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# LOWER BOUNDS FOR INTEGRAL FUNCTIONALS GENERATED BY BIPARTITE GRAPHS

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*Abstract.* We study lower estimates for integral fuctionals for which the structure of the integrand is defined by a graph, in particular, by a bipartite graph. Functionals of such kind appear in statistical mechanics and quantum chemistry in the context of Mayer's transformation and Mayer's cluster integrals. Integral functionals generated by graphs play an important role in the theory of graph limits. Specific kind of functionals generated by bipartite graphs are at the center of the famous and much studied Sidorenko's conjecture, where a certain lower bound is conjectured to hold for every bipartite graph. In the present paper we work with functionals more general and lower bounds significantly sharper than those in Sidorenko's conjecture. In his 1991 seminal paper, Sidorenko proved such sharper bounds for several classes of bipartite graphs. To obtain his result he used a certain way of "gluing" graphs. We prove his inequality for a new class of bipartite graphs by defining a different type of gluing.

*Keywords*: integral inequality; bipartite graph; graph homomorphism; Sidorenko's conjecture

*MSC 2010*: 05C35, 26D15

## 1. Introduction

Let $G$ be a bipartite graph, $G = (V_1(G), V_2(G), E(G))$, where $V_1(G)$, $V_2(G)$ is the bipartition of the vertex set, $V_1(G) \neq \emptyset$, $V_2(G) \neq \emptyset$, and $E(G)$ is the edge set of $G$. We denote $v_1(G) = |V_1(G)|$, $v_2(G) = |V_2(G)|$, $e(G) = |E(G)|$. We denote the sets of vertices of $G$ by $V_1(G) = \{u_1, u_2, \ldots, u_m\}$, $V_2(G) = \{w_1, w_2, \ldots, w_n\}$. If $\{u_l, w_j\} \in E(G)$, we write $u_l w_j \in E(G)$. Further, for any positive integer $k$ we adopt the common notation $[k] = \{1, \ldots, k\}$.

We shall work in the Lebesgue measure space $([0,1], \mathcal{L}, \mu)$ on $[0,1]$ and its powers. Throughout this paper we understand the product of measure spaces to be the completion of the tensor product (see [2], [14]). Thus, $([0,1]^p, \mathcal{L}^p, \mu^p)\overline{\otimes}([0,1]^q, \mathcal{L}^q, \mu^q) =$

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$([0,1]^{p+q}, \mathcal{L}^{p+q}, \mu^{p+q})$. Denote by $K([0,1]^2)$, $K([0,1])$ the sets of measurable, bounded, nonnegative, real-valued functions on $[0,1]^2$ and $[0,1]$, respectively. A given bipartite graph $G$ in a natural way gives rise to two integral functionals:

$$\int_{[0,1]^{m+n}} \prod_{u_l w_j \in E(G)} h(x_l, y_j) \,\mathrm{d}\mu^{m+n} \tag{1.1}$$

and

$$\int_{[0,1]^{m+n}} \prod_{u_l w_j \in E(G)} h(x_l, y_j) \prod_{l=1}^{m} f_l(x_l) \prod_{j=1}^{n} g_j(y_j) \,\mathrm{d}\mu^{m+n}, \tag{1.2}$$

where $h \in K([0,1]^2)$, $f_l, g_j \in K([0,1])$ for $l \in [m]$, $j \in [n]$. Note that we distinguish between graph vertices $u_l, w_j$ and the corresponding variables of integration $x_l$, $y_j$, $l \in [m]$, $j \in [n]$.

Throughout this paper we shall call functions $f_1, \ldots, f_m$, $g_1, \ldots, g_n$ vertex functions and terms $f_1(x_1), \ldots, f_m(x_m)$, $g_1(y_1), \ldots, g_n(y_n)$ vertex terms. We shall call $h(x_l, y_j)$ edge terms.

Integrals of the form (1.1) and (1.2) appear in combinatorics, among others in the context of graph limits. If $h$ takes values in $[0,1]$, that is, $h$ is a so-called graphon, integrals of the form (1.1) can be interpreted in terms of limiting homomorphism densities.

Intergrals (1.1) and (1.2) appear in statistical mechanics and quantum chemistry in the context of Mayer's theory of cluster expansions (see [1], [7], [10], [13]).

In this paper we study integral inequalities involving (1.1) and (1.2) and the associated bipartite graphs. In particular, we focus on bipartite graphs satisfying Sidorenko's $\mathcal{F}$-condition; that is, graphs $G$ that belong to the class $\mathcal{F}$ defined below. The class $\mathcal{F}$ was introduced by Sidorenko in [15]. The definition of $\mathcal{F}$ as well as propositions and theorems that follow involve multiple integration with respect to variables corresponding to the vertices in each of the two parts of a bipartite graph $G = (\{u_1, \ldots, u_m\}, \{w_1, \ldots, w_n\}, E(G))$. In order to make the presentation clearer, we shall use two copies of $([0,1], \mathcal{L}, \mu)$, denoted by $\Omega = ([0,1], \mathcal{L}, \mu_x)$ and $\Lambda = ([0,1], \mathcal{L}, \mu_y)$, where $\Omega$ is associated with vertex functions and variables corresponding to the part $\{u_1, \ldots, u_m\}$, $\Lambda$ is associated with $\{w_1, \ldots, w_n\}$. As $\Omega$ and $\Lambda$ are copies of $([0,1], \mathcal{L}, \mu)$, $K(\Omega) = K([0,1]) = K(\Lambda)$.

Denote by $\mathcal{F}$ the class of bipartite graphs $G = (\{u_1, \ldots, u_m\}, \{w_1, \ldots, w_n\}, E(G))$ which satisfy the following conditions (A) and (B):

(A) $e(G) \geqslant m$, $e(G) \geqslant n$.

(B) For any $h \in K([0,1]^2)$ and any functions $f, f_1, \dots, f_m \in K(\Omega)$, $g, g_1, \dots, g_n \in K(\Lambda)$ we have

$$(1.3)\quad \Big(\int \prod_{u_l w_j \in E(G)} h(x_l, y_j) \prod_{l=1}^{m} f_l(x_l) \prod_{j=1}^{n} g_j(y_j)\, \mathrm{d}\mu_x^m\, \mathrm{d}\mu_y^n\Big)\Big(\int f(x)\,\mathrm{d}\mu_x\Big)^{e(G)-m} \times \Big(\int g(y)\,\mathrm{d}\mu_y\Big)^{e(G)-n} \geqslant \Big(\int h(x,y)\Big(f(x)^{e(G)-m} g(y)^{e(G)-n} \prod_{l=1}^{m} f_l(x) \prod_{j=1}^{n} g_j(y)\Big)^{1/e(G)} \mathrm{d}\mu_x\,\mathrm{d}\mu_y\Big)^{e(G)}.$$

Integrals in (1.3) are over $[0,1]^{m+n} = \Omega^m \overline{\otimes} \Lambda^n$, $\Omega$, $\Lambda$, $\Omega \overline{\otimes} \Lambda = [0,1]^2$, respectively.

If $G \in \mathcal{F}$, we say $G$ satisfies Sidorenko's $\mathcal{F}$-condition. We shall focus on the class $\mathcal{F}$ but for convenience, we define two larger classes $\mathcal{F}_1$ and $\mathcal{F}_2$. The class $\mathcal{F}_1$ is the class of all bipartite graphs $G = (\{u_1, \dots, u_m\}, \{w_1, \dots, w_n\}, E(G))$ that satisfy the following condition:

(B1) For every $h \in K([0,1]^2)$:

$$(1.4)\quad \int_{[0,1]^{m+n}} \prod_{u_l w_j \in E(G)} h(x_l, y_j)\, \mathrm{d}\mu^{m+n} \geqslant \Big(\int_{[0,1]^2} h(x,y)\,\mathrm{d}\mu^2\Big)^{e(G)}.$$

The class $\mathcal{F}_2$ is the class of all bipartite graphs $G$ for which the following condition holds:

(B2) Inequality (1.4) is satisfied for every symmetric function $h \in K([0,1]^2)$.

In [15], Sidorenko conjectured that every bipartite graph $G$ belongs to $\mathcal{F}_1$. In the current literature the term "Sidorenko's conjecture" is understood to mean that every bipartite graph $G$ belongs to $\mathcal{F}_2$. The conjecture has been proved to hold in many special cases (see [3], [4], [5], [6], [8], [9], [11], [15], [16], [17]) but it remains open in general.

The general case notwithstanding, whenever one can prove that a given bipartite graph $G$ belongs to $\mathcal{F}$ or to $\mathcal{F}_1$ or to $\mathcal{F}_2$, one obtains a lower bound for the corresponding integral functional generated by $G$.

