V',V} \, \mathrm{d}t.$$

Letting $n \to \infty$ and recalling (3.5) and (3.6), we obtain

$$\liminf_{n\to\infty} \int_0^T E'(u_n)u_n \,\mathrm{d}t \geqslant \int_0^T \langle \chi, u\rangle_{V',V} \,\mathrm{d}t.$$

This limit together with (3.13) yield

$$\lim_{n\to\infty} \int_0^T E'(u_n)u_n \,\mathrm{d}t = \int_0^T \langle \chi, u\rangle_{V',V} \,\mathrm{d}t. \tag{3.14}$$

Let $v \in L^\infty(0,T;V)$, $\lambda \in \mathbb{R}$ and put $w_\lambda = (1-\lambda)u+\lambda v$. Using again assumption (H3) we have

$$\int_0^T \langle E'(u_n), u_n - w_\lambda\rangle_{V',V} \,\mathrm{d}t \geqslant \int_0^T \langle E'(w_\lambda), u_n - w_\lambda\rangle_{V',V} \,\mathrm{d}t,$$

which can be rewritten as

$$\begin{aligned}\lambda \int_0^T \langle E'(u_n), u - v\rangle_{V',V} \,\mathrm{d}t &\geqslant \lambda \int_0^T \langle E'(w_\lambda), u - v\rangle_{V',V} \,\mathrm{d}t \\ &\quad + \int_0^T \langle E'(w_\lambda), u_n - u\rangle_{V',V} \,\mathrm{d}t - \int_0^T \langle E'(u_n), u_n - u\rangle_{V',V} \,\mathrm{d}t.\end{aligned}$$

Letting $n \to \infty$ and recalling (3.5), (3.6) and (3.14), we obtain

$$\int_0^T \langle \chi, u - v\rangle_{V',V} \,\mathrm{d}t \geqslant \int_0^T \langle E'(w_\lambda), u - v\rangle_{V',V} \,\mathrm{d}t.$$

Let $\lambda \to 0$ and use the continuity of E' to discover

$$\int_0^T \langle \chi, v\rangle_{V',V} \,\mathrm{d}t \leqslant \int_0^T \langle E'(u), v\rangle_{V',V} \,\mathrm{d}t.$$

Since $v \in L^\infty(0,T;V)$ is arbitrary, we deduce that in fact the equality above holds, which implies

$$E'(u) = \chi.$$

Then equality (3.7) becomes for every $v \in L^2(0,T;V)$

$$\int_0^T \langle u', v\rangle_{g(t)} \,\mathrm{d}t + \int_0^T E'(u)v \,\mathrm{d}t = \int_0^T \langle f(t), v\rangle_{g(t)} \,\mathrm{d}t.$$

Thus $u(t) \in D(\nabla_{g(t)}E)$ for a.e. $t \in (0,T)$ and the function u satisfies the evolution equation of system (2.2). $\square$

Combining these lemmas, we have proved the statement of Theorem 2.1.

4. Application

Let $1 < p < \infty$ and let $N \in \mathbb{N}^*$ satisfy one of the conditions
$N = 1$ and $1 < p \leqslant 2$, or $N = p = 2$, or
$N = 2$ and $1 < p < 2$, or $N \geqslant 3$ and $2N/(N+2) \leqslant p \leqslant 2$.

Let further $\Omega \subset \mathbb{R}^N$ be an open set of class C^1 which has a finite width, that is, it lies between two parallel hyperplanes. Let $\varepsilon \in (0,1)$ and let $m\colon [0,T] \times \Omega \to [\varepsilon, 1/\varepsilon]$ be a measurable function such that for every $x \in \Omega$, $m(\cdot, x)$ is continuously differentiable and $\partial m/\partial t$ is positive on $(0,T) \times \Omega$. We consider the diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - m(t,\cdot)\Delta_p u = f & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

where Δ_p is the p-Laplace operator

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

Let $V = W_0^{1,p}(\Omega)$. By [14], Theorem 12.17, the expression

$$\|u\|_V = \|\nabla u\|_{L^p(\Omega)^N}$$

defines a norm on $W_0^{1,p}(\Omega)$ which is equivalent to the usual norm on $W_0^{1,p}(\Omega)$. Let $E\colon V \to \mathbb{R}$ be the functional defined for every $u \in V$ by

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x.$$

As well known, E is continuously differentiable, convex, coercive and we have

$$E'(u)v = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \,\mathrm{d}x \quad \forall\, u, v \in V.$$

Moreover, the derivative E' is bounded. We refer the reader to [9], Theorem 4.3 for more details about these facts. Let further

$$H = L^2(\Omega)$$

be endowed with the usual inner product and norm. If one of the preceding conditions on N and p is satisfied, then, by the Sobolev embedding theorem, we have that $W_0^{1,p}(\Omega)$ is continuously embedded into $L^2(\Omega)$.

We consider the function $g\colon [0,T] \to \mathrm{Inner}(H)$ defined for every $t \in [0,T]$ and every $v, w \in H$ by

$$\langle v, w\rangle_{g(t)} = \int_\Omega vw \frac{\mathrm{d}x}{m(t,x)}.$$

Then we have for every $t \in [0,T]$

$$Q(t) = \frac{1}{m(t,\cdot)} I_H,$$

where $I_H\colon H \to H$ denotes the identity mapping of H. We define the p-Laplace operator with the Dirichlet boundary conditions on $L^2(\Omega)$ by

$$\begin{aligned} D(\Delta_p) = \Big\{ u \in W_0^{1,p}(\Omega)\colon\, \exists\, w \in L^2(\Omega)\ \forall\, v \in W_0^{1,p}(\Omega), \\ \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega wv \,\mathrm{d}x \Big\}, \\ \Delta_p u = w. \end{aligned}$$

From Lemma 2.1, we have $D(\nabla_{g(t)} E) = D(\Delta_p)$ and

$$\nabla_{g(t)} E(u) = -m(t,\cdot)\Delta_p u$$

for every $t \in [0,T]$ and every $u \in D(\Delta_p)$. As a consequence of Theorem 2.1, we obtain the following result.

