Mathematica Bohemica

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Mathematica Bohemica, Vol. 144 (2019), No. 3, 287–297

Persistent URL: `http://dml.cz/dmlcz/147775`

TOTAL BLOW-UP OF A QUASILINEAR HEAT EQUATION WITH SLOW-DIFFUSION FOR NON-DECAYING INITIAL DATA

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Received February 25, 2018. Published online October 22, 2018.
Communicated by Jiří Šremr

Abstract. We consider solutions of quasilinear equations $u_t = \Delta u^m + u^p$ in $\mathbb{R}^N$ with the initial data u_0 satisfying $0 < u_0 < M$ and $\lim\limits_{|x|\to\infty} u_0(x) = M$ for some constant $M > 0$. It is known that if $0 < m < p$ with $p > 1$, the blow-up set is empty. We find solutions u that blow up throughout $\mathbb{R}^N$ when $m > p > 1$.

Keywords: quasilinear heat equation; total blow-up; blow-up only at space infinity

MSC 2010: 35B44, 35K59

1. Introduction

We consider the nonlinear diffusion equation:

$$\begin{cases} u_t = \Delta u^m + u^p, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x) > 0, & x \in \mathbb{R}^N \end{cases} \tag{1.1}$$

with $m > p > 1$ and $u_0 \in C(\mathbb{R}^N)$ for $N \geqslant 1$. This problem is known to admit a local time solution (see [6], [8]), but it may cease to exist in a finite time. We say that the solution of (1.1) *blows up* in finite time if there is some $T = T(u_0) < \infty$ such that

$$\limsup_{t \nearrow T} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = \infty \tag{1.2}$$

and $T(u_0)$ is called the *blow-up time* of the solution u with the initial value u_0. We define the *blow-up set* by

$$B(u_0) = \Big\{a \in \mathbb{R}^N \colon \limsup_{x\to a,\, t\nearrow T} |u(x,t)| = \infty\Big\}.$$

DOI: 10.21136/MB.2018.0026-18

Each element of $B(u_0)$ is called a *blow-up point* of u. We say that the solution u of (1.1) blows up *only at space infinity* if, in addition to (1.2), $B(u_0) = \emptyset$. In this case, the *global blow-up profile* $u(x,T) := \lim_{t\to T} u(x,t)$ is defined for every $x \in \mathbb{R}^N$.

Let us recall known results on the blow-up at space infinity. Lacey in [5] considered a one-dimensional problem $u_t = \Delta u + f(u)$ on the half-line and constructed examples of solutions that blow up only at space infinity. He also obtained results of the global blow-up profile. Giga and Umeda in [4] considered the equation $u_t = \Delta u + u^p$ on $\mathbb{R}^N$ and showed that the blow-up at space infinity occurs if the initial data u_0 satisfies

$$0 < u_0 < M \quad \text{and} \quad \lim_{|x|\to\infty} u_0(x) = M$$

for some constant $M > 0$. Shimojō in [12] considered semilinear heat equations on $\mathbb{R}^N$ and calculated the shape of global blow-up profile of solutions at the blow-up time. It is also proved that such blow-up is always complete, that means that the solution cannot extend as a weak solution after blow-up time.

For the case $0 < m < 1$, the heat conductivity mu^{m-1} becomes small as u increases. Hence, we can see that diffusion is very slow when u is large. Thus, the blow-up at space infinity must occur as the result for semilinear heat equation of [3]. This is proved by Seki for $0 < m \leqslant 1 < p$ (see [10]). He also discusses the generalization of the nonlinearity of the form $u_t = \Delta k(u) + f(u)$ including the case $0 < m \leqslant 1 < p$. On the other hand, if $m > 1$, diffusion is very fast when u is just as large. Hence, the speed of heat propagation, from the space infinity to the origin near the blow-up time, becomes much larger compared to the semilinear problem. Thus, a natural question is: "If $m \in (1,\infty)$ is sufficiently large, does the blow-up only at space infinity fail or not?". Partial answer of this problem was obtained by Seki-Suzuki-Umeda (see [11]). Their result implies that if $1 \leqslant m < p$, the blow-up only at space infinity occurs. Motivated by these results, we consider the following problem: Can the blow-up be confined to space infinity even if diffusion is so large that $m > p > 1$?