The relationship between condition (B1) and condition (B2) is obvious. The relationship between condition (B) and condition (B1) seems obvious: if we take all functions $f, f_1, \dots f_m, g, g_1, \dots, g_n$ in (B) to be constantly 1, inequality (1.3) becomes inequality (1.4). It may appear that condition (B) is stronger than condition (B1) only because it allows for vertex functions $f_l$, $g_j$. It is not so. It is important to realize (see Proposition 1.1 below) that for a bipartite graph $G$ satisfying (A), condition (B) of Sidorenko's $\mathcal{F}$-condition is stronger than conditions (B1) and (B2) not only because it allows for vertex functions $f_l$, $g_j$ but also because it provides a sharper lower bound for

$$\int \prod_{u_l w_j \in E(G)} h(x_l, y_j)\,\mathrm{d}\mu^{m+n}$$

even if vertex functions are absent. In fact, (B) gives a lower bound that is always strictly greater than the bound

$$\left(\int h(x,y)\,\mathrm{d}\mu^2\right)^{e(G)}$$

in (B1) and (B2), except for the trivial cases. To prove that, we use Proposition 1.1.

**Proposition 1.1.** *Let* $G = (\{u_1,\dots,u_m\},\{w_1,\dots,w_n\},E(G))$ *be a bipartite graph that satisfies* (A). *Then* (B) *is equivalent to each of conditions* (C1) *and* (C2)*:*

(C1) *For any* $h \in K([0,1]^2)$ *and any functions* $f_1,\dots,f_m \in K(\Omega)$, $g, g_1,\dots,g_n \in K(\Lambda)$,

$$\left(\int \prod_{u_l w_j \in E(G)} h(x_l,y_j) \prod_{l=1}^{m} f_l(x_l) \prod_{j=1}^{n} g_j(y_j)\,\mathrm{d}\mu_x^m\,\mathrm{d}\mu_y^n\right)\left(\int g(y)\,\mathrm{d}\mu_y\right)^{e(G)-n} \tag{1.5}$$
$$\geqslant \left(\int\left(\int h(x,y)\left(g(y)^{e(G)-n}\prod_{j=1}^{n} g_j(y)\right)^{1/e(G)}\mathrm{d}\mu_y\right)^{e(G)/m} \times \left(\prod_{l=1}^{m} f_l(x)\right)^{1/m}\mathrm{d}\mu_x\right)^{m}.$$

(C2) *For any* $h \in K([0,1]^2)$ *and any functions* $f, f_1,\dots,f_m \in K(\Omega)$, $g_1,\dots,g_n \in K(\Lambda)$,

$$\left(\int \prod_{u_l w_j \in E(G)} h(x_l,y_j) \prod_{l=1}^{m} f_l(x_l) \prod_{j=1}^{n} g_j(y_j)\,\mathrm{d}\mu_x^m\,\mathrm{d}\mu_y^n\right)\left(\int f(x)\,\mathrm{d}\mu_x\right)^{e(G)-m} \tag{1.6}$$
$$\geqslant \left(\int\left(\int h(x,y)\left(f(x)^{e(G)-m}\prod_{l=1}^{m} f_l(x)\right)^{1/e(G)}\mathrm{d}\mu_x\right)^{e(G)/n} \times \left(\prod_{j=1}^{n} g_j(y)\right)^{1/n}\mathrm{d}\mu_y\right)^{n}.$$

We shall prove Proposition 1.1 in Section 5. Here we want to demonstrate that conditions (C1) and (C2) which hold for any bipartite graph $G \in \mathcal{F}$ give two lower bounds for

$$\int \prod_{u_l w_j \in E(G)} h(x_l, y_j)\,\mathrm{d}\mu^{m+n}$$

and each of the bounds is essentially sharper than

$$\left(\int h(x,y)\,\mathrm{d}\mu^2\right)^{e(G)}.$$

Let $G \in \mathcal{F}$, $G = (\{u_1,\dots,u_m\},\{w_1,\dots,w_n\},E(G))$. Let $h \in K([0,1]^2)$ be given. Then $G$ satisfies (C1) and (C2). Choosing $f, f_1,\dots,f_m, g, g_1,\dots,g_n$ in (C1) and (C2) constantly equal to 1 we obtain that $G$ satisfies the following two inequalities:

$$\int \prod_{u_l w_j \in E(G)} h(x_l,y_j)\,\mathrm{d}\mu^{m+n} \geqslant \left(\int\left(\int h(x,y)\,\mathrm{d}\mu_y\right)^{e(G)/m}\mathrm{d}\mu_x\right)^m \tag{1.7}$$

and

$$\int \prod_{u_l w_j \in E(G)} h(x_l,y_j)\,\mathrm{d}\mu^{m+n} \geqslant \left(\int\left(\int h(x,y)\,\mathrm{d}\mu_x\right)^{e(G)/n}\mathrm{d}\mu_y\right)^n. \tag{1.8}$$

Since $e(G) \geqslant m$, $e(G) \geqslant n$, Jensen's inequality and (1.7) gives

$$\left(\int\left(\int h(x,y)\,\mathrm{d}\mu_y\right)^{e(G)/m}\mathrm{d}\mu_x\right)^m \geqslant \left(\int h(x,y)\,\mathrm{d}\mu^2\right)^{e(G)}, \tag{1.9}$$

and if $e(G) > m$, equality in the latter inequality holds if and only if the function

$$\varphi(x) = \int h(x,y)\,\mathrm{d}\mu_y$$

is constant a.e. in $[0,1]$. Similarly, if $e(G) > n$, we have

$$\left(\int\left(\int h(x,y)\,\mathrm{d}\mu_x\right)^{e(G)/n}\mathrm{d}\mu_y\right)^n > \left(\int h(x,y)\,\mathrm{d}\mu^2\right)^{e(G)}$$

unless

$$\psi(y) = \int h(x,y)\,\mathrm{d}\mu_x$$

is constant a.e. in $[0,1]$.

The lower bounds for integral functionals corresponding to bipartite graphs $G$ in $\mathcal{F}$, $\mathcal{F}_1$, or $\mathcal{F}_2$ can be used to obtain lower estimates for Mayer's integrals and in the variety of combinatorial contexts. In [15], Sidorenko shows how to relate condition (B) for graphs $G \in \mathcal{F}$ to Mayer's integrals. He derives a lower bound for the chromatic polynomial for $G \in \mathcal{F}$ and gives a lower bound on the number of colorings for $G \in \mathcal{F}$ with a given number of colors.

Sidorenko's conjecture, namely, that $G \in \mathcal{F}_2$ for every bipartite graph $G$, has a particularly attractive connection to homomorphism densities and it has an equivalent combinatorial formulation. Let $G = (V(G), E(G))$, $H = (V(H), E(H))$ be arbitrary graphs with sets of vertices $V(G)$ and $V(H)$, and edge sets $E(G)$ and $E(H)$, respectively. A mapping $\varphi\colon V(G) \to V(H)$ is a homomorphism if $\{u, v\} \in E(G)$ implies $\{\varphi(u), \varphi(v)\} \in E(H)$. Let $\hom(G, H)$ be the number of homomorphisms from $G$ to $H$. Then "homomorphism density" defined as

$$t(G, H) = \frac{\hom(G, H)}{v(H)^{v(G)}}$$

represents the probability that a randomly chosen mapping $\varphi\colon V(G) \to V(H)$ is a homomorphism. Sidorenko's conjecture is known to be equivalent (see [12]) to the following statement. Let a bipartite graph $G$ be given. Then for every graph $H$,

$$t(G, H) \geqslant t(K_2, H)^{e(G)}, \tag{1.10}$$

where $K_2$ is the graph consisting of a single edge.

It is worth mentioning that even if the goal is to prove $G \in \mathcal{F}_2$ for a given graph $G$, auxiliary graphs that belong to $\mathcal{F}$ might be crucial. Sidorenko's proof that $G \in \mathcal{F}$, or even that $G \in \mathcal{F}_2$, for a bipartite graph $G$ with one side of cardinality not greater than 3 would not go through without auxiliary graphs that belong to $\mathcal{F}$ and not just to $\mathcal{F}_2$ or $\mathcal{F}_1$.

In [15], Sidorenko proved that $G \in \mathcal{F}$ if $G$ is a tree, a complete bipartite graph, or an even cycle. He proved that $G \in \mathcal{F}$ if $G$ satisfies (A) and $v_1(G) \leqslant 3$ or $v_2(G) \leqslant 3$. He stated but did not present a proof of the same result with 3 replaced by 4; that is, for a bipartite graph with one of the parts containing not more than 4 vertices.

In his proofs, Sidorenko used a "gluing" technique, a "vertex gluing", to obtain new graphs in $\mathcal{F}$ by gluing copies of graphs that are known to be in $\mathcal{F}$. In this paper, we introduce a different gluing scheme, gluing along a matching. We use that gluing scheme which again allows us to prove that $G \in \mathcal{F}$ if $G$ can be obtained from elements of $\mathcal{F}$ by gluing along a matching.

