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NOTE ON α-FILTERS IN DISTRIBUTIVE NEARLATTICES

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Abstract. In this short paper we introduce the notion of α-filter in the class of distributive nearlattices and we prove that the α-filters of a normal distributive nearlattice are strongly connected with the filters of the distributive nearlattice of the annihilators.

Keywords: distributive nearlattice; annihilator; α-filter

MSC 2010: 06A12, 03G10, 06D50

1. Introduction and preliminaries

A nearlattice is a join-semilattice with greatest element in which every principal filter is a bounded lattice. These structures are a natural generalization of the implication algebras studied by Abbott in [1] and the bounded distributive lattices. The nearlattices form a variety and has been studied by Cornish and Hickman in [14] and [16], and by Chajda, Halaš, Kühr and Kolařík in [8], [9], [10] and [11]. A particular class of nearlattices are the distributive nearlattices. In [6] and [7], a full duality is developed for distributive nearlattices and some applications are given, and recently in [15], the author proposes a sentential logic associated with the class of distributive nearlattices.

On the other hand, Cornish in [13] introduced the notion of α-ideal in the class of distributive lattices and characterizes Stone lattices in terms of α-ideals. These results were extended to the Hilbert algebras in [4] and [5]. We can study a dual notion of α-ideal in the class of distributive nearlattices, i.e. the concept of α-filter. The main objective of this paper is to introduce the notion of α-filter in the variety of distributive nearlattices. We see that the α-filters of a normal distributive near-

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lattice $\mathbf{A}$ are strongly connected with the filters of the distributive nearlattice $\mathrm{R}(\mathbf{A})$ of the annihilators. This result extends those obtained by Cornish.

Let $\mathbf{A} = \langle A, \vee, 1\rangle$ be a join-semilattice with greatest element. A filter is a subset F of A such that $1 \in F$, if $a \leqslant b$ and $a \in F$, then $b \in F$ and if $a, b \in F$, then $a \wedge b \in F$ whenever $a \wedge b$ exists. If X is a nonempty subset of A, the smallest filter containing X is called the filter generated by X and will be denoted by $F(X)$. A filter G is said to be finitely generated if $G = F(X)$ for some finite nonempty subset X of A. If $X = \{a\}$, then $F(\{a\}) = [a) = \{x \in A\colon\ a \leqslant x\}$, called the principal filter of a. We denote by $\mathrm{Fi}(A)$ the set of all filters of A. A subset I of A is called an ideal if for every $a, b \in A$, if $a \leqslant b$ and $b \in I$, then $a \in I$ and for all $a, b \in I$, $a \vee b \in I$. We say that a nonempty proper ideal P is prime if for every $a, b \in A$, $a \wedge b \in I$ implies $a \in I$ or $b \in I$ whenever $a \wedge b$ exists. We denote by $\mathrm{Id}(A)$ and $\mathrm{X}(A)$ the set of all ideals and prime ideals of A, respectively. Finally, we say that a nonempty ideal I of A is maximal if it is proper and for every $J \in \mathrm{Id}(A)$, if $I \subseteq J$, then $J = I$ or $J = A$. We denote by $\mathrm{Idm}(A)$ the set of all maximal ideals of A. Note that every maximal ideal is prime.

Definition 1. Let $\mathbf{A}$ be a join-semilattice with greatest element. Then $\mathbf{A}$ is a *nearlattice* if each principal filter is a bounded lattice with respect to the induced order.

Note that the operation meet is defined only in a corresponding principal filter. We indicate this fact by indices, i.e. $\wedge_a$ denotes the meet in $[a)$. Then the operation meet is not defined everywhere. However, the nearlattices can be regarded as total algebras through a ternary operation. This fact was first proved by Hickman in [16] and independently by Chajda and Kolařík in [11]. Araújo and Kinyon in [2] found a smaller equational base.

Theorem 2 ([2]). *Let $\mathbf{A}$ be a nearlattice. Let $m\colon\ A^3 \to A$ be a ternary operation given by $m(x, y, z) = (x \vee z) \wedge_z (y \vee z)$. The following identities are satisfied:*

(1) $m(x, y, x) = x$,

(2) $m(m(x, y, z), m(y, m(u, x, z), z), w) = m(w, w, m(y, m(x, u, z), z))$,

(3) $m(x, x, 1) = 1$.

