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HIGHER ORDER RIESZ TRANSFORMS FOR THE DUNKL ORNSTEIN-UHLENBECK OPERATOR

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Dedicated to my Professor Néjib Ben Salem

Abstract. The aim of this paper is to extend the study of Riesz transforms associated to Dunkl Ornstein-Uhlenbeck operator considered by A. Nowak, L. Roncal and K. Stempak to higher order.

Keywords: Dunkl Laplacian; Dunkl Ornstein-Uhlenbeck operator; generalized Hermite polynomial; Riesz transform

MSC 2010: 26A33, 42C10, 42C20, 43A15, 47G40

1. Introduction and statement of the results

Consider the finite reflection group generated by σ_j, $j=1,\ldots,d$ (see [2]),

$$\sigma_j(x_1,\ldots,x_j,\ldots,x_d)=(x_1,\ldots,-x_j,\ldots,x_d),$$

and isomorphic to $\mathbb{Z}_2^d=\{0,1\}^d$.

The reflection σ_j is in the hyperplane orthogonal to e_j, the jth coordinate vector in $\mathbb{R}^d$. Given a root system R by $R=\{\pm\sqrt{2}e_j\colon j=1,\ldots,d\}$, and the positive root system R_+ defined by $R_+=\{\sqrt{2}e_j\colon j=1,\ldots,d\}$, we recall the nonnegative multiplicity function $k\colon R\to[0,\infty)$ which is $\mathbb{Z}_2^d$-invariant, so only values of k on R_+ are considered. Hence $k=(\alpha_1+\frac12,\ldots,\alpha_d+\frac12)$, such that $\alpha_j\geqslant-\frac12$.

Let T_j^α, $j=1,\ldots,d$, $\alpha\in[-\frac12,\infty)^d$, be the Dunkl differential-difference operators, (see [11]) defined by

$$T_j^\alpha=\partial_j f(x)+\Big(\alpha_j+\frac12\Big)\frac{f(x)-f(\sigma_j x)}{x_j},\quad f\in\mathcal{C}^1(\mathbb{R}^d),$$

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here ∂_j is the jth partial derivative and σ_j denotes the reflection in the hyperplane orthogonal to e_j, the jth coordinate vector in $\mathbb{R}^d$.

In Dunkl's theory the operator

$$\Delta_\alpha = \sum_{j=1}^{d} (T_j^\alpha)^2$$

plays the role of the Euclidean Laplacian. The explicit form is

$$\Delta_\alpha f(x) = \sum_{j=1}^{d} \Big(\frac{\partial^2 f}{\partial x_j^2}(x) + \frac{2\alpha_j + 1}{x_j} \frac{\partial f}{\partial x_j}(x) - \Big(\alpha_j + \frac{1}{2}\Big) \frac{f(x) - f(\sigma_j x)}{x_j^2} \Big).$$

We recall the definition of the Dunkl Ornstein-Uhlenbeck operator, given in [10] by

$$L_\alpha = -\Delta_\alpha + 2x\cdot\nabla.$$

Note that Δ_α, when restricted to the even subspace

$$\{f \in \mathcal{C}^1(\mathbb{R}^d)\colon \forall\, j = 1, \ldots, d,\ f(x) = f(\sigma_j x)\} \tag{1}$$

coincides with the multi-dimensional Bessel differential operator

$$\sum_{j=1}^{d} \Big(\partial_j^2 + \frac{2\alpha_j + 1}{x_j} \partial_j \Big),$$

and consequently L_α reduces to the Laguerre-type operator

$$L_\alpha = -\Delta + 2x\cdot\nabla - \sum_{j=1}^{d} \frac{2\alpha_j + 1}{x_j} \frac{\partial}{\partial x_j}. \tag{2}$$

The corresponding measure μ_α has the form

$$\mathrm{d}\mu_\alpha(x) = \prod_{j=1}^{d} |x_j|^{2\alpha_j+1} \mathrm{e}^{-x_j^2}\, \mathrm{d}x_j, \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

We denote by $L^p(\mathbb{R}^d, \mathrm{d}\mu_\alpha)$, $1 \leqslant p \leqslant \infty$, the Lebesgue space constituted of measurable functions on $\mathbb{R}^d$. By $\langle f, g\rangle_\alpha$ we mean $\int_{\mathbb{R}^d} f(x)\overline{g(x)}\, \mathrm{d}\mu_\alpha(x)$ whenever the integral makes sense.

Given $\alpha \in [-\frac{1}{2}, \infty)^d$, the associated generalized Hermite polynomials (see [1], [9], [10]) are tensor products

$$\mathcal{H}_n^\alpha(x) = \mathcal{H}_{n_1}^{\alpha_1} \times \ldots \times \mathcal{H}_{n_d}^{\alpha_d}, \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d,\ n = (n_1, \ldots, n_d) \in \mathbb{N}^d,$$

where $\mathcal{H}_{n_i}^{\alpha_i}$ are the one-dimensional generalized Hermite polynomials

$$\mathcal{H}_{2n_i}^{\alpha_i}(x_i) = (-1)^{n_i}\Big(\frac{n_i!}{\Gamma(n_i+\alpha_i+1)}\Big)^{1/2} L_{n_i}^{\alpha_i}(x_i^2),$$
$$\mathcal{H}_{2n_i+1}^{\alpha_i}(x_i) = (-1)^{n_i}\Big(\frac{n_i!}{\Gamma(n_i+\alpha_i+2)}\Big)^{1/2} x_i L_{n_i}^{\alpha_i+1}(x_i^2),$$

here $L_{n_i}^{\alpha_i}$ denotes the Laguerre polynomial of degree n_i and order α_i (see [4]).

The system $\{\mathcal{H}_n^\alpha\colon\ n\in\mathbb{N}^d\}$ is an orthonormal basis in $L^2(\mathbb{R}^d,\mathrm{d}\mu_\alpha)$ consisting of eigenfunctions of L_α (see [10]), recall that

$$L_\alpha\mathcal{H}_n^\alpha = 2|n|\mathcal{H}_n^\alpha,$$

where we denote $|n| = n_1+\ldots+n_d$.