In this paper, we give a partial answer to this problem and show that the *total blow-up*, which means that $B(u_0) = \mathbb{R}^N$, occurs.

Theorem 1.1. *Let $p > 1$ and $m - p > 2(p-1)/N$. Then problem (1.1) has a total blow-up solution with the initial value $u_0 \in C(\mathbb{R}^N)$ satisfying*

$$0 < u_0 < M \quad \textit{and} \quad \lim_{|x|\to\infty} u_0(x) = M \tag{1.3}$$

for a certain positive constant $M \in \mathbb{R}$.

This paper is organized as follows. In Section 2, we discuss the condition $m-p > 2(p-1)/N$ of Theorem 1.1 from the point of asymptotic expansion. The rigorous proof of Theorem 1.1 is given in Section 3 by constructing backward self-similar solution.

R e m a r k 1.1. For problem (1.1) with nonnegative initial data satisfying the condition $\lim\limits_{|x|\to\infty} u_0(x) = 0$, it is known that if $p > m > 1$, the blow-up set reduces to finite number of points (see [1], [13]). For $1 < p < m$, total blow-up occurs (see [2]). There is also a third possibility, $B(u_0)$ is a bounded domain for $p = m$. See also Mochizuki and Suzuki [7] for higher dimensional problem. They consider the case when the support of the initial data is compact, and that the support of the solution remains bounded if $p > m$ and it spreads out the whole space if $p < m$ at the blow-up time. The precise behavior of such solutions in one dimensional case is considered in the book [9].

2. Formal asymptotics

We shall explain why the condition $m - p > 2(p-1)/N$ yields total blow-up. We will achieve that by a formal asymptotic calculation. Let $f(u) = u^p$, then the solution of the ODE

$$U' = f(U), \quad U(0) = M, \ M > 0 \tag{2.1}$$

is written as $U(t) = \varphi\big(T(M)-t\big)$, where $\varphi(s) := \kappa s^{-1/(p-1)}$ and $\kappa := (p-1)^{-1/(p-1)}$. Here $T = T(M)$ is the blow-up time for the initial data $U(0) = M$. Substituting $t = 0$ gives $M = \varphi(T(M))$. Furthermore, by a simple calculation, we have

$$\varphi'(s) = -f(\varphi(s)), \qquad \lim_{s\to+0} \varphi(s) = \infty. \tag{2.2}$$

Let us consider (1.1) with initial data $u_0(x) = M - \varepsilon q_0(x)$, where q is a positive function satisfying $\lim\limits_{|x|\to\infty} q_0(x) = 0$ and $\varepsilon > 0$ is a small constant. The first approximation at space infinity must be the flat solution $\varphi(T-t)$. In order to calculate the second term, we shall consider a formal outer expansion

$$u(x,t) = \sum_{i=0}^{\infty} u^{(i)}(x,t)\varepsilon^i$$

and substitute this into $u_t = \Delta k(u) + f(u)$, where $k(u) = u^m$. Then

$$\begin{aligned} u_t^{(0)} &= \Delta k(u^0) + f(u^{(0)}), \\ u_t^{(1)} &= k'(u^{(0)})\Delta u^{(1)} + f'(u^{(0)})u^{(1)}. \end{aligned}$$

Observing the initial condition at space infinity, we assume $u^{(0)}(x,t) = \varphi(T-t)$ as the first approximation of the solution, hence

$$u_t^{(1)} = k'(\varphi(T-t))\Delta u^{(1)} + f'(\varphi(T-t))u^{(1)}. \tag{2.3}$$