Conversely, let $\mathbf{A} = \langle A, m, 1\rangle$ be an algebra of type $(3, 0)$ satisfying the identities (1)–(3). If we define $x \vee y = m(x, x, y)$, then $\mathbf{A}$ is a join-semilattice with greatest element. Moreover, for each $a \in A$, $[a)$ is a bounded lattice, where for every $x, y \in [a)$ their infimum is $x \wedge_a y = m(x, y, a)$. Hence, $\mathbf{A}$ is a nearlattice.

Definition 3. Let $\mathbf{A}$ be a nearlattice. Then $\mathbf{A}$ is *distributive* if each principal filter is a bounded distributive lattice with respect to the induced order.

E x a m p l e 4 ([1]). An implication algebra can be defined as a join-semilattice with greatest element such that each principal filter is a Boolean lattice. If $\mathbf{A} = \langle A, \to, 1\rangle$ is an implication algebra, then the join of two elements x and y is given by $x \vee y = (x \to y) \to y$ and for each $a \in A$, $[a) = \{x \in A\colon\ a \leqslant x\}$ is a Boolean lattice, where for $x, y \in [a)$ the meet is given by $x \wedge_a y = (x \to (y \to a)) \to a$ and $x \to a$ is the complement of x in $[a)$. Thus, every implication algebra is a distributive nearlattice.

From the results given in [14], we have the following characterization of the filter generated by a nonempty subset X in a distributive nearlattice $\mathbf{A}$:

$$F(X) = \{a \in A\colon\ \exists x_1, \ldots, x_n \in [X),\ \exists x_1 \wedge \ldots \wedge x_n,\ a = x_1 \wedge \ldots \wedge x_n\}.$$

In [3] it was shown that if $\mathbf{A}$ is a distributive nearlattice, then the set of all filters $\mathrm{Fi}(\mathbf{A}) = \langle \mathrm{Fi}(A), \veebar, \barwedge, \to, \{1\}, A\rangle$ is a Heyting algebra, where the least element is $\{1\}$, the greatest element is A, $G \veebar H = F(G \cup H)$, $G \barwedge H = G \cap H$ and

$$(\star) \qquad G \to H = \{a \in A\colon\ [a) \cap G \subseteq H\}$$

for all $G, H \in \mathrm{Fi}(A)$. So, the pseudocomplement of $F \in \mathrm{Fi}(A)$ is $F^* = F \to \{1\}$.

Theorem 5 ([9])**.** *Let* $\mathbf{A}$ *be a distributive nearlattice. Let* $I \in \mathrm{Id}(A)$ *and let* $F \in \mathrm{Fi}(A)$ *such that* $I \cap F = \emptyset$*. Then there exists* $P \in \mathrm{X}(A)$ *such that* $I \subseteq P$ *and* $P \cap F = \emptyset$*.*

The following definition given in [3] is an alternative definition of relative annihilator in distributive nearlattices different from that given in [10].

Definition 6. Let $\mathbf{A}$ be a join-semilattice with greatest element and $a, b \in A$. The *annihilators of* a *relative to* b is the set

$$a \circ b = \{x \in A\colon\ b \leqslant x \vee a\}.$$

In particular, the relative annihilator $a^{\top} = a \circ 1 = \{x \in A\colon\ x \vee a = 1\}$ is called the *annihilator of* a.

It follows that a nearlattice $\mathbf{A}$ is distributive if and only if $a \circ b \in \mathrm{Fi}(A)$ for all $a, b \in A$. Also note that by $(\star)$, we have that $[a)^* = \{x \in A\colon\ x \vee a = 1\}$, i.e. $[a)^* = a^{\top}$, which is the dual notion of annulet given by Cornish in [13]. The following result will be useful.

Lemma 7 ([3])**.** *Let* $\mathbf{A}$ *be a distributive nearlattice. Let* $a, b \in A$ *and* $I \in \mathrm{Id}(A)$*.*

(1) $I \cap a^{\top} = \emptyset$ *if only if there exists* $U \in \mathrm{Idm}(A)$ *such that* $I \subseteq U$ *and* $a \in U$*.*
(2) $U \in \mathrm{Idm}(A)$ *if only if for every* $a \in A$*,* $a \notin U$ *if only if* $U \cap a^{\top} \neq \emptyset$*.*

We are interested in a particular class of distributive nearlattices which generalize the normal lattices given in [12].

Definition 8. Let $\mathbf{A}$ be a distributive nearlattice. Then $\mathbf{A}$ is *normal* if each prime ideal is contained in a unique maximal ideal.