We define the jth partial "derivative" $\delta_{\alpha,j}$, for $1\leqslant j\leqslant d$, related to L_α, by

$$\delta_{\alpha,j} = T_j^\alpha.$$

The formal adjoint of $\delta_{\alpha,j}$ in $L^2(\mathbb{R}^d,\mathrm{d}\mu_\alpha)$ is

$$\delta_{\alpha,j}^* = -T_j^\alpha + 2x_j.$$

This precisely means that

$$\langle\delta_{\alpha,j}f,g\rangle_\alpha = \langle f,\delta_{\alpha,j}^*g\rangle_\alpha,\quad f,g\in\mathcal{C}_c^1(\mathbb{R}^d).$$

A direct computation shows that

$$L_\alpha + (2|\alpha|+2d) = \frac{1}{2}\sum_{j=1}^{d}(\delta_{\alpha,j}^*\delta_{\alpha,j}+\delta_{\alpha,j}\delta_{\alpha,j}^*).$$

We recall that for $1\leqslant j\leqslant d$ (see [7])

$$\delta_{\alpha,j}\mathcal{H}_n^\alpha = m(n_j,\alpha_j)\mathcal{H}_{n-e_j}^\alpha,$$

where

$$m(n_j,\alpha_j) = \begin{cases}\sqrt{2n_j} & \text{if } n_j \text{ is even,}\\ \sqrt{2n_j+4\alpha_j+2} & \text{if } n_j \text{ is odd,}\end{cases}$$

by convention, $\mathcal{H}_{n-e_j}^\alpha\equiv 0$ if $n_j=0$.

Note that for each j the system $\{\delta_{\alpha,j}\mathcal{H}_n^\alpha\colon\ n_j \geqslant 1\}$ is orthogonal in $L^2(\mathbb{R}^d, \mathrm{d}\mu_\alpha)$.

The self-adjoint extension of L_α initially considered on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ is given by the operator

$$\mathcal{L}_\alpha f = \sum_{n\in\mathbb{N}^d} 2|n|\langle f, \mathcal{H}_n^\alpha\rangle_\alpha \mathcal{H}_n^\alpha,$$

and defined on the domain

$$\mathrm{Dom}(\mathcal{L}_\alpha) = \Big\{ f \in L^2(\mathbb{R}^d, \mathrm{d}\mu_\alpha)\colon \sum_{n\in\mathbb{N}^d} |2|n|\langle f, \mathcal{H}_n^\alpha\rangle_\alpha|^2 < \infty \Big\}.$$

The spectrum of $\mathcal{L}_\alpha$ is the discrete set $\{2m\colon\ m \in \mathbb{N}\}$, and the spectral decomposition of $\mathcal{L}_\alpha$ is

$$\mathcal{L}_\alpha f = \sum_{m=0}^{\infty} 2m\mathcal{P}_m^\alpha f, \quad f \in \mathrm{Dom}(\mathcal{L}_\alpha),$$

where the spectral projections are

$$\mathcal{P}_m^\alpha f = \sum_{|n|=m} \langle f, \mathcal{H}_n^\alpha\rangle_\alpha \mathcal{H}_n^\alpha.$$

Observe that since zero is an eigenvalue of $\mathcal{L}_\alpha$, then denoting by Π_0 the orthogonal projection operator onto the orthogonal complement of the subspace spanned by the constant functions, it is also given

$$\Pi_0 f = f - \int_{\mathbb{R}^d} f(y)\,\mathrm{d}\mu_\alpha(y).$$

We have for $M \in \mathbb{N}^*$,

$$\mathcal{L}_\alpha^{-M/2}\Pi_0 f = \sum_{m=1}^{\infty} (2m)^{-M/2}\mathcal{P}_m^\alpha f,$$

and this operator is bounded on $L^2(\mathbb{R}^d, \mathrm{d}\mu_\alpha)$.

The Riesz transforms related to the Dunkl harmonic oscillator and to the Dunkl Ornstein-Uhlenbeck operator have been intensively studied in recent years by many authors, see e.g. [6], [7], [8], and references therein. In [7] Nowak, Roncal and Stempak introduced the Riesz transforms of order one related to the Dunkl Ornstein-Uhlenbeck operator L_α and they proved that these transforms are L^p bounded with $1 < p < \infty$ in the one-dimensional setting. The aim of this paper is to present an extension of this result to the Riesz-Dunkl transforms of order M with $M \in \mathbb{N}^*$. We note that for technical reasons, we have considered the $\mathbb{Z}_2^d$ group case.

According to a general principle, see [3], we now define higher order Riesz-Dunkl transforms in the following way: let $\tau = (\tau_1, \ldots, \tau_d) \in \mathbb{N}^d$ be a multi-index and $\alpha = (\alpha_1, \ldots, \alpha_d) \in [-\frac{1}{2}, \infty)^d$. Then for $M \in \mathbb{N}^*$, the family of the Riesz-Dunkl transforms $(\mathcal{R}_\tau^\alpha)$ of order M such that $|\tau| = \tau_1 + \ldots + \tau_d = M$ (the length of τ) is given by

$$\mathcal{R}^{\alpha,M} = (\mathcal{R}_\tau^\alpha)_{|\tau|=M} = (\delta_\alpha^\tau \mathcal{L}_\alpha^{-M/2} \Pi_0)_{|\tau|=M},$$

where

$$\delta_\alpha^\tau = \delta_{\alpha,1}^{\tau_1} \ldots \delta_{\alpha,d}^{\tau_d}.$$

In the one-dimensional case, to prove our main result Theorem 1, we split a function f into its even, and odd parts f_e and f_o and we observe that if the order m is odd then the Riesz-Dunkl transform of order $m \in \mathbb{N}^*$ $\mathcal{R}_m^\alpha f_\mathrm{e}$ is odd and $\mathcal{R}_m^\alpha f_\mathrm{o}$ is even, and if the order m is even then $\mathcal{R}_m^\alpha f_\mathrm{e}$ is even and $\mathcal{R}_m^\alpha f_\mathrm{o}$ is odd.