The paper is organized as follows. In Section 2, we define gluing along a matching and state our main theorem, Theorem 2.1. In Section 3, we give examples of graphs that can be proved to belong to $\mathcal{F}$ via gluing along a matching. The proof of Theorem 2.1 is given in Section 4. Section 5 contains a proof of Proposition 1.1 and final remarks.

## 2. Gluing copies of a bipartite graph along a matching and the main theorem

We begin by defining gluing along matchings for bipartite graphs. Let $m$, $n$, $s$ be integers such that $s \geqslant 1$, $s \leqslant m$, $s \leqslant n$. Let $G_1 = (\{u_1^1, \ldots, u_s^1, \ldots, u_m^1\}, \{w_1^1, \ldots, w_s^1, \ldots, w_n^1\}, E(G_1))$ be a bipartite graph such that

$$u_j^1 w_j^1 \in E(G_1) \text{ for } j \in [s], \quad u_l^1 w_j^1 \notin E(G_1) \text{ if } l, j \in [s],\ l \neq j. \tag{2.1}$$

That is, the induced subgraph of $G_1$ on the set of vertices $\{u_1^1, \ldots, u_s^1\} \cup \{w_1^1, \ldots, w_s^1\}$ is a matching with edges $\{u_1^1 w_1^1, \ldots, u_s^1 w_s^1\}$. Let $k \geqslant 2$ be an integer. Consider $k$ labeled copies of $G_1$:

$$G_1, G_2, \ldots, G_k,$$

the graph $G_1$ itself being the first of them, such that for each $i = 1, \ldots, k$:

$$G_i = (\{u_1^i, \ldots, u_s^i, \ldots u_m^i\}, \{w_1^i, \ldots, w_s^i, \ldots, w_n^i\}, E(G_i)),$$

where

$$u_l^i w_j^i \in E(G_i) \quad \text{if and only if} \quad u_l^1 w_j^1 \in E(G_1),\ l \in [m],\ j \in [n].$$

In each copy $G_i$, $i = 1, \ldots, k$, we identify vertices $u_1^i, \ldots, u_s^i$ with $u_1^1, \ldots, u_s^1$, respectively, and we identify vertices $w_1^i, \ldots, w_s^i$ with $w_1^1$, …, $w_s^1$, respectively. Because of its special role in this construction, we will call $G_1$ the base graph. For simplicity of presentation, after identifying "gluing" the vertices, we relabel $u_1^1, \ldots, u_s^1$, $w_1^1, \ldots, w_s^1$ as $u_1, \ldots, u_s$, $w_1, \ldots, w_s$. Hence, for $i = 1, \ldots, k$

$$\begin{aligned} V_1(G_i) &= \{u_1, \ldots, u_s, u_{s+1}^i, \ldots, u_m^i\}, \\ V_2(G_i) &= \{w_1, \ldots, w_s, w_{s+1}^i, \ldots, w_m^i\}, \\ E(G_i) &= \{u_p w_p \colon p \in [s]\} \cup \{u_p w_j^i \colon p \in [s], j \in \{s+1, \ldots, n\}, u_p w_j^1 \in E(G_1)\} \\ &\quad \cup \{u_l^i w_p \colon p \in [s], l \in \{s+1, \ldots, m\}, u_l^1 w_p \in E(G_1)\} \\ &\quad \cup \{u_l^i w_j^i \colon l \in \{s+1, \ldots, m\}, j \in \{s+1, \ldots, n\}, u_l^1 w_j^1 \in E(G_1)\}. \end{aligned} \tag{2.2}$$

Unless $s = m = n$, $G_i$, $i \in [k]$, are distinct labeled graphs isomorphic to $G_1$. The "glued" copies $G_1, G_2, \ldots, G_k$ define a new graph $G'$:

$$G' = (V_1(G'), V_2(G'), E(G')),$$

where

$$V_1(G') = \{u_1, \dots, u_s\} \cup \bigcup_{i=1}^{k} \{u^i_{s+1}, \dots, u^i_m\}, \tag{2.3}$$
$$V_2(G') = \{w_1, \dots, w_s\} \cup \bigcup_{i=1}^{k} \{w^i_{s+1}, \dots, w^i_n\},$$
$$E(G') = \bigcup_{i=1}^{k} E(G_i).$$

We say that $G'$ is obtained by *gluing along a (induced) matching* of $s$ edges in $k$ copies $G_1, \dots, G_k$ of $G_1$ and write $G' = G_1 + G_2 + \dots + G_k$. If the base graph $G_1$ satisfying (2.1) for a given $s$ is clear from the context, we say simply that $G'$ is obtained by *gluing* $k$ copies $G_1, \dots, G_k$.

The main result of the present paper is the following theorem.

**Theorem 2.1.** *Let $k$, $m$, $n$, $s$ be integers such that $k \geqslant 2$, $s \geqslant 1$, $s \leqslant m$, $s \leqslant n$. Let $G_1 = (\{u^1_1, \dots, u^1_s, \dots, u^1_m\}, \{w^1_1, \dots, w^1_s, \dots, w^1_n\}, E(G_1))$ be a bipartite graph that satisfies conditions* (2.1)*. Let $G' = G_1 + G_2 + \dots + G_k$ be the graph obtained by gluing $k$ copies of $G_1$ as defined above. Assume $G_1 \in \mathcal{F}$. Then $G' \in \mathcal{F}$.*

In his paper [15], Sidorenko defines a different kind of gluing copies of a bipartite graph $G_1 \in \mathcal{F}$. He chooses special vertices in the left and the right part of $G_1$ in such a way that no two special vertices are adjacent. He takes $k$ copies $G_1, \dots, G_k$ and identifies the special vertices in each copy with their counterparts in $G_1$. Sidorenko proves that if $G_1 \in \mathcal{F}$, the graph obtained by such gluing $k$ copies is again in $\mathcal{F}$. His gluing process is very different than ours: he disallows edges where we require them.

In the next section, we give several examples of graphs for which we prove the Sidorenko $\mathcal{F}$-condition by gluing along a matching.

## 3. Examples

In this section, we present several examples that illustrate Theorem 2.1. Most of the graphs we consider below are Cartesian products of an edge $K_2$ and a bipartite graph. Kim, Lee, Lee [8] proved that the Cartesian product of a tree $T$ and a bipartite graph $H$ satisfies Sidorenko's conjecture provided $H$ satisfies Sidorenko's conjecture. Our examples show slightly stronger results for the Cartesian products considered. There has been recent work regarding connections between Cartesian products and validity of Sidorenko's conjecture using methods different from ours in [4], [5] and [9].

In Example 3.1, we show that Sidorenko's $\mathcal{F}$-condition holds for the Cartesian products of $K_2$ and $C_{2^\tau}$, $\tau \geqslant 2$. Since the 3-dimensional hypercube $Q_3$ is isomorphic to the Cartesian product of $K_2$ and $C_4$, Example 3.1 establishes the validity of Sidorenko's $\mathcal{F}$-condition for $Q_3$. In Example 3.2, we show Sidorenko's $\mathcal{F}$-condition for the Cartesian product of $K_2$ and the star $S_{n+1}$, $n \geqslant 1$. Example 3.3 shows Theorem 2.1 applied to the case of gluing three edges.

To introduce our first example, let us start with a definition. Let $s \geqslant 1$ be an integer. We define a bipartite graph $J_s = (V(J_s), E(J_s))$ as follows: $V(J_s) = \{(i,j)\colon i = 0, 1, \ldots, s,\ j = 0, 1\}$ and two vertices $(i_1, j_1)$ and $(i_2, j_2)$ are joined by an edge if $j_1 = j_2$ and $i_2 = i_1 + 1$, $i = 0, \ldots, s-1$, or $i_1 = i_2$ and $j_1 \neq j_2$. We can informally think about $J_s$ as $s$ copies of $C_4$ consecutively glued together by edges.

Let us recall the Caley graph of the abelian group $\mathbb{Z}_t \times \mathbb{Z}_2$, $t \geqslant 2$ an integer. Let $S = \{(\pm 1, 0), (0, \pm 1)\}$ be a symmetric subset of $\mathbb{Z}_t \times \mathbb{Z}_2$. The Cayley graph $G = \mathrm{Cay}(\mathbb{Z}_t \times \mathbb{Z}_2, S)$ of the abelian group $\mathbb{Z}_t \times \mathbb{Z}_2$ relative to $S$ has $\mathbb{Z}_t \times \mathbb{Z}_2$ as vertices and two vertices $v, u \in \mathbb{Z}_t \times \mathbb{Z}_2$ form an edge if and only if $v - u \in S$.

**Example 3.1.** Let $\tau \geqslant 2$ be an integer. We claim the Cayley graph $G = \mathrm{Cay}(\mathbb{Z}_{2^\tau} \times \mathbb{Z}_2, S) \in \mathcal{F}$. Note that this Cayley graph is isomorphic to the Cartesian product of $K_2$ and $C_{2^\tau}$.