Let $q(x,t) = \mathrm{e}^{\Phi(t)\Delta} q_0$ be a solution of $q_t = k'(\varphi(T-t))\Delta q$ with the initial condition $q(x,0) = q_0(x) \in L^1(\mathbb{R}^N)$. In other words,

$$q(x,t) = \mathrm{e}^{\Phi(t)\Delta} q_0, \quad \Phi(t) = \int_0^t k'(\varphi(T-\tau))\,\mathrm{d}\tau.$$

Here we employ the notation

$$(\mathrm{e}^{s\Delta}\, q_0)(x) := \int_{\mathbb{R}^N} G(x-y,s) q_0(y)\,\mathrm{d}y$$

where G is the fundamental solution of the heat equation in $\mathbb{R}^N$:

$$G(x,s) := \frac{1}{(4\pi s)^{N/2}} \exp\Big(-\frac{|x|^2}{4s}\Big).$$

Then the solution of (2.3) is represented as $u^{(1)}(x,t) = -f(\varphi(T-t))q(x,t)$. This can be easily checked from the following calculation.

$$\begin{aligned}
u_t^{(1)} &= -f(\varphi(T-t))q_t - \frac{\mathrm{d}f(\varphi(T-t))}{\mathrm{d}t} q \\
&= -f(\varphi(T-t))q_t + f'(\varphi(T-t))\varphi'(T-t)q \\
&= -f(\varphi(T-t))k'(\varphi(T-t))\Delta q - f'(\varphi(T-t))f(\varphi(T-t))q \\
&= k'(\varphi(T-t))\Delta u^{(1)} + f'(\varphi(T-t))u^{(1)},
\end{aligned}$$

where we applied (2.2) and substitute $s = T-t$. By a formal asymptotic expansion, together with $\varphi'(T-t) = -f(\varphi(T-t))$ again, we get

$$u(x,t) = \varphi(T-t) - \varepsilon f(\varphi(T-t))q(x,t) + O(\varepsilon^2) = \varphi(T-t+\varepsilon q(x,t))$$

provided that $|x|$ is sufficiently large so that $T-t \gg q(x,t)$. We shall discuss a sufficient condition for this approach. Note that $\Phi(t)$ is proportional to $(T-t)^{(p-m)/(p-1)} - T^{(p-m)/(p-1)}$, which implies $\Phi(T) = \infty$ if $m > p$. Assume, for simplicity, that the support of q_0 is compact. Then by applying the inequality

$$\sup_{x\in\mathbb{R}^N} |q(x,t)| \leqslant \frac{1}{(4\pi\Phi(t))^{N/2}} \int_{\mathbb{R}^N} q_0(x)\,\mathrm{d}x,$$

we get the following sufficient condition for $T-t \gg q(x,t)$:

$$T-t \gg O((T-t)^{N(m-p)/(2(p-1))}) = O(\Phi(t)^{-N/2}) \geqslant q(x,t).$$

Since we are interested in what happens as $t \to T_-$, we need the restriction below, which appeared in Theorem 1.1.

$$1 < \frac{N(m-p)}{2(p-1)} \;\Leftrightarrow\; m-p > \frac{2}{N}(p-1).$$

Under this condition, we obtain the following approximation:

$$u(x,t) \approx \varphi(T-t+\varepsilon \mathrm{e}^{\Phi(t)\Delta} q_0) \quad \text{if } t \approx T$$

provided that $|x|$ is sufficiently large so that $T-t \gg q(x,t)$. Here $a \approx b$ means that there exist two constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$, where a and b are two positive functions. Taking a limit $t \to T$ and regarding $\mathrm{e}^{\Phi(T)\Delta} q_0 \equiv 0$, we expect that the total blow-up occurs when $m-p > 2(p-1)/N$. On the other hand, the above formal calculation suggests that $m-p < 2(p-1)/N$ yields the blow-up only at space infinity, and the global profile must be

$$u(x,T) \approx \varphi(\varepsilon \mathrm{e}^{\Phi(T)\Delta} q_0) \quad \text{if } t \approx T. \tag{2.4}$$

Note that $\Phi(T) < \infty$ if $m-p < 2(p-1)/N$. This conjecture (2.4) is proved rigorously in [12] for the semi-linear problem ($m=1$), by constructing suitable sub-super solutions.