Due to these symmetries we consider the operators $\mathcal{R}_{\mathrm{e},m}^\alpha$ and $\mathcal{R}_{\mathrm{o},m}^\alpha$ on $L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$ emerging naturally from restrictions of $\mathcal{R}_m^\alpha$ to the subspaces of $L^2(\mathbb{R}, \mathrm{d}\mu_\alpha)$ of even and odd functions, respectively.

The L^p-boundedness of the even and odd Riesz-Dunkl operators follows from the L^p-boundedness of the Riesz-Laguerre-type transforms and of shift and multiplier operators depending on m.

In the $\mathbb{Z}_2^d$ group case we investigate a natural variant of the Dunkl Ornstein-Uhlenbeck operator by means of the Dunkl gradient rather than the Euclidean one, then we obtain higher order Riesz-Dunkl transforms which are L^2-contractions. The L^p-boundedness of these Riesz-Dunkl transforms is proved in the one-dimensional case.

The paper is organized as follows. In Section 2 we give the expansions of higher order Riesz transforms associated with the Dunkl Ornstein-Uhlenbeck operator of $f = \sum\limits_{n\in\mathbb{N}^d} \langle f, \mathcal{H}_n^\alpha \rangle \mathcal{H}_n^\alpha$ on $L^2(\mathbb{R}^d, \mathrm{d}\mu_\alpha)$ and we study the L^2-boundedness of this transform.

In Section 3, for the one-dimensional case, we establish L^p-boundedness of shift operators, we define and study the Riesz-Laguerre-type transforms of order $m \in \mathbb{N}^*$. After that, we prove our main result.

Finally in Section 4, we discuss higher order Riesz transforms related to the alternative Dunkl Ornstein-Uhlenbeck operator by the methods developed in the previous section.

2. Higher order Riesz transforms associated with the Dunkl Ornstein-Uhlenbeck operator

Let $\tau = (\tau_1, \dots, \tau_d) \in \mathbb{N}^d$ be a multi-index and $\alpha = (\alpha_1, \dots, \alpha_d) \in [-\frac{1}{2}, \infty)^d$, we denote by δ_α^τ the operator

$$\delta_\alpha^\tau = \delta_{\alpha,1}^{\tau_1} \dots \delta_{\alpha,d}^{\tau_d}.$$

It is natural to define the Riesz transform of order $M \in \mathbb{N}^*$ for the Dunkl Ornstein-Uhlenbeck operator by

$$\mathcal{R}^{\alpha,M} = (\mathcal{R}_\tau^\alpha)_{|\tau|=M} = (\delta_\alpha^\tau \mathcal{L}_\alpha^{-M/2} \Pi_0)_{|\tau|=M},$$

where $|\tau| = \tau_1 + \dots + \tau_d$ is the length of τ.

In order to study the higher order Riesz transforms $\mathcal{R}^{\alpha,M}$ of order $M \in \mathbb{N}^*$, we shall see how δ_α^τ acts on $\mathcal{H}_n^\alpha$.

We begin by observing that

$$\delta_{\alpha,j}^{\tau_j} \mathcal{H}_n^\alpha = m(n_j, \alpha_j, \tau_j) \mathcal{H}_{n-\tau_j e_j}^\alpha$$

by the convention that $\mathcal{H}_{n-\tau_j e_j}^\alpha \equiv 0$ if $n_j < \tau_j$, so we take $m(n_j, \alpha_j, \tau_j) = 0$ if $n_j < \tau_j$.

Otherwise $m(n_j, \alpha_j, \tau_j)$ is given by the next lemma.

Lemma 1. (i) *If $\tau_j = 1$, then $m(n_j, \alpha_j, 1) = m(n_j, \alpha_j)$ given by*

$$m(n_j, \alpha_j) = \begin{cases} \sqrt{2n_j} & \text{if } n_j \text{ is even,} \\ \sqrt{2n_j + 4\alpha_j + 2} & \text{if } n_j \text{ is odd.} \end{cases}$$

(ii) *If $2 \leqslant \tau_j \leqslant n_j$ and τ_j even, then*

$$m(n_j, \alpha_j, \tau_j) = \begin{cases} \sqrt{2^{\tau_j} n_j (n_j - 2) \dots (n_j - \tau_j + 2)(n_j + 2\alpha_j)} \\ \qquad \times \sqrt{(n_j + 2\alpha_j - 2) \dots (n_j + 2\alpha_j - \tau_j + 2)} & \text{if } n_j \text{ is even,} \\ \sqrt{2^{\tau_j} (n_j - 1)(n_j - 3) \dots (n_j - \tau_j + 1)(n_j + 2\alpha_j + 1)} \\ \qquad \times \sqrt{(n_j + 2\alpha_j - 1) \dots (n_j + 2\alpha_j - \tau_j + 3)} & \text{if } n_j \text{ is odd.} \end{cases}$$

(iii) *If $3 \leqslant \tau_j \leqslant n_j$ and τ_j odd, then*

$$m(n_j, \alpha_j, \tau_j) = \begin{cases} \sqrt{2^{\tau_j} n_j (n_j - 2) \dots (n_j - \tau_j + 1)(n_j + 2\alpha_j)} \\ \qquad \times \sqrt{(n_j + 2\alpha_j - 2) \dots (n_j + 2\alpha_j - \tau_j + 3)} & \text{if } n_j \text{ is even,} \\ \sqrt{2^{\tau_j} (n_j - 1)(n_j - 3) \dots (n_j - \tau_j + 2)(n_j + 2\alpha_j + 1)} \\ \qquad \times \sqrt{(n_j + 2\alpha_j - 1) \dots (n_j + 2\alpha_j - \tau_j + 2)} & \text{if } n_j \text{ is odd.} \end{cases}$$

P r o o f. We have in [7] that for $1 \leqslant j \leqslant d$

$$\delta_{\alpha,j}\mathcal{H}_n^\alpha = m(n_j,\alpha_j)\mathcal{H}_{n-e_j}^\alpha,$$

where

$$m(n_j,\alpha_j) = \begin{cases} \sqrt{2n_j} & \text{if } n_j \text{ is even,} \\ \sqrt{2n_j+4\alpha_j+2} & \text{if } n_j \text{ is odd,} \end{cases}$$

by convention, $\mathcal{H}_{n-e_j}^\alpha \equiv 0$ if $n_j = 0$. So we obtain (i).