Let $\tau \geqslant 2$. Sidorenko [15] proved that $C_4 \in \mathcal{F}$. Using Theorem 2.1 for two copies of $C_4$ glued by a single edge, we conclude $J_2 \in \mathcal{F}$. Further, again by Theorem 2.1 for two copies of $J_2$ glued by a single edge incident to vertices of degree 2, we conclude $J_4 \in \mathcal{F}$. Inductively, $J_{2^\nu} \in \mathcal{F}$, $\nu = 1, 2, \ldots$. Note that $C_4$ is isomorphic to the graph $J_1$.

Next consider two copies of $J_{2^{\tau-1}}$ and denote them by $J^1_{2^{\tau-1}}$, $J^2_{2^{\tau-1}}$. We label vertices of those copies as $V(J^1_{2^{\tau-1}}) = \{(i,j)^1\colon i = 0, 1, \ldots, 2^{\tau-1},\ j = 0, 1\}$ and $V(J^2_{2^{\tau-1}}) = \{(i,j)^2\colon i = 0, 1, \ldots, 2^{\tau-1},\ j = 0, 1\}$. We glue $J^1_{2^{\tau-1}}$ and $J^2_{2^{\tau-1}}$ by gluing two edges: the edge joining vertices $(0,0)^1$ and $(0,1)^1$ with the edge joining vertices $(0,0)^2$ and $(0,1)^2$, and the edge joining vertices $(2^{\tau-1}, 0)^1$ and $(2^{\tau-1}, 1)^1$ with the edge joining vertices $(2^{\tau-1}, 0)^2$ and $(2^{\tau-1}, 1)^2$. The resulting graph is $G = \mathrm{Cay}(\mathbb{Z}_{2^\tau} \times \mathbb{Z}_2, S)$. Since $J_{2^{\tau-1}} \in \mathcal{F}$, we conclude the Cayley graph $G = \mathrm{Cay}(\mathbb{Z}_{2^\tau} \times \mathbb{Z}_2, S) \in \mathcal{F}$.

Since all even cycles belong to $\mathcal{F}$ (see [15]), we can repeat similar constructions as in Example 3.1 for other even cycles in place of $C_4$.

The Cayley graph $\mathrm{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_2, S)$ defined above is isomorphic to $Q_3$. Hence, $Q_3 \in \mathcal{F}$. In other words, we proved that $Q_3$ satisfies Sidorenko's $\mathcal{F}$-condition. The Sidorenko's conjecture for $Q_3$ was first proved by Hatami in [6].

Before we state Example 3.2, we define for any positive integer $n$ the star $S_{n+1}$ to be the bipartite graph on $n+1$ vertices with one vertex of degree $n$ on one side of the bipartition of the vertex set of $S_{n+1}$ and the remaining $n$ vertices of degree 1 on the other side of the bipartition. The book graph $B_n$ is defined as the Cartesian product of $K_2$ and the star $S_{n+1}$.

**Example 3.2.** Let $n \geqslant 1$. We will show that the book graph $B_n \in \mathcal{F}$, that is, $B_n$ satisfies Sidorenko's $\mathcal{F}$-condition.

To see $B_n \in \mathcal{F}$, we need to realize that $B_n$ consists of $n$ copies of $C_4$ glued by a single edge, the 'spine' edge. Note that $C_4 \in \mathcal{F}$ by Sidorenko [15]. We consider $n$ copies of $C_4$ and choose a single edge in each copy of $C_4$. Gluing those copies of $C_4$ along that single edge, Theorem 2.1 implies that $B_n \in \mathcal{F}$.

We can repeat similar gluing as in Example 3.2 for other even cycles. More precisely, let $k \geqslant 2$. Consider $k$ copies of a fixed even cycle $C$ and glue those copies along a single 'spine' edge. Theorem 2.1 implies that the resulting graph is in $\mathcal{F}$. Note that the resulting graph is a Cartesian product of $K_2$ with $C$ only if $C = C_4$.

To demonstrate the strength of Theorem 2.1, let us present one additional example related to the Cartesian product of $K_2$ and a single graph. In that example we will use gluing along three edges.

**Example 3.3.** Let $H$ be the bipartite graph obtained by joining two copies of the cycle $C_4$ at a common vertex. We will show that the Cartesian product of $K_2$ and $H$ belongs to $\mathcal{F}$.

Let us consider two copies of $J_4$ defined above. Consider the three pairs of edges in these copies of $J_4$ corresponding to the three edges with end vertices $(0,0)$ and $(0,1)$, $(2,0)$ and $(2,1)$, $(4,0)$ and $(4,1)$. If we glue the two copies of $J_4$ along the three pairs of edges corresponding to the edges listed above which form an induced matching, we obtain a graph isomorphic to the Cartesian product of $K_2$ and $H$. By Theorem 2.1, the Cartesian product of $K_2$ and $H$ belongs to $\mathcal{F}$.

## 4. Proof of Theorem 2.1

Take $k$, $m$, $n$, $s$, $G_i$, $i = 1, \ldots, k$, $G'$ as in Theorem 2.1. Assume that the base graph $G_1 \in \mathcal{F}$. To simplify notation, we will denote the base graph by $G$ or by $G_1$ whenever convenient:

$$G = G_1 = (\{u_1, \ldots, u_s, u^1_{s+1}, \ldots, u^1_m\}, \{w_1, \ldots, w_s, w^1_{s+1}, \ldots, w^1_n\}, E(G_1)).$$

To prove Theorem 2.1, we have to show that $G' \in \mathcal{F}$; that is, $G'$ satisfies (A) and (B). To show (A) note that by the definition of $G'$ and by condition (2.1), we have

$$v_1(G') = kv_1(G) - s(k-1), \quad v_2(G') = kv_2(G) - s(k-1), \qquad (4.1)$$
$$e(G') = ke(G) - s(k-1).$$

Since $G$ satisfies (A), $e(G) \geqslant v_1(G)$, $e(G) \geqslant v_2(G)$ which combined with (4.1) implies $e(G') \geqslant v_1(G')$, $e(G') \geqslant v_2(G')$. Hence, $G'$ satisfies (A).

We need to show that $G'$ satisfies (B). Denote $m' = v_1(G')$, $n' = v_2(G')$. Take $h \in K([0,1]^2)$ and arbitrary vertex functions corresponding to vertices of $G'$: $f_1, \dots, f_s, f^i_{s+1}, \dots, f^i_{m'} \in K(\Omega)$, $g_1, \dots, g_s, g^i_{s+1}, \dots, g^i_{n'} \in K(\Lambda)$, $i = 1, \dots, k$.

To simplify presentation, we shall prove (B) in two steps.

*Step 1:* In this step we will make an additional assumption

$$f(x) = 1 \text{ a.e. in } [0,1], \quad g(y) = 1 \text{ a.e. in } [0,1]. \tag{4.2}$$

We have to prove inequality (1.3) for $G'$. The integrand of the left-hand side of (1.3) will be easier to work with if for $i = 1, \dots, k$ we introduce $P_i$ to be the product of all edge terms corresponding to edges in $G_i$, except for $h(x_l, y_l)$, $l \in [s]$, times the product of all vertex functions corresponding to vertices in $G_i$ except for $f_1(x_1), \dots, f_s(x_s)$, $g_1(y_1), \dots, g_s(y_s)$. Hence, for $i = 1, \dots, k$

$$\begin{aligned} P_i = &\prod_{\substack{u^i_l w^i_j \in E(G_i) \\ s+1 \leqslant l \leqslant m \\ s+1 \leqslant j \leqslant n}} h(x^i_l, y^i_j) \prod_{\substack{u_l w^i_j \in E(G_i) \\ l \in [s] \\ s+1 \leqslant j \leqslant n}} h(x_l, y^i_j) \prod_{\substack{u^i_l w_j \in E(G_i) \\ s+1 \leqslant l \leqslant m \\ j \in [s]}} h(x^i_l, y_j) \\ &\times \prod_{l=s+1}^{m} f^i_l(x^i_l) \prod_{j=s+1}^{n} g^i_j(y^i_j). \end{aligned} \tag{4.3}$$

Let $\widetilde{P}_i$, $i = 1, \dots, k$, be the same as $P_i$ except that all variables $\{x^i_{s+1}, \dots, x^i_m\}$, $\{y^i_{s+1}, \dots, y^i_n\}$ are replaced by their counterparts corresponding to $G_1$: $\{x^1_{s+1}, \dots, x^1_m\}$, $\{y^1_{s+1}, \dots, y^1_n\}$. Thus, by (4.3) and (2.2)

$$\begin{aligned} \widetilde{P}_i = &\prod_{\substack{u^i_l w^i_j \in E(G_i) \\ l,j \geqslant s+1}} h(x^1_l, y^1_j) \prod_{\substack{u_l w^i_j \in E(G_i) \\ l \in [s] \\ j \geqslant s+1}} h(x_l, y^1_j) \prod_{\substack{u^i_l w_j \in E(G_i) \\ l \geqslant s+1 \\ j \in [s]}} h(x^1_l, y_j) \\ &\times \prod_{l=s+1}^{m} f^i_l(x^1_l) \prod_{j=s+1}^{n} g^i_j(y^1_j). \end{aligned} \tag{4.4}$$