3. Total blow-up for quasilinear equation

Our aim of this section is to construct a backward self-similar total blow-up solution of problem (1.1) with the initial value $u_0 \in C(\mathbb{R}^N)$ satisfying (1.3).

Assume the solution u of (1.1) blows up in finite time and let $T > 0$ be its blow-up time. We introduce a simple change of variable as described in Section 2:

$$u(x,t) = \varphi(T-t+h(x,t)). \tag{3.1}$$

From this and $\lim_{s\to 0} \varphi(s) = \infty$, we can see that the blow-up of the solution $u(x,t)$ for (1.1) as $t \to T$ corresponds to the extinction of the solution $h(x,t)$ as $t \to T$. By a simple calculation together with (3.1) and (2.2),

$$\partial_t \varphi(T-t+h) = \varphi'(T-t+h)(h_t - 1), \quad f(\varphi(T-t+h)) = -\varphi'(T-t+h).$$

By substituting (3.1) into $\Delta u^m = m(m-1)u^{m-2}|\nabla u|^2 + mu^{m-1}\Delta u$, we have

$$\begin{aligned}
&\Delta\varphi^m(T-t+h)\\
&\quad= m(m-1)\varphi^{m-2}(T-t+h)|\varphi'(T-t+h)\nabla h|^2\\
&\qquad+ m\varphi^{m-1}(T-t+h)(\varphi'(T-t+h)\Delta h + \varphi''(T-t+h)|\nabla h|^2)\\
&\quad= m(m-1)\varphi^{m-2}(T-t+h)|\varphi'(T-t+h)\nabla h|^2\\
&\qquad+ m\varphi^{m-1}(T-t+h)(\Delta h - f'(\varphi(T-t+h))|\nabla h|^2)\varphi'(T-t+h).
\end{aligned}$$

Here we apply the relation $\varphi''(s) = -f'(\varphi(s))\varphi'(s)$, which can be shown by differentiating (2.2). Substituting (3.1) into (1.1) and dividing it by $\varphi'(T-t+h)$, we obtain

$$h_t = m\varphi^{m-1}(T-t+h)\Big(\Delta h + \Big((m-1)\frac{\varphi'(T-t+h)}{\varphi(T-t+h)} - f'(\varphi(T-t+h))\Big)|\nabla h|^2\Big).$$

Applying $\varphi'(s)/\varphi(s) = -s^{-1}/(p-1)$ and $f'(\varphi(s)) = ps^{-1}/(p-1)$, we get the equation

$$h_t = \frac{m\kappa^{m-1}}{(T-t+h)^{(m-1)/(p-1)}}\Big(\Delta h - \frac{(m+p-1)|\nabla h|^2}{(p-1)(T-t+h)}\Big) \tag{3.2}$$

with the initial data $h(\cdot,0) = \varphi^{-1}(u_0) - T$.

Next we introduce new space and time variables and a function

$$w(y,\sigma) := \frac{h(x,t)}{T-t}, \quad y := (T-t)^\beta x, \quad \sigma = \log\frac{1}{T-t},$$

where $\beta := (m-p)/(2(p-1))$ and h is the solution of (3.2). By the chain rule, together with

$$y_t(x,t) = -\mathrm{e}^\sigma\beta y(x,t), \quad y_x(x,t) = \mathrm{e}^{-\beta\sigma}, \quad \sigma_t(t) = \mathrm{e}^\sigma,$$

we obtain

$$h_t(x,t) = \partial_t((T-t)w(y,\sigma)) = -\beta y\cdot\nabla w(y,\sigma) + w_\sigma(y,\sigma) - w(y,\sigma)$$

and

$$\nabla h(x,t) = \mathrm{e}^{-(\beta+1)\sigma}\nabla w(y,\sigma), \quad \Delta h(x,t) = \mathrm{e}^{-(2\beta+1)\sigma}\Delta w(y,\sigma).$$