Theorem 9 ([3]). *Let $\mathbf{A}$ be a distributive nearlattice. The following conditions are equivalent:*

(1) *$\mathbf{A}$ is normal,*
(2) *$(a \vee b)^{\top} = a^{\top} \veebar b^{\top}$ for all $a, b \in A$.*

2. α-filters

In this section we study the notion of α-filter in the class of distributive nearlattices. First, we see some characteristics of annihilators. Let $\mathbf{A}$ be a distributive nearlattice, $a \in A$ and we consider the set

$$a^{\top\top} = \{y \in A\colon \ \forall x \in a^{\top}, y \vee x = 1\}.$$

Lemma 10. *Let $\mathbf{A}$ be a distributive nearlattice. The following properties are satisfied for every $a, b \in A$:*

(1) $[a) \subseteq a^{\top\top}$.
(2) $a^{\top\top\top} = a^{\top}$.
(3) *$a \leqslant b$ implies $a^{\top} \subseteq b^{\top}$.*
(4) *$a^{\top} \subseteq b^{\top}$ if only if $b^{\top\top} \subseteq a^{\top\top}$.*
(5) *$(a \wedge b)^{\top} = a^{\top} \cap b^{\top}$ whenever $a \wedge b$ exists.*
(6) $(a \vee b)^{\top\top} = a^{\top\top} \cap b^{\top\top}$.

Proof. We prove only the assertions (2), (4) and (6).

(2) Let $y \in a^{\top\top\top}$. Thus, for every $x \in a^{\top\top}$ we have $y \vee x = 1$. In particular, $a \in a^{\top\top}$ and $y \vee a = 1$. Therefore $y \in a^{\top}$. The reciprocal is similar.

(4) Suppose that $a^{\top} \subseteq b^{\top}$. Let $y \in b^{\top\top}$. If $x \in a^{\top}$, then $x \in b^{\top}$ and $y \vee x = 1$. So, $y \in a^{\top\top}$ and $b^{\top\top} \subseteq a^{\top\top}$. Conversely, suppose that $b^{\top\top} \subseteq a^{\top\top}$ and let $x \in a^{\top}$. Since $b \in b^{\top\top}$, $b \in a^{\top\top}$ and $b \vee x = 1$. Therefore $x \in b^{\top}$ and $a^{\top} \subseteq b^{\top}$.

(6) Since $a, b \leqslant a \vee b$, we have $(a \vee b)^{\top\top} \subseteq a^{\top\top}, b^{\top\top}$ and $(a \vee b)^{\top\top} \subseteq a^{\top\top} \cap b^{\top\top}$. Let $y \in a^{\top\top} \cap b^{\top\top}$ and suppose that $y \notin (a \vee b)^{\top\top}$. Then there is $x \in (a \vee b)^{\top}$ such that $y \vee x < 1$ and by Theorem 5, there exists $P \in \mathrm{X}(A)$ such that $y \vee x \in P$. So, $x, y \in P$. Since $y \in a^{\top\top} \cap b^{\top\top}$, we have that for every $z \in a^{\top}$, $y \vee z = 1$ and for every $w \in b^{\top}$, $y \vee w = 1$. On the other hand, as $x \in (a \vee b)^{\top}$, it follows that

$a \vee b \vee x = 1$ and $a \vee x \in b^{\top}$. Consequently, $y \vee a \vee x = 1$. We have two cases: if $P \cap a^{\top} \neq \emptyset$, then there is $t \in a^{\top}$ such that $t \in P$. Thus, $y \vee t = 1 \in P$, which is a contradiction. If $P \cap a^{\top} = \emptyset$, then by Lemma 7 there exists $U \in \mathrm{Idm}(A)$ such that $P \subseteq U$ and $a \in U$. So, $x, y, a \in U$ and $y \vee a \vee x = 1 \in U$, which is a contradiction. Therefore, we conclude that $(a \vee b)^{\top\top} = a^{\top\top} \cap b^{\top\top}$. □

If $\mathbf{A}$ is a distributive nearlattice, then an element $a \in A$ is *dense* if $a^{\top} = \{1\}$. We denote by $D(A)$ the set of all dense elements of A. By Lemma 10, it is easy to prove that $D(A) \in \mathrm{Id}(A)$ and $a^{\top\top} \in \mathrm{Fi}(A)$ for all $a \in A$. The following result gives an equivalence of the implication algebras in terms of annihilators.

Theorem 11. *Let* $\mathbf{A}$ *be a distributive nearlattice. The following conditions are equivalent:*

(1) $\mathbf{A}$ *is an implication algebra,*
(2) $[a) \veebar a^{\top} = A$ *for all* $a \in A$.