Denote the geometric mean of $\widetilde{P}_1, \dots, \widetilde{P}_k$ by $P$:

$$P = (\widetilde{P}_1 \dots \widetilde{P}_k)^{1/k}. \tag{4.5}$$

Next, denote by $C(x)$, $D(y)$ the products of vertex functions corresponding to all vertices on the left-hand side and all vertices on the right-hand side in $G'$, respectively, with all variables corresponding to the left-hand and the right-hand side vertices replaced by $x$ and $y$, respectively. That is

$$C(x) = \left( \prod_{i=1}^{k} \prod_{l=s+1}^{m} f^i_l(x) \right) \prod_{p=1}^{s} f_p(x), \quad D(y) = \left( \prod_{i=1}^{k} \prod_{j=s+1}^{n} g^i_j(y) \right) \prod_{p=1}^{s} g_p(y). \tag{4.6}$$

By (4.3) and (4.6), to prove (B) for $G'$ when (4.2) holds, it suffices to show

$$\int\Big(\prod_{p=1}^{s} h(x_p,y_p)f_p(x_p)g_p(y_p)\Big)P_1P_2\dots P_k\,\mathrm{d}\mu_x^{m'}\,\mathrm{d}\mu_y^{n'} \geqslant \Big(\int h(x,y)C(x)^{1/e(G')}D(y)^{1/e(G')}\,\mathrm{d}\mu_x\,\mathrm{d}\mu_y\Big)^{e(G')}. \tag{4.7}$$

Observe that the integrand on the left-hand side depends on $m'+n'$ variables

$$\{x_1,\dots,x_s\}\cup\bigcup_{i=1}^{k}\{x_{s+1}^i,\dots,x_m^i\}\cup\{y_1,\dots,y_s\}\cup\bigcup_{i=1}^{k}\{y_{s+1}^i,\dots,y_n^i\}.$$

Notice next that for a given $i$, the only factor in the integrand that depends on $x_{s+1}^i,\dots,x_m^i$, $y_{s+1}^i,\dots,y_n^i$ is $P_i$. We apply the Fubini theorem to the left-hand side of (4.7) by integrating first with respect to $\Big(\prod_{i=1}^{k}\prod_{l=s+1}^{m}\mathrm{d}x_l^i\Big)\Big(\prod_{i=1}^{k}\prod_{j=s+1}^{n}\mathrm{d}y_j^i\Big)$. We obtain

$$\text{LHS of (4.7)} = \int\prod_{p=1}^{s} h(x_p,y_p)f_p(x_p)g_p(y_p) \times\prod_{i=1}^{k}\Big(\int P_i\,\mathrm{d}x_{s+1}^i\dots\mathrm{d}x_m^i\,\mathrm{d}y_{s+1}^i\dots\mathrm{d}y_n^i\Big)\mathrm{d}x_1\dots\mathrm{d}x_s\,\mathrm{d}y_1\dots\mathrm{d}y_s. \tag{4.8}$$

Because of the length of our formulas, we will use the notation "LHS of (4.7)" to mean "the left-hand side of formula (4.7)". Denote for $i=1,\dots,k$:

$$B_i(x_1,\dots,x_s,y_1,\dots,y_s)=\int P_i\,\mathrm{d}x_{s+1}^i\dots\mathrm{d}x_m^i\,\mathrm{d}y_{s+1}^i\dots\mathrm{d}y_n^i.$$

By the Fubini theorem, each $B_i$ is defined a.e. in $[0,1]^{2s}$, measurable, integrable (as all our functions are bounded), nonnegative. By changing the dummy variables of integration in the definition of $B_i$ to $x_{s+1}^1,\dots,x_m^1$, $y_{s+1}^1,\dots,y_n^1$ we obtain for $i=1,\dots,k$:

$$B_i=\int\widetilde{P}_i\,\mathrm{d}x_{s+1}^1\dots\mathrm{d}x_m^1\,\mathrm{d}y_{s+1}^1\dots\mathrm{d}y_n^1 \tag{4.9}$$

for almost all $(x_1,\dots,x_s,y_1,\dots,y_s)\in[0,1]^{2s}$. By (4.9) and (4.8), we get

$$\text{LHS of (4.7)} = \int\prod_{p=1}^{s} h(x_p,y_p)f_p(x_p)g_p(y_p)\prod_{i=1}^{k}B_i\,\mathrm{d}x_1\dots\mathrm{d}x_s\,\mathrm{d}y_1\dots\mathrm{d}y_s. \tag{4.10}$$

We will use the following lemma:

**Lemma 4.1.** *With $B_i$, $\widetilde{P}_i$, $i = 1, \ldots, k$, defined as above, we have*

$$\prod_{i=1}^{k} B_i \geqslant \left( \int (\widetilde{P}_1 \ldots \widetilde{P}_k)^{1/k} \, \mathrm{d}x_{s+1}^1 \ldots \mathrm{d}x_m^1 \, \mathrm{d}y_{s+1}^1 \ldots \mathrm{d}y_n^1 \right)^k = \left( \int P \, \mathrm{d}x_{s+1}^1 \ldots \mathrm{d}x_m^1 \, \mathrm{d}y_{s+1}^1 \ldots \mathrm{d}y_n^1 \right)^k \tag{4.11}$$

*for almost all* $(x_1, \ldots, x_s, y_1, \ldots y_s) \in [0,1]^{2s}$.

P r o o f. The equality in (4.11) follows simply from the definition of $P$. The inequality in (4.11) is equivalent to

$$\prod_{i=1}^{k} B_i^{1/k} \geqslant \int (\widetilde{P}_1 \ldots \widetilde{P}_k)^{1/k} \, \mathrm{d}x_{s+1}^1 \ldots \mathrm{d}x_m^1 \, \mathrm{d}y_{s+1}^1 \ldots \mathrm{d}y_n^1$$

a.e. in $[0,1]^{2s}$, which in turn is equivalent to

$$\prod_{i=1}^{k} \left( \int \widetilde{P}_i \, \mathrm{d}x_{s+1}^1 \ldots \mathrm{d}x_m^1 \, \mathrm{d}y_{s+1}^1 \ldots \mathrm{d}y_n^1 \right)^{1/k} \geqslant \int (\widetilde{P}_1 \ldots \widetilde{P}_k)^{1/k} \, \mathrm{d}x_{s+1}^1 \ldots \mathrm{d}x_m^1 \, \mathrm{d}y_{s+1}^1 \ldots \mathrm{d}y_n^1 \tag{4.12}$$

a.e. in $[0,1]^{2s}$.

We prove (4.12) using the generalized Hölder inequality. We can apply the inequality to the left-hand side of (4.12) as $\underbrace{1/k + \ldots + 1/k}_{k \text{ times}} = 1$. We obtain

$$\begin{aligned} &\prod_{i=1}^{k} \left( \int \widetilde{P}_i \, \mathrm{d}x_{s+1}^1 \ldots \mathrm{d}_m^1 \, \mathrm{d}y_{s+1}^1 \ldots \mathrm{d}y_n^1 \right)^{1/k} \\ &\quad = \prod_{i=1}^{k} \left( \int \left(\widetilde{P}_i^{1/k}\right)^k \mathrm{d}x_{s+1}^1 \ldots \mathrm{d}x_m^1 \, \mathrm{d}y_{s+1}^1 \ldots \mathrm{d}y_n^1 \right)^{1/k} \\ &\quad \geqslant \int \prod_{i=1}^{k} (\widetilde{P}_i)^{1/k} \, \mathrm{d}x_{s+1}^1 \ldots \mathrm{d}x_m^1 \, \mathrm{d}y_{s+1}^1 \ldots \mathrm{d}y_n^1, \end{aligned}$$

which gives (4.12). Hence, the lemma is proved. □

The lemma implies that (4.7) holds if the following inequality is satisfied:

$$\begin{aligned}(4.13)\quad &\int \prod_{p=1}^{s} h(x_p, y_p) f_p(x_p) g_p(y_p)\\ &\qquad \times \left(\int P \,\mathrm{d}x_{s+1}^1 \dots \mathrm{d}x_m^1 \,\mathrm{d}y_{s+1}^1 \dots \mathrm{d}y_n^1\right)^k \mathrm{d}x_1 \dots \mathrm{d}x_s \,\mathrm{d}y_1 \dots \mathrm{d}y_s\\ &\quad \geqslant \left(\int h(x,y) C(x)^{1/e(G')} D(y)^{1/e(G')} \,\mathrm{d}x\,\mathrm{d}y\right)^{e(G')}.\end{aligned}$$