To prove (ii) and (iii) we give some computations of $\delta_{\alpha,j}^{\tau_j}\mathcal{H}_n^\alpha$:

If n_j is even we can see that

$$\begin{aligned} \delta_{\alpha,j}\mathcal{H}_n^\alpha &= \sqrt{2n_j}\mathcal{H}_{n-e_j}^\alpha \\ \delta_{\alpha,j}^2\mathcal{H}_n^\alpha &= \sqrt{2^2 n_j(n_j+2\alpha_j)}\mathcal{H}_{n-2e_j}^\alpha \\ \delta_{\alpha,j}^3\mathcal{H}_n^\alpha &= \sqrt{2^3 n_j(n_j+2\alpha_j)(n_j-2)}\mathcal{H}_{n-3e_j}^\alpha. \end{aligned}$$

On the other hand, if n_j is odd we show that

$$\begin{aligned} \delta_{\alpha,j}\mathcal{H}_n^\alpha &= \sqrt{2(n_j+2\alpha_j+1)}\mathcal{H}_{n-e_j}^\alpha \\ \delta_{\alpha,j}^2\mathcal{H}_n^\alpha &= \sqrt{2^2(n_j+2\alpha_j+1)(n_j-1)}\mathcal{H}_{n-2e_j}^\alpha \\ \delta_{\alpha,j}^3\mathcal{H}_n^\alpha &= \sqrt{2^3(n_j+2\alpha_j+1)(n_j-1)(n_j+2\alpha_j-1)}\mathcal{H}_{n-3e_j}^\alpha. \end{aligned}$$

Thus, by iteration method, we deduce the results. □

Lemma 2. *For $\tau = (\tau_1,\ldots,\tau_d) \in \mathbb{N}^d$ and $\alpha = (\alpha_1,\ldots,\alpha_d) \in [-\frac{1}{2},\infty)^d$, we have*

$$\delta_\alpha^\tau\mathcal{H}_n^\alpha = (\delta_{\alpha,1}^{\tau_1}\delta_{\alpha,2}^{\tau_2}\ldots\delta_{\alpha,d}^{\tau_d})\mathcal{H}_n^\alpha = \mathcal{M}(n,\alpha,\tau)\mathcal{H}_{n-\sum_{j=1}^d \tau_j e_j}^\alpha,$$

where

$$\mathcal{M}(n,\alpha,\tau) = \prod_{j=1}^d m(n_j,\alpha_j,\tau_j).$$

Also, for $\tau = (\tau_1,\ldots,\tau_d) \in \mathbb{N}^d$, we have

$$0 \leqslant \mathcal{M}(n,\alpha,\tau) \leqslant C(|n|+2|\alpha|+1)^{|\tau|/2},$$

where $|\alpha| = \sum\limits_{j=1}^d |\alpha_j|$ and C is a positive constant independent of significant quantities.

And $\mathcal{M}(n,\alpha,\tau)$ vanishes if and only if there exists $1 \leqslant j \leqslant d$ such that $n_j - \tau_j < 0$.

P r o o f. A direct composition gives

$$(\delta_{\alpha,1}^{\tau_1}\delta_{\alpha,2}^{\tau_2}\ldots\delta_{\alpha,d}^{\tau_d})\mathcal{H}_n^\alpha=\prod_{j=1}^d m(n_j,\alpha_j,\tau_j)\mathcal{H}^\alpha_{n-\sum_{j=1}^d\tau_je_j},$$

where $m(n_j,\alpha_j,\tau_j)$ is defined as in the previous lemma.

For $1\leqslant j\leqslant d$, we see that each factor under the square root in the expression of $m(n_j,\alpha_j,\tau_j)$ is bounded by $|n|+2|\alpha|+1$ and there is τ_j factors, so

$$m(n_j,\alpha_j,\tau_j)\leqslant C(|n|+2|\alpha|+1)^{\tau_j/2}.$$

We deduce that

$$\mathcal{M}(n,\alpha,\tau)\leqslant C(|n|+2|\alpha|+1)^{|\tau|/2}.$$

□

The higher order Riesz-Dunkl transform $\mathcal{R}_\tau^\alpha$ of $\mathcal{H}_n^\alpha$ is defined by

$$\mathcal{R}_\tau^\alpha\mathcal{H}_n^\alpha=\frac{\mathcal{M}(n,\alpha,\tau)}{(2|n|)^{|\tau|/2}}\mathcal{H}^\alpha_{n-\sum_{j=1}^d\tau_je_j}.$$

So the higher order Riesz-Dunkl transform $\mathcal{R}_\tau^\alpha$ of $f=\sum\limits_{n\in\mathbb{N}^d}\langle f,\mathcal{H}_n^\alpha\rangle_\alpha\mathcal{H}_n^\alpha$ in $L^2(\mathbb{R}^d,\mathrm{d}\mu_\alpha)$ is given by

$$\mathcal{R}_\tau^\alpha f=\sum_{n\in\mathbb{N}^d,|n|>0}\frac{\mathcal{M}(n,\alpha,\tau)}{(2|n|)^{|\tau|/2}}\langle f,\mathcal{H}_n^\alpha\rangle_\alpha\mathcal{H}^\alpha_{n-\sum_{j=1}^d\tau_je_j}. \tag{3}$$

From equality (3) and Lemma 2, the L^2-boundedness can easily be seen.

Remark 1. We note that $\mathcal{R}_\tau^\alpha$ is not a contraction on $L^2(\mathbb{R}^d,\mathrm{d}\mu_\alpha)$ if $\alpha\in[-\frac{1}{2},\infty)^d$.