Proof. (1) $\Rightarrow$ (2): Suppose that $\mathbf{A}$ is an implication algebra. By the results developed in [1], we know that $\mathrm{X}(A) = \mathrm{Idm}(A)$. Let $a \in A$. Obviously $[a) \veebar a^{\top} \subseteq A$. We prove the other inclusion. Let $c \in A$ and suppose that $c \notin [a) \veebar a^{\top}$. So, by Theorem 5 there exists $P \in \mathrm{X}(A)$ such that $c \in P$ and $P \cap ([a) \veebar a^{\top}) = \emptyset$. Then $a \notin P$ and $P \cap a^{\top} = \emptyset$. Thus, P is maximal and by Lemma 7 it follows that $P \cap a^{\top} \neq \emptyset$, which is a contradiction. Therefore $[a) \veebar a^{\top} = A$.

(1) $\Rightarrow$ (2): Let $a \in A$ and $b \in [a)$ such that $b \neq a$ and $b \neq 1$. Let us prove that b has a complement in $[a)$. We know that $a \in [b) \veebar b^{\top} = F([b) \cup b^{\top})$. If only there is $x \in [b)$ such that $a = x$, then $b \leqslant x = a$ and $b = a$, which is a contradiction. On the other hand, if only there is $x \in b^{\top}$ such that $a = x$, then $x \vee b = a \vee b = 1$. Since $a \leqslant b$, it follows that $a \vee b = b$ and $b = 1$, which is a contradiction. Thus, there exists $x \in [b)$ and there exists $y \in b^{\top}$ such that $x \wedge y$ exists and $a = x \wedge y$. Then

$$a = a \wedge b = (x \wedge y) \wedge b = (x \wedge b) \wedge y = b \wedge y,$$

i.e. $a = b \wedge y$. Moreover, $y \in b^{\top}$ and $b \vee y = 1$. As $y \in [a)$, then y is the complement of b in $[a)$ and $\mathbf{A}$ is an implication algebra. □

Let $\mathbf{A}$ be a normal distributive nearlattice and we consider the family

$$\mathrm{R}(A) = \{a^{\top} \colon a \in A\}.$$

Let $\overline{m} \colon \mathrm{R}(A)^3 \to \mathrm{R}(A)$ be a map given by $\overline{m}(a^{\top}, b^{\top}, c^{\top}) = (a^{\top} \veebar c^{\top}) \barwedge (b^{\top} \veebar c^{\top})$. By Theorems 9 and 2 and Lemma 10, it follows that the structure

$$\mathrm{R}(\mathbf{A}) = \langle \mathrm{R}(A), \overline{m}, A \rangle$$

is a distributive nearlattice.

Corollary 12. *Let* $\mathbf{A}$ *be a normal distributive nearlattice. Then the relation* $\theta^\top$ *on* A *defined by*

$$(*) \qquad (a,b) \in \theta^\top \quad \textit{if only if} \quad a^\top = b^\top$$

is a congruence on $\mathbf{A}$.

Corollary 13. *Let* $\mathbf{A}$ *be a normal distributive nearlattice and* $\theta^\top$ *be the congruence given by* $(*)$. *Then* $\mathrm{R}(\mathbf{A})$ *is isomorphic to* $\mathbf{A}/\theta^\top$.

Proof. Let $\varrho\colon A \to \mathrm{R}(A)$ be the map defined by $\varrho(a) = a^\top$. By Theorem 9 and Lemma 10 we have that $\varrho(m(a,b,c)) = \overline{m}(\varrho(a), \varrho(b), \varrho(c))$, where the ternary operation $m(a,b,c)$ is given by Theorem 2. So, ϱ is an homomorphism onto such that $\theta^\top = \mathrm{Ker}(\varrho)$. It follows by Isomorphism Theorem. $\square$

Example 14. Let $\mathbf{A}$ be the normal distributive nearlattice from Figure 1. Then $\mathrm{R}(A) = \{1^\top, a^\top, b^\top, c^\top\}$. On the other hand, the congruence $\theta^\top$ is given by the partition $\{1\}$, $\{b\}$, $\{a,d\}$ and $\{c,e\}$. Hence, $\mathrm{R}(\mathbf{A})$ and $\mathbf{A}/\theta^\top$ are isomorphic.