Corollary 4.1. *For every $f \in L^2(0,T; L^2(\Omega))$ and every $u_0 \in W_0^{1,p}(\Omega)$, there exists a unique $u \in W^{1,2}(0,T; L^2(\Omega)) \cap L^\infty(0,T; W_0^{1,p}(\Omega))$ such that $u(t) \in D(\Delta_p)$ for almost every $t \in (0,T)$, which is the solution of problem* (4.1).

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References

[1] *N. U. Ahmed, X. Xiang*: Existence of solutions for a class of nonlinear evolution equations with nonmonotone perturbations. Nonlinear Anal., Theory Methods Appl. *22* (1994), 81–89. zbl MR doi
[2] *H. Amann*: Linear and Quasilinear Parabolic Problems. Vol. 1: Abstract Linear Theory. Monographs in Mathematics 89, Birkhäuser, Basel, 1995. zbl MR doi
[3] *W. Arendt, R. Chill*: Global existence for quasilinear diffusion equations in isotropic nondivergence form. Ann. Sc. Norm. Super. Pisa, Cl. Sci. *9* (2010), 523–539. zbl MR doi
[4] *W. Arendt, A. Grabosch, G. Greiner, U. Moustakas, R. Nagel, U. Schlotterbeck, U. Groh, H. P. Lotz, F. Neubrander*: One-Parameter Semigroups of Positive Operators (R. Nagel, ed.). Lecture Notes in Mathematics 1184, Springer, Berlin, 1986. zbl MR doi

[5] *H. Attouch, A. Damlamian*: Strong solutions for parabolic variational inequalities. Nonlinear Anal., Theory, Methods Appl. *2* (1978), 329–353. zbl MR doi
[6] *V. Barbu*: Nonlinear Semigroups and Differential Equations in Banach Spaces. Editura Academiei Republicii Socialiste România, Bucharest; Noordhoff International Publishing, Leyden, 1976. zbl MR
[7] *S. Boussandel*: Global existence and maximal regularity of solutions of gradient systems. J. Differ. Equations *250* (2011), 929–948. zbl MR doi
[8] *H. Brézis*: Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland Mathematics Studies 5, Notas de Matemática (50). North-Holland Publishing, Amsterdam; American Elsevier Publishing, New York, 1973. (In French.) zbl MR
[9] *R. Chill, E. Fašangová*: Gradient Systems. Lecture Notes of the 13th International Internet Seminar, MatfyzPress, Praha, 2010.
[10] *S. Fučík, A. Kufner*: Nonlinear Differential Equations. Studies in Applied Mechanics 2, Elsevier, Amsterdam, 1980. zbl MR doi
[11] *M. R. Hestenes*: Applications of the theory of quadratic forms in Hilbert space to the calculus of variations. Pac. J. Math. *1* (1951), 525–581. zbl MR doi
[12] *N. Hirano*: Nonlinear evolution equations with nonmonotonic perturbations. Nonlinear Anal., Theory Methods Appl. *13* (1989), 599–609. zbl MR doi
[13] *O. A. Ladyženskaja, V. A. Solonnikov, N. N. Ural'ceva*: Linear and Quasilinear Equations of Parabolic Type. Translations of Mathematical Monographs 23, American Mathematical Society, Providence, 1968. zbl MR
[14] *G. Leoni*: A First Course in Sobolev Spaces. Graduate Studies in Mathematics 105, American Mathematical Society, Providence, 2009. zbl MR doi
[15] *G. M. Lieberman*: Second Order Parabolic Differential Equations. World Scientific, Singapore, 1996. zbl MR doi
[16] *J.-L. Lions*: Quelques méthodes de résolution des problèmes aux limites non linéaires. Etudes mathematiques, Dunod, Paris, 1969. (In French.) zbl MR
[17] *S. Littig, J. Voigt*: Porous medium equation and fast diffusion equation as gradient systems. Czech. Math. J. *4* (2015), 869–889. zbl MR doi
[18] *M. Ôtani*: On the existence of strong solutions for $\frac{\mathrm{d}u}{\mathrm{d}t}(t)+\partial\psi^1(u(t))-\partial\psi^2(u(t))\ni f(t)$. J. Fac. Sci., Univ. Tokyo, Sect. I A *24* (1977), 575–605. zbl MR
[19] *M. Ôtani*: Nonmonotone perturbations for nonlinear parabolic equations associated with subdifferential operators, Cauchy problems. J. Differ. Equations *46* (1982), 268–289. zbl MR doi
[20] *A. Pazy*: Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Mathematical Sciences 44, Springer, New York, 1983. zbl MR doi
[21] *T. Roubíček*: Nonlinear Partial Differential Equations with Applications. ISNM. International Series of Numerical Mathematics 153, Birkhäuser, Basel, 2013. zbl MR doi
[22] *R. E. Showalter*: Monotone Operators in Banach Space and Nonlinear Partial Differential Equations. Mathematical Surveys and Monographs 49, American Mathematical Society, Providence, 1997. zbl MR doi
[23] *E. Zeidler*: Nonlinear Functional Analysis and Its Applications. II/B: Nonlinear Monotone Operators. Springer, New York, 1990. zbl MR doi

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