3. $\mathbb{Z}_2$ -Higher order Riesz transforms associated with the Dunkl Ornstein-Uhlenbeck operator

Our main result, Theorem 1 below, is an extension to higher order of Nowak, Roncal and Stempak's L^p results given in [7] for the Riesz transform $\mathcal{R}_1^\alpha$ related to the Dunkl Ornstein-Uhlenbeck operator in one-dimension setting.

Theorem 1. *Let $d=1$ and assume that $\alpha\geqslant-\frac{1}{2}$. Then for each $1<p<\infty$ and $m\in\mathbb{N}^*$, the Riesz-Dunkl transform $\mathcal{R}_m^\alpha$ of order m, associated with the Dunkl Ornstein-Uhlenbeck operator, defined on $L^2(\mathbb{R},\mathrm{d}\mu_\alpha)$ by (3), extends to a bounded operator on $L^p(\mathbb{R},\mathrm{d}\mu_\alpha)$.*

First of all we recall some results in the one-dimensional setting and in the case when the order of the Riesz-Dunkl transform is one.

By the change of variable $x \mapsto x^2$ on $\mathbb{R}_+$, the authors in [7] translate some results from the classical Laguerre setting to the so called "squared" Laguerre setting.

For $\alpha \geqslant -\frac{1}{2}$, the restriction of μ_α to $\mathbb{R}_+$ will be denoted by the same symbol. The Dunkl Ornstein-Uhlenbeck operator (2) in this case is

$$\mathbb{L}_\alpha = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{2\alpha + 1 - 2x^2}{x}\frac{\mathrm{d}}{\mathrm{d}x},$$

which is positive and symmetric in $L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$. The Laguerre polynomials $L_n^\alpha(x^2)$, $n \in \mathbb{N}$, are eigenfunctions of $\mathbb{L}_\alpha$,

$$\mathbb{L}_\alpha L_n^\alpha(x^2) = 4n L_n^\alpha(x^2),$$

and the set $\{\mathbb{L}_\alpha L_n^\alpha(x^2)\colon\ n \in \mathbb{N}\}$ forms an orthogonal basis in $L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$.

Also the authors in [7] considered the polynomials

$$\varphi_n^\alpha(x) = \Big(\frac{2n!}{\Gamma(n+\alpha+1)}\Big)^{1/2} L_n^\alpha(x^2)$$

and

$$\psi_n^\alpha(x) = \Big(\frac{2n!}{\Gamma(n+\alpha+2)}\Big)^{1/2} x L_n^{\alpha+1}(x^2),$$

which form two orthonormal bases in $L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$.

These polynomials φ_n^α and ψ_n^α coincide, up to constant factors independent of n and α, with the generalized Hermite polynomials $\mathcal{H}_{2n}^\alpha$ and $\mathcal{H}_{2n+1}^\alpha$, respectively.

The definition of the first order Riesz-Dunkl transform is inherited from the classical Laguerre setting given by [5], and induced by the mapping

$$\mathcal{R}_\varphi^\alpha\colon\ \varphi_n^\alpha \to -\psi_{n-1}^\alpha, \quad n \in \mathbb{N},$$

where $\psi_{-1}^\alpha \equiv 0$.

Muckenhoupt proved in [5] the following:

Theorem 2. *Let $\alpha \geqslant -\frac{1}{2}$ and $1 < p < \infty$. Then*

$$\|\mathcal{R}_\varphi^\alpha f\|_{L^p(\mathbb{R}_+, \mathrm{d}\mu_\alpha)} \leqslant C\|f\|_{L^p(\mathbb{R}_+, \mathrm{d}\mu_\alpha)},$$

with a constant C independent of $f \in L^2 \cap L^p(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$.

In [7] the authors give the adjoint operator of $\mathcal{R}^\alpha_\varphi$, taken in $L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$, by the mapping

$$\mathcal{R}^\alpha_\psi\colon\ \psi^\alpha_n \to -\varphi^\alpha_{n+1}, \quad n \in \mathbb{N},$$

they proved by Theorem 2 and duality that for $1 < p < \infty$

$$\|\mathcal{R}^\alpha_\psi f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)} \leqslant C\|f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)}, \tag{4}$$

with a constant C independent of $f \in L^2 \cap L^p(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$.

Also, they translate the multiplier theorem below, given in [3], to the squared Laguerre setting after restricting it to one dimension and taking $\beta = 1$,

Theorem 3. *Let $1 < p < \infty$ and $\alpha \geqslant -\frac{1}{2}$. Assume that h is an analytic function in a neighborhood of the origin. Let $\{\xi(n)\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\xi(n) = h(n^{-1})$ for $n \geqslant n_0 \geqslant 0$. Then the multiplier operator given by*

$$\mathcal{M}_\xi\colon\ \varphi^\alpha_n \to \xi(n)\varphi^\alpha_n$$

satisfies

$$\|\mathcal{M}_\xi f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)} \leqslant C\|f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)},$$

with a constant C independent of $f \in L^2 \cap L^p(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$.

In our context, in order to prove our Theorem 1 we consider the right and left shift operators of order m, for $m \geqslant 1$, related to the system $\{\varphi^\alpha_n\}$, respectively denoted by

$$\mathcal{S}_{r,m}\colon\ \varphi^\alpha_n \to \varphi^\alpha_{n+m}$$

and

$$\mathcal{S}_{l,m}\colon\ \varphi^\alpha_n \to \varphi^\alpha_{n-m},$$

where $\varphi^\alpha_{n-m} \equiv 0$ if $n - m < 0$.

We establish L^p-boundedness of these shift operators, which may be regarded as an extension of Theorem 6.3 stated in [7].

Theorem 4. *Let $1 < p < \infty$ and $\alpha \geqslant -\frac{1}{2}$. Then the shift operators of order $m \in \mathbb{N}^*$ defined above satisfy*

$$\|\mathcal{S}_{l,m} f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)} \leqslant C\|f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)}$$

and

$$\|\mathcal{S}_{r,m} f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)} \leqslant C\|f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)},$$

with a constant C independent of $f \in L^2 \cap L^p(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$.