Substituting these into (3.2), we have

$$\begin{aligned}
&-\beta y\cdot\nabla w(y,\sigma) + w_\sigma(y,\sigma) - w(y,\sigma)\\
&\qquad= \frac{m\kappa^{m-1}}{(1+w(y,\sigma))^{(m-1)/(p-1)}}\mathrm{e}^{((m-1)/(p-1)-(2\beta+1))\sigma}\\
&\qquad\quad\times\Big(\Delta w(y,\sigma) - \frac{m+p-1}{p-1}\frac{|\nabla w(y,\sigma)|^2}{1+w(y,\sigma)}\Big).
\end{aligned}$$

Therefore, the function w satisfies the rescaled equation

$$w_\sigma = \frac{m\kappa^{m-1}}{(1+w)^{2\beta+1}}\Big(\Delta w - \frac{m+p-1}{p-1}\frac{|\nabla w|^2}{1+w}\Big) + (\beta y\cdot\nabla w + w) \tag{3.3}$$

for $y \in \mathbb{R}^N$ and $s > 0$. We can easily see that

$$\lim_{\sigma\to\infty} \|\mathrm{e}^{-\sigma} w(\cdot,\sigma)\|_{L^\infty(\mathbb{R}^N)} = 0 \quad \text{if and only if} \quad B(u_0) = \mathbb{R}^N. \tag{3.4}$$

The simplest example of a solution of (3.3) is a constant $w \equiv 0$, which corresponds to a flat solution $u(x,t) = U(t)$ of the original problem (1.1). Here $U(t)$ is the solution of (2.1). Another typical example is the self-similar solution. In our case, it has the form $h(x,t) = (T-t)g((T-t)^\beta x)$, where $g = g(y)$ satisfies

$$\Delta g - \frac{m+p-1}{p-1}\frac{|\nabla g|^2}{1+g} + \frac{(1+g)^{2\beta+1}}{m\kappa^{m-1}}(\beta y{\cdot}\nabla g + g) = 0 \tag{3.5}$$

with $y = (T-t)^\beta x$. In other words, a solution h is self-similar if its rescaled function $w(y,\sigma)$ is independent of σ. If we assume that $g(y)$ is a radial function, $g = g(r)$ is the solution of the following ordinary differential equation:

$$g_{rr} + \frac{N-1}{r}g_r - \frac{m+p-1}{p-1}\frac{g_r^2}{1+g} + \frac{(1+g)^{2\beta+1}}{m\kappa^{m-1}}(\beta r g_r + g) = 0, \tag{3.6}$$

$$g(0) = \mu, \quad g_r(0) = 0, \tag{3.7}$$

where $r = |y|$ and $\mu > 0$ is a constant.

Let us note that equation (3.6) has a trivial solution $g \equiv 0$, as well as the spatially homogeneous solution $g \equiv -1$. Let us also note that problem (3.6)–(3.7) admits a solution $g(r)$ with asymptotic behavior:

$$g(r) = \mu - \frac{\mu(1+\mu)^{2\beta+1}}{2m\kappa^{m-1}N}r^2 + o(r^2) \quad \text{as } r \to 0. \tag{3.8}$$

This asymptotics is obtained by solving an approximated ordinary differential equation:

$$g_{rr} + \frac{(1+\mu)^{2\beta+1}}{m\kappa^{m-1}} g \approx 0 \quad \text{for } r \approx 0,$$

which comes from the even symmetric assumption $g_r(0) = 0$ and $g(0) = \mu$.

We must find a value μ with the corresponding solution of the above problem (3.6)–(3.7) that is nonnegative and decreasing at space infinity.