If the integral on the right-hand side of (4.13) is equal to 0, (4.13) holds. Assume the integral is not 0. By (4.1), $e(G') = ke(G) - s(k-1)$. Hence, (4.13) is equivalent to the following inequality:

$$\begin{aligned}(4.14)\quad &\Bigg(\int \prod_{p=1}^{s} h(x_p, y_p) f_p(x_p) g_p(y_p)\\ &\qquad \times \left(\int P \,\mathrm{d}x_{s+1}^1 \dots \mathrm{d}x_m^1 \,\mathrm{d}y_{s+1}^1 \dots \mathrm{d}y_n^1\right)^k \mathrm{d}x_1 \dots \mathrm{d}x_s \,\mathrm{d}y_1 \dots \mathrm{d}y_s\Bigg)^{1/k}\\ &\qquad \times \left(\left(\int h(x,y) C(x)^{1/e(G')} D(y)^{1/e(G')} \,\mathrm{d}x\,\mathrm{d}y\right)^s\right)^{(k-1)/k}\\ &\quad \geqslant \left(\int h(x,y) C(x)^{1/e(G')} D(y)^{1/e(G')} \,\mathrm{d}x\,\mathrm{d}y\right)^{e(G)}.\end{aligned}$$

Observe that

$$\begin{aligned}&\left(\int h(x,y) C(x)^{1/e(G')} D(y)^{1/e(G')} \,\mathrm{d}x\,\mathrm{d}y\right)^s\\ &\quad = \prod_{p=1}^{s} \int h(x_p,y_p) C(x_p)^{1/e(G')} D(y_p)^{1/e(G')} \,\mathrm{d}x_p\,\mathrm{d}y_p\\ &\quad = \int \prod_{p=1}^{s} h(x_p,y_p) C(x_p)^{1/e(G')} D(y_p)^{1/e(G')} \,\mathrm{d}x_1 \dots \mathrm{d}x_s \,\mathrm{d}y_1 \dots \mathrm{d}y_s.\end{aligned}$$

Substituting the latter integral into the left-hand side of (4.14) and applying Hölder's inequality with exponents $1/k, (k-1)/k$ gives

$$\begin{aligned}(4.15)\quad &\text{LHS of (4.14)}\\ &\quad \geqslant \int \left(\prod_{p=1}^{s} h(x_p, y_p) f_p(x_p) g_p(y_p)\right)^{1/k} \int P \,\mathrm{d}x_{s+1}^1 \dots \mathrm{d}x_m^1 \,\mathrm{d}y_{s+1}^1 \dots \mathrm{d}y_n^1\\ &\quad \times \left(\prod_{p=1}^{s} h(x_p,y_p) C(x_p)^{1/e(G')} D(y_p)^{1/e(G')}\right)^{(k-1)/k} \mathrm{d}x_1 \dots \mathrm{d}x_s \,\mathrm{d}y_1 \dots \mathrm{d}y_s.\end{aligned}$$

Hence, to prove (4.14) it suffices to show that

$$\text{RHS of (4.15)} \geqslant \text{RHS of (4.14)}. \tag{4.16}$$

Denote by $H_G$ the product of all edge functions corresponding to edges in $G$. By definition (4.4), (4.5) of $P$,

$$P\prod_{p=1}^{s} h(x_p,y_p) = H_G \prod_{i=1}^{k}\prod_{l=s+1}^{m} f_l^i(x_l^1)^{1/k} \prod_{i=1}^{k}\prod_{j=s+1}^{n} g_j^i(y_j^1)^{1/k}.$$

Hence,

(4.17)

$$\begin{aligned}\text{RHS of (4.15)} = \int H_G \prod_{p=1}^{s} f_p(x_p)^{1/k} g_p(y_p)^{1/k} C(x_p)^{(k-1)/ke(G')} D(y_p)^{(k-1)/ke(G')}\\ \times \prod_{i=1}^{k}\prod_{l=s+1}^{m} f_l^i(x_l^1)^{1/k} \prod_{i=1}^{k}\prod_{j=s+1}^{n} g_j^i(y_j^1)^{1/k}\, \mathrm{d}\mu_x^m\, \mathrm{d}\mu_y^n.\end{aligned}$$

Set

$$\gamma = \frac{k-1}{ke(G')}.$$

Equality (4.17) can be rewritten as

$$\text{RHS of (4.15)} = \int H_G \prod_{p=1}^{s} \tilde{f}_p(x_p) \prod_{l=s+1}^{m} \tilde{f}_l(x_l^1) \prod_{p=1}^{s} \widetilde{g}_p(y_p) \prod_{j=s+1}^{n} \widetilde{g}_j(y_j^1)\, \mathrm{d}\mu_x^m\, \mathrm{d}\mu_y^n, \tag{4.18}$$

where by (4.17), (4.6), $\tilde{f}_1,\dots,\tilde{f}_m$, $\widetilde{g}_1,\dots,\widetilde{g}_n$ are as follows. For every fixed $l_*, j_* \in \{1,\dots,s\}$

$$\begin{aligned}\tilde{f}_{l_*}(x_{l_*}) &= f_{l_*}(x_{l_*})^{1/k} C(x_{l_*})^{\gamma} = f_{l_*}(x_{l_*})^{1/k} \prod_{p=1}^{s} f_p(x_{l_*})^{\gamma} \prod_{i=1}^{k}\prod_{l=s+1}^{m} f_l^i(x_{l_*})^{\gamma},\\ \widetilde{g}_{j_*}(y_{j_*}) &= g_{j_*}(y_{j_*})^{1/k} D(y_{j_*})^{\gamma} = g_{j_*}(y_{j_*})^{1/k} \prod_{p=1}^{s} g_p(y_{j_*})^{\gamma} \prod_{i=1}^{k}\prod_{j=s+1}^{n} g_j^i(y_{j_*})^{\gamma}.\end{aligned} \tag{4.19}$$

For every $l_* \in \{s+1,\dots,m\}$, $j_* \in \{s+1,\dots,n\}$

$$\tilde{f}_{l_*}(x_{l_*}^1) = \prod_{i=1}^{k} f_{l_*}^i(x_{l_*}^1)^{1/k}, \quad \widetilde{g}_{j_*}(y_{j_*}^1) = \prod_{i=1}^{k} g_{j_*}^i(y_{j_*}^1)^{1/k}. \tag{4.20}$$

Since $G = G_1 \in \mathcal{F}$, $G$ satisfies (B). We obtain

$$\int H_G \prod_{p=1}^{s} \tilde{f}_p(x_p) \prod_{l=s+1}^{m} \tilde{f}_l(x_l^1) \prod_{p=1}^{s} \widetilde{g}_p(y_p) \prod_{j=s+1}^{n} \widetilde{g}_j(y_j^1)\, \mathrm{d}\mu_x^m\, \mathrm{d}\mu_y^n \geqslant \Big(\int h(x,y) \prod_{l=1}^{m} \tilde{f}_l(x)^{1/e(G)} \prod_{j=1}^{n} \widetilde{g}_j(y)^{1/e(G)}\, \mathrm{d}\mu_x\, \mathrm{d}\mu_y\Big)^{e(G)}. \tag{4.21}$$

By (4.21), (4.18), to prove (4.14) it suffices to show that

$$\text{RHS of (4.21)} \geqslant \text{ RHS of (4.14)}.$$

In fact, we have RHS of (4.21) = RHS of (4.14).

To prove this equality, it suffices to show

$$\Big(\prod_{l=1}^{m} \tilde{f}_l(x)\Big)^{1/e(G)} = C(x)^{1/e(G')}, \quad \Big(\prod_{j=1}^{n} \widetilde{g}_j(y)\Big)^{1/e(G)} = D(y)^{1/e(G')}, \tag{4.22}$$

or equivalently

$$\prod_{l=1}^{m} \tilde{f}_l(x) = C(x)^{e(G)/e(G')}, \quad \prod_{j=1}^{n} \widetilde{g}_j(y) = D(y)^{e(G)/e(G')}.$$

By (4.19), (4.22), (4.6) and simple arithmetic applied to exponents we obtain

$$\prod_{l=1}^{m} \tilde{f}_l(x) = C(x)^{s\gamma+(1/k)}, \quad \prod_{j=1}^{n} \widetilde{g}_j(y) = D(y)^{s\gamma+(1/k)}.$$

To finish the proof of (4.22), it suffices to show that

$$s\gamma + \frac{1}{k} = \frac{e(G)}{e(G')},$$

which follows easily from the definition of $\gamma$ and (4.1). Hence, (4.14) is proved and so is (4.7). The proof of Step 1 is complete.