P r o o f. If $m \leqslant n$, we can see that

$$\mathcal{S}_{l,m}\colon\ \varphi_n^\alpha \to \varphi_{n-m}^\alpha,$$

so

$$\mathcal{S}_{l,m}(\varphi_n^\alpha) = (\mathcal{S}_l)^m(\varphi_n^\alpha),$$

where $\mathcal{S}_l$ is the left shift operator of order 1 given in [7] and verifies that

$$\|\mathcal{S}_l f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)} \leqslant C\|f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)}.$$

We deduce that

$$\|\mathcal{S}_{l,m} f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)} \leqslant C_m\|f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)},$$

where C_m is a positive constant depending on m.

Similarly we have

$$\mathcal{S}_{r,m}\colon\ \varphi_n^\alpha \to \varphi_{n+m}^\alpha$$

so

$$\mathcal{S}_{r,m}(\varphi_n^\alpha) = (\mathcal{S}_r)^m(\varphi_n^\alpha)$$

with $\mathcal{S}_r$ the right shift operator of order 1 which verifies that

$$\|\mathcal{S}_r f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)} \leqslant C\|f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)}.$$

We deduce that

$$\|\mathcal{S}_{r,m} f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)} \leqslant C(m)\|f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)},$$

where $C(m)$ is a positive constant depending on m. □

Now we define the operators $\mathcal{R}_{\varphi,m}^\alpha$ and $\mathcal{R}_{\psi,m}^\alpha$, for $m \geqslant 1$, induced, respectively, by the mappings

$$\mathcal{R}_{\varphi,m}^\alpha\colon\ \varphi_n^\alpha \to (-1)^m\psi_{n-m}^\alpha, \quad n \in \mathbb{N},$$

where $\psi_{n-m}^\alpha \equiv 0$ if $m > n$, and

$$\mathcal{R}_{\psi,m}^\alpha\colon\ \psi_n^\alpha \to (-1)^m\varphi_{n+m}^\alpha, \quad n \in \mathbb{N}.$$

We establish L^p-boundedness of these transforms in the theorem below.

Theorem 5. *Let $1 < p < \infty$, $\alpha \geqslant -\frac{1}{2}$ and $m \in \mathbb{N}^*$. Then*

$$\|\mathcal{R}_{\varphi,m}^\alpha f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)} \leqslant C\|f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)}$$

and

$$\|\mathcal{R}_{\psi,m}^\alpha f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)} \leqslant C\|f\|_{L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)},$$

with a constant C independent of $f \in L^2 \cap L^p(\mathbb{R}_+,\mathrm{d}\mu_\alpha)$.

P r o o f. We have, for $n \geqslant m$

$$\mathcal{R}^{\alpha}_{\varphi,m}(\varphi^{\alpha}_n) = (-1)^{m-1}\mathcal{R}^{\alpha}_{\varphi}\mathcal{S}_{l,m-1}(\varphi^{\alpha}_n).$$

We can deduce the L^p-boundedness of $\mathcal{R}^{\alpha}_{\varphi,m}$ by Theorem 2 and Theorem 4.

On the other hand

$$\mathcal{R}^{\alpha}_{\psi,m}(\psi^{\alpha}_n) = (-1)^{m-1}\mathcal{S}_{r,m-1}\mathcal{R}^{\alpha}_{\psi}(\psi^{\alpha}_n),$$

so the L^p-boundedness of $\mathcal{R}^{\alpha}_{\psi,m}$ is a consequence of Theorem 4 and inequality (4). □

We are now in a position to prove Theorem 1.

P r o o f of Theorem 1. In the one-dimensional setting for $\alpha \geqslant -\frac{1}{2}$ and for the Riesz-Dunkl transform of order $m \in \mathbb{N}^*$, defined on $L^2(\mathbb{R}, \mathrm{d}\mu_\alpha)$ by

$$\mathcal{R}^{\alpha,m} = \mathcal{R}^{\alpha}_m = \delta^m_\alpha \mathcal{L}^{-m/2}_\alpha \Pi_0,$$

and for

$$f = \sum_{n\in\mathbb{N}} \langle f, \mathcal{H}^{\alpha}_n\rangle_\alpha \mathcal{H}^{\alpha}_n,$$

we have

$$\mathcal{R}^{\alpha}_m f = \sum_{n>0} \frac{\mathcal{M}(n,\alpha,m)}{(2n)^{m/2}} \langle f, \mathcal{H}^{\alpha}_n\rangle_\alpha \mathcal{H}^{\alpha}_{n-m}. \tag{5}$$

Given $f \in L^2 \cap L^p(\mathbb{R}, \mathrm{d}\mu_\alpha)$, we decompose it into its even and odd parts,

$$f = f_{\mathrm{e}} + f_{\mathrm{o}}.$$

Then to prove Theorem 1 it is sufficient to show the L^p estimates

$$\|\mathcal{R}^{\alpha}_m f_{\mathrm{e}}\|_{L^p(\mathbb{R},\mathrm{d}\mu_\alpha)} \leqslant C\|f_{\mathrm{e}}\|_{L^p(\mathbb{R},\mathrm{d}\mu_\alpha)}$$

and

$$\|\mathcal{R}^{\alpha}_m f_{\mathrm{o}}\|_{L^p(\mathbb{R},\mathrm{d}\mu_\alpha)} \leqslant C\|f_{\mathrm{o}}\|_{L^p(\mathbb{R},\mathrm{d}\mu_\alpha)}.$$

Since the generalized Hermite polynomial $\mathcal{H}^{\alpha}_n$ is even if n is even and odd for n odd, expansions of f_{e} and f_{o} are given only by even and odd $\mathcal{H}^{\alpha}_n$, respectively.

In view of (5), we observe that if the order m is odd, then $\mathcal{R}^{\alpha}_m f_{\mathrm{e}}$ is odd and $\mathcal{R}^{\alpha}_m f_{\mathrm{o}}$ is even.