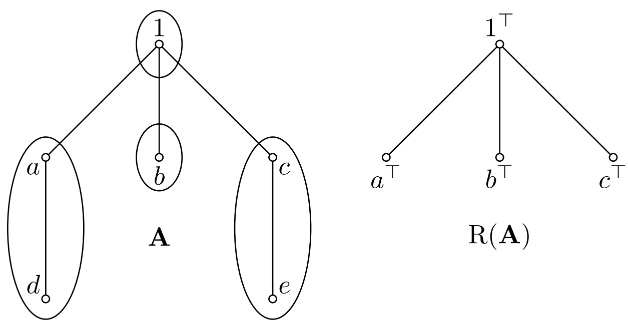

Figure 1.

Definition 15. Let $\mathbf{A}$ be a distributive nearlattice and $F \in \mathrm{Fi}(A)$. We say that F is an *α-filter* if $a^{\top\top} \subseteq F$ for all $a \in F$.

We denote by $\mathrm{Fi}_\alpha(A)$ the set of all α-filters of A.

Example 16. If $\mathbf{A}$ is a normal distributive nearlattice, then $\mathrm{Ker}(\theta^\top)$ is an α-filter.

Example 17. If $\mathbf{A}$ is a distributive nearlattice, then $a^\top$ is an α-filter for all $a \in A$. Let $x \in a^\top$. We prove that $x^{\top\top} \subseteq a^\top$. If $y \in x^{\top\top}$, then $x^\top \subseteq y^\top$ and since $a^\top$ is a filter, we have $x \vee y \in a^\top$ and $x \vee y \vee a = 1$, i.e. $y \vee a \in x^\top$. So, $y \vee a \in y^\top$ and $y \vee a = 1$. It follows that $y \in a^\top$ and $a^\top$ is an α-filter.

R e m a r k 18. Not every filter is an α-filter. In Example 14, we consider the filter $F=\{1,a,b\}$. Thus, $a^{\top\top}=\{1,a,d\}$ and $a^{\top\top}\nsubseteq F$.

Theorem 19. *Let* $\mathbf{A}$ *be a distributive nearlattice and* $F\in \mathrm{Fi}(A)$. *The following conditions are equivalent:*

(1) *F is an α-filter.*
(2) *If $a^{\top}=b^{\top}$ and $a\in F$, then $b\in F$ for all $a,b\in A$.*
(3) *$F=\bigcup\{a^{\top\top}\colon a\in F\}$.*

P r o o f. (1) $\Rightarrow$ (2): Let $a,b\in A$ such that $a^{\top}=b^{\top}$ and $a\in F$. Then $a^{\top\top}=b^{\top\top}$ and since F is an α-filter, $a^{\top\top}\subseteq F$. Then $b\in b^{\top\top}$ and $b^{\top\top}\subseteq F$, i.e. $b\in F$.

(2) $\Rightarrow$ (3): Since $a\in a^{\top\top}$ for all $a\in A$, we have $F\subseteq \bigcup\{a^{\top\top}\colon a\in F\}$. We see the other inclusion. If $x\in\bigcup\{a^{\top\top}\colon a\in F\}$, then there is $b\in F$ such that $x\in b^{\top\top}$. So, $b^{\top}\subseteq x^{\top}$ and $x^{\top\top}\subseteq b^{\top\top}$. Then by Lemma 10, $x^{\top\top}=x^{\top\top}\cap b^{\top\top}=(x\vee b)^{\top\top}$ and $x^{\top}=(x\vee b)^{\top}$. As $x\vee b\in F$, by hypothesis we have $x\in F$.

(3) $\Rightarrow$ (1): Let $b\in F$. If $x\in b^{\top\top}$, then $x\in\bigcup\{a^{\top\top}\colon a\in F\}$ and $x\in F$. Therefore $b^{\top\top}\subseteq F$ and F is an α-filter. $\square$

Theorem 20. *Let* $\mathbf{A}$ *be a normal distributive nearlattice and* $F\in\mathrm{Fi}(A)$. *Then*

$$\alpha(F)=\{x\in A\colon \exists\, a\in F,\ a^{\top}\subseteq x^{\top}\}$$

is the smallest α-filter containing F.