*Step 2*: In this step, $f$ and $g$ are arbitrary, $f \in K(\Omega)$, $g \in K(\Lambda)$. We will use the same notation as in Step 1. To prove (B) for $G'$, we have to show that

$$\begin{aligned}\int \Big(\prod_{p=1}^{s} h(x_p, y_p) f_p(x_p) g_p(y_p)\Big) P_1 P_2 \dots P_k\, \mathrm{d}\mu_x^{m'}\, \mathrm{d}\mu_y^{n'} \\ \times \Big(\int f\, \mathrm{d}\mu_x\Big)^{e(G')-m'} \Big(\int g\, \mathrm{d}\mu_y\Big)^{e(G')-n'} \\ \geqslant \Big(\int h(x,y) f(x)^{(e(G')-m')/e(G')} g(y)^{(e(G')-n')/e(G')} \\ \times C(x)^{1/e(G')} D(y)^{1/e(G')}\, \mathrm{d}\mu_x\, \mathrm{d}\mu_y\Big)^{e(G')}.\end{aligned} \tag{4.23}$$

Note that by (4.1)

$$e(G') - m' = k(e(G) - m), \quad e(G') - n' = k(e(G) - n), \\ e(G') = ke(G) - s(k-1). \tag{4.24}$$

Denote

$$\alpha = \frac{k(e(G) - m)}{ke(G) - s(k-1)}, \quad \beta = \frac{k(e(G) - n)}{ke(G) - s(k-1)}. \tag{4.25}$$

If the integral on the right-hand side of (4.23) vanishes, (4.23) is satisfied. Assume the integral is positive. In that case, by Lemma 4.1, (4.24), (4.25), inequality (4.23) holds provided the following inequality is satisfied:

$$\begin{aligned} &\Big(\int \prod_{p=1}^{s} h(x_p, y_p) f_p(x_p) g_p(y_p) \Big(\int P \, \mathrm{d}x_{s+1}^1 \ldots \mathrm{d}x_m^1 \, \mathrm{d}y_{s+1}^1 \ldots \mathrm{d}y_n^1\Big)^k \\ &\quad \times \mathrm{d}x_1 \ldots \mathrm{d}x_s \, \mathrm{d}y_1 \ldots \mathrm{d}y_s\Big) \Big(\int f \, \mathrm{d}\mu_x\Big)^{k(e(G)-m)} \Big(\int g \, \mathrm{d}\mu_y\Big)^{k(e(G)-n)} \\ &\geqslant \Big(\int h(x,y) f(x)^\alpha g(y)^\beta C(x)^{1/e(G')} D(y)^{1/e(G')} \, \mathrm{d}\mu_x \, \mathrm{d}\mu_y\Big)^{ke(G)-s(k-1)}. \end{aligned} \tag{4.26}$$

Inequality (4.26) is equivalent to

$$\begin{aligned} &\Big(\int \prod_{p=1}^{s} h(x_p, y_p) f(x_p) g(y_p) \Big(\int P \, \mathrm{d}x_{s+1}^1 \ldots \mathrm{d}x_m^1 \, \mathrm{d}y_{s+1}^1 \ldots \mathrm{d}y_m^1\Big)^k \\ &\quad \times \mathrm{d}x_1 \ldots \mathrm{d}x_s \, \mathrm{d}y_1 \ldots \mathrm{d}y_s\Big)^{1/k} \Big(\int f \, \mathrm{d}\mu_x\Big)^{e(G)-m} \Big(\int g \, \mathrm{d}\mu_y\Big)^{e(G)-n} \\ &\quad \times \Big(\Big(\int h(x,y) f(x)^\alpha g(y)^\beta C(x)^{1/e(G')} D(y)^{1/e(G')} \, \mathrm{d}\mu_x \, \mathrm{d}\mu_y\Big)^s\Big)^{(k-1)/k} \\ &\geqslant \Big(\int h(x,y) f(x)^\alpha g(y)^\beta C(x)^{1/e(G')} D(y)^{1/e(G')} \, \mathrm{d}\mu_x \, \mathrm{d}\mu_y\Big)^{e(G)}. \end{aligned} \tag{4.27}$$

As in Step 1, we notice

$$\begin{aligned} &\Big(\int h(x,y) f(x)^\alpha g(y)^\beta C(x)^{1/e(G')} D(y)^{1/e(G')} \, \mathrm{d}\mu_x \, \mathrm{d}\mu_y\Big)^s \\ &\quad = \int \prod_{p=1}^{s} h(x_p, y_p) f(x_p)^\alpha g(y_p)^\beta C(x_p)^{1/e(G')} D(y_p)^{1/e(G')} \\ &\quad \times \mathrm{d}x_1 \ldots \mathrm{d}x_s \, \mathrm{d}y_1 \ldots \mathrm{d}y_s. \end{aligned} \tag{4.28}$$

We apply Hölder's inequality with the exponents $(k-1)/k$, $1/k$ to the first and the last factor of the left-hand side of (4.27). We obtain that (4.27) holds provided the following inequality is satisfied:

$$\begin{aligned}(4.29)\qquad \int\Big(\prod_{p=1}^{s} & h(x_p,y_p)f_p(x_p)^{1/k}g_p(y_p)^{1/k}C(x_p)^{\gamma}D(x_p)^{\gamma} \\ & \times f(x_p)^{\alpha(k-1)/k}g(y_p)^{\beta(k-1)/k}\Big)P\,\mathrm{d}\mu_x^m\,\mathrm{d}\mu_y^n \\ & \times\Big(\int f\,\mathrm{d}\mu_x\Big)^{e(G)-m}\Big(\int g\,\mathrm{d}\mu_y\Big)^{e(G)-n} \\ & \geqslant\Big(\int h(x,y)f(x)^{\alpha}g(y)^{\beta}C(x)^{1/e(G')}D(y)^{1/e(G')}\,\mathrm{d}\mu_x\,\mathrm{d}\mu_y\Big)^{e(G)}.\end{aligned}$$

Reasoning as in Step 1, we deduce that the left-hand side of (4.29) can be rewritten as

$$\begin{aligned}(4.30)\ \text{LHS of (4.29)} = \int & H_G\prod_{p=1}^{s}\tilde{\tilde{f}}_p(x_p)\prod_{l=s+1}^{m}\tilde{\tilde{f}}_l(x_l^1)\prod_{p=1}^{s}\tilde{\tilde{g}}_p(y_p) \\ & \times\prod_{j=s+1}^{n}\tilde{\tilde{g}}_j(y_j^1)\,\mathrm{d}\mu_x^m\,\mathrm{d}\mu_y^n\Big(\int f\,\mathrm{d}\mu_x\Big)^{e(G)-m}\Big(\int g\,\mathrm{d}\mu_y\Big)^{e(G)-n},\end{aligned}$$

where

$$\begin{aligned}(4.31)\qquad & \tilde{\tilde{f}}_l(x_l)=\tilde{f}_l(x_l)f(x_l)^{\alpha(k-1)/k},\quad \tilde{\tilde{g}}_j(y_j)=\tilde{g}_j(y_j)g(y_j)^{\beta(k-1)/k},\quad l,j\in[s], \\ & \tilde{\tilde{f}}_l(x_l^1)=\tilde{f}_l(x_l^1),\quad l=s+1,\dots,m, \\ & \tilde{\tilde{g}}_j(y_j)=\tilde{g}_j(y_j^1),\quad j=s+1,\dots,n.\end{aligned}$$

Since $G$ satisfies (B), we have by (4.30)

$$\begin{aligned}(4.32)\qquad \text{LHS of (4.29)} \geqslant \Big(\int & h(x,y)f(x)^{(e(G)-m)/e(G)}g(y)^{(e(G)-n)/e(G)} \\ & \times\prod_{l=1}^{m}\tilde{\tilde{f}}_l(x)^{1/e(G)}\prod_{j=1}^{n}\tilde{\tilde{g}}_j(y)^{1/e(G)}\,\mathrm{d}\mu_x\,\mathrm{d}\mu_y\Big)^{e(G)}.\end{aligned}$$

To prove (4.29) it suffices to show that
(4.33)

$$\text{RHS of (4.32)} = \Big(\int h(x,y)f(x)^{\alpha}g(y)^{\beta}C(x)^{1/e(G')}D(y)^{1/e(G')}\,\mathrm{d}\mu_x\,\mathrm{d}\mu_y\Big)^{e(G)}.$$

By (4.31) and (4.22), we have

$$\prod_{l=1}^{m} \tilde{\tilde{f}}_l(x)^{1/e(G)} = \prod_{l=1}^{m} \tilde{f}_l(x)^{1/e(G)} \prod_{l=1}^{s} f(x)^{\alpha(k-1)/ke(G)} = C(x)^{1/e(G')} f(x)^{s\alpha(k-1)/ke(G)},$$
$$\prod_{j=1}^{n} \tilde{\tilde{g}}_j(y)^{1/e(G)} = \prod_{j=1}^{n} \tilde{g}_j(y)^{1/e(G)} \prod_{j=1}^{s} g(y)^{\beta(k-1)/ke(G)} = D(y)^{1/e(G')} g(y)^{s\beta(k-1)/ke(G)}.$$

It remains to show that

$$\frac{e(G)-m}{e(G)} + \frac{s\alpha(k-1)}{ke(G)} = \alpha, \quad \frac{e(G)-n}{e(G)} + \frac{s\beta(k-1)}{ke(G)} = \beta,$$

which follows easily from (4.25). Therefore (4.29) holds, which implies that $G'$ satisfies (B). Hence $G' \in \mathcal{F}$. The proof of Theorem 2.1 is complete.