And if the order m is even, then $\mathcal{R}^{\alpha}_m f_{\mathrm{e}}$ is even and $\mathcal{R}^{\alpha}_m f_{\mathrm{o}}$ is odd.

Due to these symmetries we consider the operators $\mathcal{R}^{\alpha}_{\mathrm{e},m}$ and $\mathcal{R}^{\alpha}_{\mathrm{o},m}$ on $L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$ emerging naturally from restrictions of $\mathcal{R}^{\alpha}_{m}$ to the subspaces of $L^2(\mathbb{R}, \mathrm{d}\mu_\alpha)$ of even and odd functions, respectively.

Observe that by relation (5) we have:

(i) If m is even, then

$$\mathcal{R}^{\alpha}_{\mathrm{e},m}\colon\ \varphi^{\alpha}_{n} \to \frac{\mathcal{M}(2n,\alpha,m)}{(4n)^{m/2}}\varphi^{\alpha}_{n-m/2},$$

and

$$\mathcal{R}^{\alpha}_{\mathrm{o},m}\colon\ \psi^{\alpha}_{n} \to \frac{\mathcal{M}(2n+1,\alpha,m)}{(4n+2)^{m/2}}\psi^{\alpha}_{n-m/2}.$$

Thus we can see that

$$\mathcal{R}^{\alpha}_{\mathrm{e},m}(\varphi^{\alpha}_{n}) = \mathcal{M}_{\xi_1}\mathcal{S}_{l,m/2}(\varphi^{\alpha}_{n})$$

with

$$\xi_1(n) = \frac{\mathcal{M}(2n,\alpha,m)}{(4n)^{m/2}}.$$

And

$$\mathcal{R}^{\alpha}_{\mathrm{o},m}(\psi^{\alpha}_{n}) = \mathcal{R}^{\alpha}_{\varphi,m/2+1}\mathcal{M}_{\xi_2}\mathcal{R}^{\alpha}_{\psi}(\psi^{\alpha}_{n})$$

with

$$\xi_2(n) = (-1)^{m/2}\frac{\mathcal{M}(2n+1,\alpha,m)}{(4n+2)^{m/2}}.$$

Consequently, the relevant L^p estimate follows by relation (4) accordingly with Theorems 3, 4 and 5.

(ii) On the other hand, if m is odd, then

$$\mathcal{R}^{\alpha}_{\mathrm{e},m}\colon\ \varphi^{\alpha}_{n} \to (-1)^{(m+1)/2}\frac{\mathcal{M}(2n,\alpha,m)}{(4n)^{m/2}}\psi^{\alpha}_{n-(m+1)/2}$$

and

$$\mathcal{R}^{\alpha}_{\mathrm{o},m}\colon\ \psi^{\alpha}_{n} \to (-1)^{(m-1)/2}\frac{\mathcal{M}(2n+1,\alpha,m)}{(4n+2)^{m/2}}\varphi^{\alpha}_{n-(m-1)/2}.$$

Thus we can see that

$$\mathcal{R}^{\alpha}_{\mathrm{e},m}(\varphi^{\alpha}_{n}) = \mathcal{R}_{\varphi,(m+1)/2}\mathcal{M}_{\xi_3}(\varphi^{\alpha}_{n}),$$

with

$$\xi_3(n) = \frac{\mathcal{M}(2n,\alpha,m)}{(4n)^{m/2}}$$

and

$$\mathcal{R}^\alpha_{o,m}(\psi^\alpha_n) = \mathcal{M}_{\xi_4}\mathcal{S}_{l,(m+1)/2}\mathcal{R}^\alpha_\psi(\psi^\alpha_n),$$

with

$$\xi_4(n) = (-1)^{(m+1)/2}\frac{\mathcal{M}(2n+1,\alpha,m)}{(4n+2)^{m/2}}.$$

Thus we see again that the relevant L^p estimate follows by relation (4) and Theorems 3, 4 and 5. □

Remark 2. We conjecture in our context that an analogue of Theorem 1 holds for arbitrary dimension d and $\alpha \in [-\frac{1}{2},\infty)^d$.

4. Higher order Riesz transforms associated with the alternative Dunkl Ornstein-Uhlenbeck operator

In this section we consider the alternative Dunkl Ornstein-Uhlenbeck operator given in [7] by

$$\widetilde{L}_\alpha = -\Delta_\alpha + 2x\cdot\nabla_\alpha,$$

where the Dunkl gradient ∇_α is defined by

$$\nabla_\alpha = (T^\alpha_1,\dots,T^\alpha_d).$$

The authors in [7] define, in the $\mathbb{Z}_2^d$ group case, the Riesz-Dunkl transforms of order one associated with $\widetilde{L}_\alpha$. These transforms are contractions in $L^2(\mathbb{R}^d, \mathrm{d}\mu_\alpha)$, which is not true in the case of L_α.

Similarly as L_α, when restricted to the even subspace (1), $\widetilde{L}_\alpha$ coincides with the Laguerre-type operator (2), and for $\alpha = (-\frac{1}{2},\dots,-\frac{1}{2})$ it reduces to the classic Ornstein-Uhlenbeck operator. We recall that

$$\widetilde{L}_\alpha = \sum_{j=1}^d \delta^*_{\alpha,j}\delta_{\alpha,j}.$$

It follows that $\widetilde{L}_\alpha$ is formally symmetric and nonnegative in $L^2(\mathbb{R}^d, \mathrm{d}\mu_\alpha)$.

Also, we have

$$\widetilde{L}_\alpha\mathcal{H}^\alpha_n = \Big(2|n| + \sum_{\{j:\, n_j \text{ odd}\}}(4\alpha_j+2)\Big)\mathcal{H}^\alpha_n = \Big(\sum_{j=1}^d [m(n_j,\alpha_j)]^2\Big)\mathcal{H}^\alpha_n.$$

Let $\widetilde{\mathcal{L}}_\alpha$ be the self-adjoint extension of $\widetilde{L}_\alpha$ whose spectral decomposition is given by $\mathcal{H}^\alpha_n$.