P r o o f. It is clear that $F\subseteq\alpha(F)$. Let $x,y\in A$ such that $x\leqslant y$ and $x\in\alpha(F)$. Then by Lemma 10, $x^{\top}\subseteq y^{\top}$ and there exists $a\in F$ such that $a^{\top}\subseteq x^{\top}$. So, $a^{\top}\subseteq y^{\top}$ and $y\in\alpha(F)$. Let $x,y\in\alpha(F)$ and suppose that $x\wedge y$ exists. Then there exist $a,b\in F$ such that $a^{\top}\subseteq x^{\top}$ and $b^{\top}\subseteq y^{\top}$. Since F is a filter, $m(a,b,x\wedge y)\in F$, where the ternary operation $m(a,b,x\wedge y)$ is given by Theorem 2. On the other hand, $a^{\top}\veebar(x\wedge y)^{\top}\subseteq x^{\top}$ and $b^{\top}\veebar(x\wedge y)^{\top}\subseteq y^{\top}$. As $\mathbf{A}$ is normal,

$$m(a,b,x\wedge y)^{\top}=\overline{m}(a^{\top},b^{\top},(x\wedge y)^{\top})\subseteq x^{\top}\barwedge y^{\top}=(x\wedge y)^{\top}.$$

Thus, $m(a,b,x\wedge y)^{\top}\subseteq(x\wedge y)^{\top}$ and $x\wedge y\in\alpha(F)$. Then $\alpha(F)$ is a filter. Let $x\in\alpha(F)$. We see that $x^{\top\top}\subseteq\alpha(F)$. If $y\in x^{\top\top}$, then $x^{\top}\subseteq y^{\top}$. Since $x\in\alpha(F)$, there exists $a\in F$ such that $a^{\top}\subseteq x^{\top}$. So, $a^{\top}\subseteq y^{\top}$ and $y\in\alpha(F)$. Then $x^{\top\top}\subseteq\alpha(F)$ and $\alpha(F)$ is an α-filter. Let $H\in\mathrm{Fi}_{\alpha}(A)$ such that $F\subseteq H$. If $x\in\alpha(F)$, then there exists $a\in F$ such that $a^{\top}\subseteq x^{\top}$, i.e. $x^{\top\top}\subseteq a^{\top\top}$. As $a\in H$ and H is an α-filter, we have $a^{\top\top}\subseteq H$. Consequently, $x\in H$ and $\alpha(F)\subseteq H$. $\square$

R e m a r k 21. Let $\mathbf{A}$ be a normal distributive nearlattice.

(1) Note that the map $\alpha\colon \mathrm{Fi}(A) \to \mathrm{Fi}(A)$ of Theorem 20 is a closure operator and the α-filters are closed elements with respect to α.

(2) A proper α-filter contains non-dense elements. Indeed, if F is a proper α-filter and $x \in F \cap D(A)$, then $F = \alpha(F)$ and $x^{\top} = \{1\}$. Thus, there exists $a \in F$ such that $a^{\top} \subseteq x^{\top}$. So, $a^{\top} = \{1\}$ and $a^{\top\top} = A$. On the other hand, since F is an α-filter, $a^{\top\top} \subseteq F$, i.e. $A = F$ which is a contradiction.

Now, we define the operations of infimum $\overline{\sqcap}$, supremum $\underline{\sqcup}$, and implication $\Rightarrow$ in $\mathrm{Fi}_{\alpha}(A)$ as:

$$F \,\overline{\sqcap}\, G = F \cap G, \quad F \,\underline{\sqcup}\, G = \alpha(F \veebar G), \quad F \Rightarrow G = \alpha(F \to G)$$

for each pair $F, G \in \mathrm{Fi}_{\alpha}(A)$. By Theorem 20, we have that $F \,\overline{\sqcap}\, G$, $F \,\underline{\sqcup}\, G$, $F \Rightarrow G \in \mathrm{Fi}_{\alpha}(A)$ for all $F, G \in \mathrm{Fi}_{\alpha}(A)$. Consider the structure

$$\mathrm{Fi}_{\alpha}(\mathbf{A}) = \langle \mathrm{Fi}_{\alpha}(A), \underline{\sqcup}, \overline{\sqcap}, \Rightarrow, \{1\}, A \rangle.$$

Theorem 22. *Let* $\mathbf{A}$ *be a normal distributive nearlattice. Then* $\mathrm{Fi}_{\alpha}(\mathbf{A})$ *is a Heyting algebra.*

P r o o f. It is easy to verify that $\langle \mathrm{Fi}_{\alpha}(A), \underline{\sqcup}, \overline{\sqcap}, \{1\}, A \rangle$ is a bounded lattice. Let $F, H, K \in \mathrm{Fi}_{\alpha}(A)$. Suppose that $F \,\overline{\sqcap}\, H \subseteq K$. If $x \in F$, then $[x) \cap H \subseteq F \,\overline{\sqcap}\, H \subseteq K$. Thus, $[x) \cap H \subseteq K$ and $x \in H \to K$. Hence, $x \in H \Rightarrow K$ and $F \subseteq H \Rightarrow K$.