Proposition 3.1. *Let $p > 1$ and $m - p > 2(p-1)/N$. Then problem* (3.6)–(3.7) *has a strictly positive monotone solution satisfying $g(\infty) = 0$ if $\mu > 0$ is sufficiently small.*

If we assume this Proposition, by (3.1), the corresponding solution u of problem (1.1) is written in the form:

$$u_s(x,t) = \varphi((T-t)(1+g((T-t)^\beta x))), \quad \beta > 0.$$

Combining this with $\varphi(0) = \infty$, we obtain $u_s(x,T) = \infty$ for any $x \in \mathbb{R}^N$. Thus $B(u_s(\cdot,0)) = \mathbb{R}^N$. Furthermore, condition (1.3) of the initial value can be easily checked and our result is obtained. Now we shall prove the existence of strictly positive solution $g = g(r)$ for problem (3.6)–(3.7).

Lemma 3.1. *Let $g = g(r)$ be the solution of problem* (3.6)–(3.7). *If $g > 0$ on an interval $[0, R_0)$, then g is strictly decreasing on $[0, R_0)$.*

P r o o f. Define

$$r_0 = \sup\{r > 0\colon\ g \text{ is strictly decreasing on } [0,r]\}$$

and assume $r_0 < R_0$. Then the definition of r_0 implies $g_r(r_0) = 0$ (both $g_r(r_0) > 0$ and $g_r(r_0) < 0$ easily lead to a contradiction) and (3.6) implies $g_{rr}(r_0) < 0$. This in turn means that g is strictly decreasing on a right neighborhood of r_0, a contradiction with the definition of r_0. Hence $r_0 \geqslant R_0$. $\square$

By Lemma 3.1, one can distinguish the following two cases:

(a) $g > 0$ on $[0,\infty)$ and g is strictly decreasing on $[0,\infty)$.

(b) There exists $R \in (0,\infty)$ such that $g > 0$ on $[0,R)$ and $g(R) = 0$. This implies that g is strictly decreasing on $[0,R)$; thus, by continuity, it is strictly decreasing on $[0,R]$. In particular, $g_r(R) < 0$.

Now we exclude the second case (b) using the following lemma.

Lemma 3.2. *Assume that $\beta N > (1+\mu)^{2\beta+1}$. Let $g = g(r)$ be the solution of problem* (3.6)–(3.7). *Then $g > 0$ on $[0,\infty)$.*

P r o o f. The decay rate of the solution is given by the solution of $\beta r\overline{g}_r + \overline{g} = 0$, which is the dominant term of the ODE (3.6). Thus, we introduce a function

$$v := -\frac{\beta r g_r}{g}\colon\ [0,R) \to [0,\infty). \tag{3.9}$$

By the definition of R, the function v is a nonnegative function and is well-defined. Assume that $R < \infty$. Then case (b) of Lemma 3.1 implies that $\lim\limits_{r\to R} v(r) = \infty$.

Differentiating (3.9) and using (3.6), we get

$$\begin{aligned}
v_r &= -\frac{\beta r}{g}\Big(g_{rr} + \frac{1}{r}g_r\Big) + \beta r\Big(\frac{g_r}{g}\Big)^2 \\
&= \beta(N-2)\frac{g_r}{g} + \beta r\Big(\frac{g_r}{g}\Big)^2 - \frac{m+p-1}{p-1}\frac{\beta r g_r^2}{g(1+g)} + \frac{\beta r(1+g)^{2\beta+1}}{m\kappa^{m-1}}(1-v) \\
&= -(N-2)\frac{v}{r} + \frac{v^2}{\beta r} - \frac{m+p-1}{p-1}\frac{g}{1+g}\frac{v^2}{\beta r} + \frac{\beta r(1+g)^{2\beta+1}}{m\kappa^{m-1}}(1-v) \\
&= -(N-2)\frac{v}{r} + \Big(1 - \frac{m+p-1}{p-1}\frac{g}{1+g}\Big)\frac{v^2}{\beta r} + \frac{\beta r(1+g)^{2\beta+1}}{m\kappa^{m-1}}(1-v).
\end{aligned}$$