## 5. Final remarks

Theorem 2.1 states that the way of "gluing" that we defined in Section 2 preserves Sidorenko's $\mathcal{F}$-condition. It is natural to ask if our way of gluing preserves the $\mathcal{F}_i$-condition; that is, the membership in the class $\mathcal{F}_i$ for $i = 1, 2$.

**Theorem 5.1.** *Let $i \in \{1, 2\}$ be fixed. Let $k$, $m$, $n$, $s$ be integers such that $k \geqslant 2$, $s \geqslant 1$, $s \leqslant m$, $s \leqslant n$. Let $G_1 = (\{u_1^1, \ldots, u_s^1, \ldots, u_m^1\}, w_1^1, \ldots, w_s^1, \ldots, w_n^1\}, E(G_1))$ be a bipartite graph that satisfies conditions* (2.1). *Let $G' = G_1 + G_2 + \ldots + G_k$ be the graph obtained by gluing $k$ copies of $G_1$ as defined in Section 2. Assume $G_1 \in \mathcal{F}_i$. Then $G' \in \mathcal{F}_i$.*

Proof. Let $h \in K([0,1]^2)$ be fixed. As before we denote for simplicity $G = G_1$. Denote by $H_G, H_{G'}$ the products of all edge functions corresponding to $h$ and the edges in $G, G'$, respectively. Assume that $G \in \mathcal{F}_1$. (The proof for $G \in \mathcal{F}_2$ is the same.) That is, assume

$$\int H_G \, \mathrm{d}\mu_x^m \, \mathrm{d}\mu_y^n \geqslant \left( \int h(x,y) \, \mathrm{d}\mu_x \, \mathrm{d}\mu_y \right)^{e(G)}. \tag{5.1}$$

We need to prove that

$$\int H_{G'} \, \mathrm{d}\mu_x^{m'} \, \mathrm{d}\mu_y^{n'} \geqslant \left( \int h(x,y) \, \mathrm{d}\mu_x d\mu_y \right)^{e(G')}. \tag{5.2}$$

Without any loss of generality, we may assume that $G$ has no isolated vertices. Indeed, $H_G$, $H_{G'}$ depend only on the edges of $G$ and $G'$ and are blind to isolated vertices.

Hence, assume $G$ has no isolated vertices and therefore $G$ satisfies (A). Now we can in essence repeat the proof of Theorem 2.1, Step 1, in the case when all functions $f, f_1, \ldots, f_{m'}$, $g, g_1, \ldots, g_{n'}$ are all equal constantly to 1. The only place in Theorem 2.1, Step 1, where we need that $G$ satisfies Sidorenko's $\mathcal{F}$-condition is (4.21). In our case, though, (4.21) reduces to our weaker assumption (5.1) as by (4.19), (4.20), $\tilde{f}_1, \ldots, \tilde{f}_m$, $\widetilde{g}_1, \ldots, \widetilde{g}_n$ are all constantly equal to 1. Hence, we obtain (4.7) which in our case coincides with (5.2). The proof of Theorem 5.1 is complete. □

A brief sketch of the proof of Proposition 1.1 is in Sidorenko [15]. For completeness of the presentation, we give a more detailed proof below.

Proof of Proposition 1.1. Let $G = (\{u_1, \ldots, u_m\}, \{w_1, \ldots, w_n\}, E(G))$ be a bipartite graph which satisfies (A). We shall prove that (B) is equivalent to (C1). The case of (B) equivalent to (C2) is symmetric.

Assume $G$ satisfies (B). For any $h \in K([0,1]^2)$, $f_1, \ldots, f_m \in K(\Omega)$, $g, g_1, \ldots, g_n \in K(\Lambda)$ choose $f$ as

$$f(x) = \left(\int h(x,y)\Big(g(y)^{e(G)-n}\prod_{j=1}^{n} g_j(y)\Big)^{1/e(G)} \mathrm{d}\mu_y\right)^{e(G)/m} \left(\prod_{l=1}^{m} f_l(x)\right)^{1/m}. \tag{5.3}$$

We easily calculate that for such $f$ we have

$$\text{RHS of (1.3)} = \left(\int f(x)\,\mathrm{d}\mu_x\right)^{e(G)}, \quad \text{RHS of (1.5)} = \left(\int f(x)\,\mathrm{d}\mu_x\right)^{m}. \tag{5.4}$$

We assume $\int f(x)\,\mathrm{d}\mu_x > 0$. Otherwise (1.5) holds. Divide both sides of (1.3) by $\left(\int f(x)\,\mathrm{d}\mu_x\right)^{e(G)-m}$. We obtain (1.5). Since $h$, $f_1, \ldots, f_m$, $g, g_1, \ldots, g_n$ were arbitrary, (C1) holds. Hence (B) implies (C1).

To prove the converse implication, assume $G$ satisfies (C1). Choose any $h \in K([0,1]^2)$, $f, f_1, \ldots, f_m \in K(\Omega)$, $g, g_1, \ldots, g_n \in K(\Lambda)$. To prove (B) it suffices to show that

$$\begin{aligned}&\left(\int\left(\int h(x,y)\Big(g(y)^{e(G)-n}\prod_{j=1}^{n} g_j(y)\Big)^{1/e(G)} \mathrm{d}\mu_y\right)^{e(G)/m}\right.\\ &\qquad\left.\times\left(\prod_{l=1}^{m} f_l(x)\right)^{1/m} \mathrm{d}\mu_x\right)^{m}\left(\int f(x)\,\mathrm{d}\mu_x\right)^{e(G)-m}\\ &\geqslant \left(\int h(x,y)\Big(f(x)^{e(G)-m} g(y)^{e(G)-n}\prod_{l=1}^{m} f_l(x)\prod_{j=1}^{n} g_j(y)\Big)^{1/e(G)} \mathrm{d}\mu_x\,\mathrm{d}\mu_y\right)^{e(G)}.\end{aligned} \tag{5.5}$$

Inequality (5.5) is equivalent to

$$(5.6)\quad \left(\int\left(\int h(x,y)\Big(g(y)^{e(G)-n}\prod_{j=1}^{n}g_j(y)\prod_{l=1}^{m}f_l(x)\Big)^{1/e(G)}\,\mathrm{d}\mu_y\right)^{e(G)/m}\mathrm{d}\mu_x\right)^{m/e(G)} \times\left(\int f(x)\,\mathrm{d}\mu_x\right)^{(e(G)-m)/e(G)} \geqslant \left(\int h(x,y)\Big(f(x)^{e(G)-m}g(y)^{e(G)-n}\prod_{l=1}^{m}f_l(x)\prod_{j=1}^{n}g_j(y)\Big)^{1/e(G)}\,\mathrm{d}\mu_x\,\mathrm{d}\mu_y\right).$$

Note that condition (A) gives $e(G)\geqslant m$, $e(G)\geqslant n$. Hence, we can assume $e(G)>0$ as well as $e(G)-m\geqslant 0$. If $e(G)-m=0$, (5.6) holds trivially. Assume $e(G)-m>0$ and apply Hölder's inequality to the left-hand side of (5.6) with exponents $m/e(G)$, $(e(G)-m)/e(G)$. We obtain

$$\text{LHS of (5.6)}\geqslant\int\left(\int h(x,y)\Big(g(y)^{e(G)-n}\prod_{j=1}^{n}g_j(y)\prod_{l=1}^{m}f_l(x)\Big)^{1/e(G)}\,\mathrm{d}\mu_y\right) \times f(x)^{(e(G)-m)/e(G)}\,\mathrm{d}\mu_x = \text{RHS of (5.6)}.$$

Hence, (5.6) and (5.5) hold and (B) is proved. The proof of Proposition 1.1 is complete. □

The last remark concerns measure spaces $\Omega$ and $\Lambda$ and their products. We assumed for simplicity $\Omega=\Lambda=([0,1],\mathcal{L},\mu)$ and used $\overline{\otimes}$ as the product. All proofs in this paper adapt with minor changes to the case where $\Omega$, $\Lambda$ are arbitrary probability spaces and the product $\overline{\otimes}$ is replaced by the tensor product $\otimes$. Of course, when we talk about the class $\mathcal{F}_2$ where $h$ is symmetric, we have to assume $\Omega=\Lambda$.

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