Reciprocally, we assume that $F \subseteq H \Rightarrow K$. Let $x \in F \,\overline{\sqcap}\, H$. So, $x \in F \subseteq H \Rightarrow K$ and there exists $a \in H \to K$ such that $a^{\top} \subseteq x^{\top}$. It follows that $x \vee a \in [a) \cap H \subseteq K$ and $x^{\top} = x^{\top} \veebar a^{\top} = (x \vee a)^{\top}$, i.e. $x^{\top} = (x \vee a)^{\top}$ and $x \vee a \in K$. By Theorem 19, we have $x \in K$. Therefore, $F \,\overline{\sqcap}\, H \subseteq K$ and $\mathrm{Fi}_{\alpha}(\mathbf{A})$ is a Heyting algebra. $\square$

Let $\mathbf{A}$ be a nearlattice. Following the results developed in [15], we introduce the next notation. For each natural number n we define inductively for every $a_1, \ldots, a_n, b \in A$, the element $m^{n-1}(a_1, \ldots, a_n, b)$ as follows:

(1) $m^0(a_1, b) = m(a_1, a_1, b)$,

(2) for $n > 1$, $m^{n-1}(a_1, \ldots, a_n, b) = m(m^{n-2}(a_1, \ldots, a_{n-1}, b), a_n, b)$.

Then $m^{n-1}(a_1, \ldots, a_n, b) = (a_1 \vee b) \wedge_b \ldots \wedge_b (a_n \vee b)$ and in particular, $m^0(a_1, b) = a_1 \vee b$ and $m^1(a_1, a_2, b) = m(a_1, a_2, b)$, where the operation $m(a_1, a_2, b)$ is given by Theorem 2. We are able to formulate our main result.

Theorem 23. *Let* $\mathbf{A}$ *be a normal distributive nearlattice. Then* $\mathrm{Fi}_{\alpha}(\mathbf{A})$ *is isomorphic to the Heyting algebra* $\mathrm{Fi}(\mathrm{R}(\mathbf{A}))$.

P r o o f. We consider the map $\psi\colon \mathrm{Fi}_\alpha(A) \to \mathrm{Fi}(\mathrm{R}(A))$ defined by

$$\psi(F) = \{a^\top\colon\ a \in F\}.$$

We prove that ψ is well-defined. Let $F \in \mathrm{Fi}_\alpha(A)$. It is clear that $1^\top \in \psi(F)$. Let $a^\top, b^\top \in \mathrm{R}(A)$ such that $a^\top \subseteq b^\top$ and $a^\top \in \psi(F)$. Then $b^{\top\top} \subseteq a^{\top\top}$ and $a \in F$. Thus, $b \in a^{\top\top}$ and as F is an α-filter, $a^{\top\top} \subseteq F$. So, $b \in F$ and $b^\top \in \psi(F)$. Let $a^\top, b^\top \in \psi(F)$ and suppose that $a^\top \barwedge b^\top$ exists in $\mathrm{R}(A)$, i.e. there is $c \in A$ such that $a^\top \barwedge b^\top = c^\top$. Then $a, b \in F$ and as F is a filter, $m(a,b,c) \in F$. It follows that

$$m(a,b,c)^\top = \overline{m}(a^\top, b^\top, c^\top) = (a^\top \barwedge b^\top) \veebar c^\top = c^\top$$

and $c^\top \in \psi(F)$. Thus, $a^\top \barwedge b^\top \in \psi(F)$ and $\psi(F) \in \mathrm{Fi}(\mathrm{R}(A))$.