From (3.8) and (3.9), we see that

$$v(r) = \frac{\beta(1+\mu)^{2\beta+1}}{m\kappa^{m-1}N}r^2 + o(r^2) \quad \text{as } r \to 0.$$

We will use this asymptotics in order to estimate the function v from above. Next we shall check that the function $\overline{v}(r) := \beta(1+\mu)^{2\beta+1}/m\kappa^{m-1}Nr^2$ is a super-solution of the above ODE provided that

$$1 \leqslant \beta N\frac{(1+g)^{2\beta+1}}{(1+\mu)^{2\beta+1}} + \frac{m+p-1}{p-1}\frac{g}{1+g} \tag{3.10}$$

for all $r \in [0, R)$. In fact, under condition (3.10), we get

$$\begin{aligned}
\overline{v}_r &+ (N-2)\frac{\overline{v}}{r} - \Big(1 - \frac{m+p-1}{p-1}\frac{g}{1+g}\Big)\frac{\overline{v}^2}{\beta r} - \frac{\beta r(1+g)^{2\beta+1}}{m\kappa^{m-1}}(1-\overline{v}) \\
&= \frac{N\overline{v}}{r}\Big(1 - \frac{(1+g)^{2\beta+1}}{(1+\mu)^{2\beta+1}}\Big) - \Big(1 - \frac{m+p-1}{p-1}\frac{g}{1+g} - \beta N\frac{(1+g)^{2\beta+1}}{(1+\mu)^{2\beta+1}}\Big)\frac{\overline{v}^2}{\beta r} \\
&\geqslant -\Big(1 - \frac{m+p-1}{p-1}\frac{g}{1+g} - \beta N\frac{(1+g)^{2\beta+1}}{(1+\mu)^{2\beta+1}}\Big)\frac{\overline{v}^2}{\beta r} \geqslant 0.
\end{aligned}$$

Here we used the relations $\overline{v}_r = 2\overline{v}/r$ together with

$$\frac{\beta r(1+g)^{2\beta+1}}{m\kappa^{m-1}} = \frac{N\overline{v}}{r}\frac{(1+g)^{2\beta+1}}{(1+\mu)^{2\beta+1}}$$

and the inequality $g(r) \leqslant \mu$ for $r \in [0, R]$. Condition (3.10) is satisfied because the function g is nonnegative on $[0, R)$ and $\beta N > (1+\mu)^{2\beta+1}$. Therefore, by the comparison argument, $v \leqslant \overline{v}$ for all $r \in [0, R)$ and $\lim_{r\to r_1} v(r) \leqslant \overline{v}(R) < \infty$. This yields a contradiction. □

P r o o f of Proposition 3.1. Let $p > 1$ and $m - p > 2(p-1)/N$, then $\beta N > 1$. By Lemma 3.2, problem (3.6)–(3.7) has a positive solution if we choose $\mu > 0$ sufficiently small such that $\beta N > (1+\mu)^{2\beta+1}$. Lemma 3.1 implies that this solution is strictly decreasing. Furthermore, since there exists no positive spatially homogeneous solution of equation (3.6), we obtain $g(\infty) = 0$. Hence we obtain the result. □

A c k n o w l e d g e m e n t s. Masahiko Shimojō was supported by JSPS KAKENHI Grant-in-Aid for Young Scientists (B) (No. 16K17634). Amy Poh Ai Ling was supported by the Research Grant for Encouragement of Students, Graduate School of Natural Science and Technology, Okayama University. Our gratitude goes to Professor Hiroshi Matano of Tokyo University for his valuable comments. In addition, the authors would like to express thanks to anonymous reviewers for their fruitful suggestions.

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