Let $\tau=(\tau_1,\ldots,\tau_d)\in\mathbb{N}^d$ be a multi-index and $\alpha=(\alpha_1,\ldots,\alpha_d)\in[-\frac{1}{2},\infty)^d$, we denote by δ_α^τ the operator

$$\delta_\alpha^\tau=\delta_{\alpha,1}^{\tau_1}\ldots\delta_{\alpha,d}^{\tau_d}.$$

It is natural to define the Riesz transform of order $M\in\mathbb{N}^*$ for the alternative Dunkl Ornstein-Uhlenbeck $\widetilde{L}_\alpha$ operator by

$$\widetilde{\mathcal{R}}^{\alpha,M}=(\widetilde{\mathcal{R}}_\tau^\alpha)_{|\tau|=M}=(\delta_\alpha^\tau\widetilde{\mathcal{L}}_\alpha^{-M/2}\Pi_0)_{|\tau|=M},$$

where $|\tau|=\tau_1+\ldots+\tau_d$ is the length of τ.

So the higher order Riesz-Dunkl transform $\widetilde{\mathcal{R}}_\tau^\alpha$ of $f=\sum_{n\in\mathbb{N}^d}\langle f,\mathcal{H}_n^\alpha\rangle_\alpha\mathcal{H}_n^\alpha$ on $L^2(\mathbb{R}^d,\mathrm{d}\mu_\alpha)$ is given by

$$\widetilde{\mathcal{R}}_\tau^\alpha f=\sum_{\substack{n\in\mathbb{N}^d\\|n|>0}}\frac{\mathcal{M}(n,\alpha,\tau)}{\Big(\sum_{j=1}^d[m(n_j,\alpha_j)]^2\Big)^{|\tau|/2}}\langle f,\mathcal{H}_n^\alpha\rangle_\alpha\mathcal{H}_{n-\sum_{j=1}^d\tau_je_j}^\alpha. \tag{6}$$

From formula (6) and Lemma 2, the L^2-boundedness can easily be seen directly.

Remark 3. By Plancherel's theorem the mapping

$$f\to\Big(\sum_{|\tau|=M}|\widetilde{\mathcal{R}}_\tau^\alpha f|^2\Big)^{1/2}$$

is a contraction on $L^2(\mathbb{R}^d,\mathrm{d}\mu_\alpha)$.

We now state an analogue of Theorem 1 in the context of $\widetilde{L}_\alpha$.

Theorem 6. *Let $d=1$ and assume that $\alpha\geqslant-\frac{1}{2}$. Then for each $1<p<\infty$ and $m\in\mathbb{N}^*$, the Riesz-Dunkl transform $\widetilde{\mathcal{R}}_m^\alpha$ of order m, associated with the alternative Dunkl Ornstein-Uhlenbeck operator, defined on $L^2(\mathbb{R},\mathrm{d}\mu_\alpha)$ by (6), extends to a bounded operator on $L^p(\mathbb{R},\mathrm{d}\mu_\alpha)$.*

Proof. We proceed as in the proof of Theorem 1 and arrive at the operators $\widetilde{\mathcal{R}}_{\mathrm{e},m}^\alpha$ and $\widetilde{\mathcal{R}}_{\mathrm{o},m}^\alpha$ on $L^2(\mathbb{R}_+,\mathrm{d}\mu_\alpha)$. Then to prove this theorem, it is sufficient to show the L^p estimates for these two operators.

We recall that in one-dimensional setting, for $\alpha\geqslant-\frac{1}{2}$ and for the Riesz-Dunkl transform of order $m\in\mathbb{N}^*$, defined on $L^2(\mathbb{R}_+,\mathrm{d}\mu_\alpha)$ by

$$\widetilde{\mathcal{R}}^{\alpha,m}=\widetilde{\mathcal{R}}_m^\alpha=\delta_\alpha^m\widetilde{\mathcal{L}}_\alpha^{-m/2}\Pi_0,$$

and for

$$f = \sum_{n\in\mathbb{N}} \langle f, \mathcal{H}_n^\alpha \rangle_\alpha \mathcal{H}_n^\alpha,$$

we have

$$\widetilde{\mathcal{R}}_m^\alpha f = \sum_{n>0} \frac{\mathcal{M}(n,\alpha,m)}{[m(n,\alpha)]^m} \langle f, \mathcal{H}_n^\alpha \rangle_\alpha \mathcal{H}_{n-m}^\alpha. \tag{7}$$

Notice that by (7) we have:

(i) If m is even, then

$$\widetilde{\mathcal{R}}_{\mathrm{e},m}^\alpha = \mathcal{M}_{\xi_1} \mathcal{S}_{l,m/2}$$

with

$$\xi_1(n) = \frac{\mathcal{M}(2n,\alpha,m)}{[m(2n,\alpha)]^m}.$$

And

$$\widetilde{\mathcal{R}}_{\mathrm{o},m}^\alpha = \mathcal{R}_{\varphi,m/2+1}^\alpha \mathcal{M}_{\xi_2} \mathcal{R}_\psi^\alpha$$

with

$$\xi_2(n) = (-1)^{m/2} \frac{\mathcal{M}(2n+1,\alpha,m)}{[m(2n+1,\alpha)]^m}.$$

(ii) If m is odd, then

$$\widetilde{\mathcal{R}}_{\mathrm{e},m}^\alpha = \mathcal{R}_{\varphi,(m+1)/2} \mathcal{M}_{\xi_3}$$

with

$$\xi_3(n) = \frac{\mathcal{M}(2n,\alpha,m)}{[m(2n,\alpha)]^m}.$$

And

$$\widetilde{\mathcal{R}}_{\mathrm{o},m}^\alpha = \mathcal{M}_{\xi_4} \mathcal{S}_{l,(m+1)/2} \mathcal{R}_\psi^\alpha$$

with

$$\xi_4(n) = (-1)^{(m+1)/2} \frac{\mathcal{M}(2n+1,\alpha,m)}{[m(2n+1,\alpha)]^m}.$$

Consequently, the relevant L^p estimate follows by relation (4) accordingly with Theorems 3, 4 and 5. □

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