Let $F, H \in \mathrm{Fi}_\alpha(A)$. It is immediate that $\psi(F \,\overline{\sqcap}\, H) = \psi(F) \barwedge \psi(H)$. We see that $\psi(F \,\underline{\sqcup}\, H) = \psi(F) \veebar \psi(H)$. Let $x^\top \in \psi(F \,\underline{\sqcup}\, H)$. Then $x \in \alpha(F \veebar H)$ and there exists $a \in F \veebar H$ such that $a^\top \subseteq x^\top$. So, there exist $x_1, \ldots, x_n \in F \cup H$ such that $x_1 \wedge \ldots \wedge x_n$ exists and $a = x_1 \wedge \ldots \wedge x_n$. Then $x_1^\top, \ldots, x_n^\top \in \psi(F) \cup \psi(H)$. On the other hand, $a^\top = (x_1 \wedge \ldots \wedge x_n)^\top = x_1^\top \barwedge \ldots \barwedge x_n^\top$ and $a^\top \in \psi(F) \veebar \psi(H)$. Since $\psi(F) \veebar \psi(H)$ is a filter, we have $x^\top \in \psi(F) \veebar \psi(H)$ and $\psi(F \,\underline{\sqcup}\, H) \subseteq \psi(F) \veebar \psi(H)$. Conversely, if $x^\top \in \psi(F) \veebar \psi(H)$, then there exist $x_1^\top, \ldots, x_n^\top \in \psi(F) \cup \psi(H)$ such that $x_1^\top \barwedge \ldots \barwedge x_n^\top$ exists and $x^\top = x_1^\top \barwedge \ldots \barwedge x_n^\top$. It follows that $x_1, \ldots, x_n \in F \cup H$ and $m^{n-1}(x_1, \ldots, x_n, x) \in F \veebar H$. So,

$$m^{n-1}(x_1, \ldots, x_n, x)^\top = \overline{m}^{n-1}(x_1^\top, \ldots, x_n^\top, x^\top) = (x_1^\top \barwedge \ldots \barwedge x_n^\top) \veebar x^\top = x^\top$$

and $m^{n-1}(x_1, \ldots, x_n, x)^\top \subseteq x^\top$. Thus, $x \in \alpha(F \veebar H) = F \,\underline{\sqcup}\, H$, i.e. $x^\top \in \psi(F \,\underline{\sqcup}\, H)$ and $\psi(F) \veebar \psi(H) \subseteq \psi(F \,\underline{\sqcup}\, H)$. Therefore, $\psi(F \,\underline{\sqcup}\, H) = \psi(F) \veebar \psi(H)$.

Now, we prove that $\psi(F \Rightarrow H) = \psi(F) \to \psi(H)$. Let $x^\top \in \psi(F \Rightarrow H)$. Then $x \in F \Rightarrow H = \alpha(F \to H)$ and there exists $a \in F \to H$ such that $a^\top \subseteq x^\top$. So, $[a) \cap F \subseteq H$. We see that $x^\top \in \psi(F) \to \psi(H)$, i.e. $[x^\top) \cap \psi(F) \subseteq \psi(H)$. If $y^\top \in [x^\top) \cap \psi(F)$, then $x^\top \subseteq y^\top$ and $y \in F$. Thus, $a \vee y \in [a) \cap F$ and $a \vee y \in H$. On the other hand, since $a^\top \subseteq y^\top$, we have $y^\top = (a \vee y)^\top$. As $a \vee y \in H$ and H is an α-filter, by Theorem 19, $y \in H$. Then $y^\top \in \psi(H)$ and $x^\top \in \psi(F) \to \psi(H)$. So, $\psi(F \Rightarrow H) \subseteq \psi(F) \to \psi(H)$. We prove the other inclusion. Let $x^\top \in \psi(F) \to \psi(H)$, i.e. $[x^\top) \cap \psi(F) \subseteq \psi(H)$. Then $x^\top \in \psi(F \Rightarrow H)$ if and only if $x \in \alpha(F \to H)$ if and only if there exists $a \in F \to H$ such that $a^\top \subseteq x^\top$. We see that $x \in F \to H$. If $y \in [x) \cap F$, then by Lemma 10, $x^\top \subseteq y^\top$ and $y \in F$, i.e. $y^\top \in [x^\top) \cap \psi(F)$. Since $[x^\top) \cap \psi(F) \subseteq \psi(H)$, we have $y^\top \in \psi(H)$ and $y \in H$. Then $[x) \cap F \subseteq H$ and $x \in F \to H$. Thus, $x^\top \in \psi(F \Rightarrow H)$ and $\psi(F \Rightarrow H) = \psi(F) \to \psi(H)$.

Let $\pi\colon \mathrm{Fi}(\mathrm{R}(A)) \to \mathrm{Fi}_\alpha(A)$ be the map given by $\pi(G) = \{a\colon\ a^\top \in G\}$. By Lemma 10, it follows that $\pi(G) \in \mathrm{Fi}_\alpha(A)$. So, ψ and π are the inverses of each other and ψ is 1-1 and onto. Therefore ψ is an isomorphism. $\square$

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