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Shahram Rezaei

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ANNIHILATORS OF LOCAL HOMOLOGY MODULES

Shahram Rezaei, Tehran

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Abstract. Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R and M a nonzero Artinian R-module of Noetherian dimension n with $\operatorname{hd}(\mathfrak{a}, M) = n$. We determine the annihilator of the top local homology module $\mathrm{H}_n^{\mathfrak{a}}(M)$. In fact, we prove that

$$\operatorname{Ann}_R(\mathrm{H}_n^{\mathfrak{a}}(M)) = \operatorname{Ann}_R(N(\mathfrak{a}, M)),$$

where $N(\mathfrak{a}, M)$ denotes the smallest submodule of M such that $\operatorname{hd}(\mathfrak{a}, M/N(\mathfrak{a}, M)) < n$. As a consequence, it follows that for a complete local ring $(R, \mathfrak{m})$ all associated primes of $\mathrm{H}_n^{\mathfrak{a}}(M)$ are minimal.

Keywords: local homology; Artinian modules; annihilator

MSC 2010: 13D45, 13E05

1. Introduction

Throughout this paper we assume that $(R, \mathfrak{m})$ is a commutative Noetherian local ring, $\mathfrak{a}$ is an ideal of R and M is an R-module. Cuong and Nam in [5] defined the local homology modules $\mathrm{H}_i^{\mathfrak{a}}(M)$ with respect to $\mathfrak{a}$ by

$$\mathrm{H}_i^{\mathfrak{a}}(M) = \varprojlim_n \operatorname{Tor}_i^R(R/\mathfrak{a}^n, M).$$

This definition is dual to Grothendieck's definition of local cohomology modules and coincides with the definition of Greenless and May in [9] for an Artinian R-module M. For basic results about local homology we refer the reader to [5], [6] and [16]; for local cohomology we refer to [4].

In this paper we study the top local homology module $\mathrm{H}_n^{\mathfrak{a}}(M)$, where M is a nonzero Artinian R-module of Noetherian dimension n and $\mathfrak{a}$ is an arbitrary ideal of R. The module $\mathrm{H}_n^{\mathfrak{a}}(M)$ is called a top local homology module because

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$\max\{i\colon \mathrm{H}_i^{\mathfrak{a}}(M)\neq 0\}\leqslant n$ by [5], Proposition 4.8. The problem of finding annihilators of local cohomology modules has been studied by several authors; see for example [1], [2] and [3]. In [3], the authors proved that if $(R,\mathfrak{m})$ is a complete Noetherian local ring and M is a finitely generated R-module then $\mathrm{Ann}_R(\mathrm{H}_{\mathfrak{m}}^{\dim M}(M)) = T_R(M)$, where $T_R(M)$ is the largest submodule of M such that $\dim T_R(M) < \dim(M)$. This result was later extended to noncomplete Noetherian local rings by Bahmanpour in [2]. Also, for an ideal $\mathfrak{a}$ (not necessarily $\mathfrak{a}=\mathfrak{m}$) in an arbitrary Noetherian ring R (not necessarily local), in [1] Atazadeh et al. proved that $\mathrm{Ann}_R(\mathrm{H}_{\mathfrak{a}}^{\dim M}(M)) = T_R(\mathfrak{a},M)$ where $T_R(\mathfrak{a},M)$ is the largest submodule of M such that $\mathrm{cd}(\mathfrak{a},T_R(\mathfrak{a},M)) < \mathrm{cd}(\mathfrak{a},M)$.

Here we determine the annihilator of the top local homology modules. In fact, the following is the main result of this paper.

Theorem 1.1. *Let $(R,\mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R and M a nonzero Artinian R-module of Noetherian dimension n with $\mathrm{hd}(\mathfrak{a},M)=n$. Then*

$$\mathrm{Ann}_R(\mathrm{H}_n^{\mathfrak{a}}(M)) = \mathrm{Ann}_R(N(\mathfrak{a},M)).$$

where $N(\mathfrak{a},M)$ denotes the smallest submodule of M such that

$$\mathrm{hd}(\mathfrak{a}, M/N(\mathfrak{a},M)) < n.$$

By using the above result we describe the annihilator of the top local homology modules in terms of a secondary representation of M, as follows:

Theorem 1.2. *Let $(R,\mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R and M a nonzero Artinian module of Noetherian dimension n with $\mathrm{hd}(\mathfrak{a},M)=n$. Let $M = N_1 + N_2 + \ldots + N_t$ be a secondary representation of M as an $\widehat{R}$-module where N_j is a $\mathfrak{P}_j$-secondary submodule of M. Then*

$$\mathrm{Ann}_R(\mathrm{H}_n^{\mathfrak{a}}(M)) = \mathrm{Ann}_R\biggl(\sum_{\mathrm{cd}(\mathfrak{a}\widehat{R},\widehat{R}/\mathfrak{P}_j)=n} N_j\biggr).$$

As an application of the above results, we will show that for a complete local ring $(R,\mathfrak{m})$ we have $\mathrm{Ass}_R(\mathrm{H}_n^{\mathfrak{a}}(M)) = \min \mathrm{Ass}_R(\mathrm{H}_n^{\mathfrak{a}}(M))$.

A nonzero R-module M is called secondary if the multiplication map by any element a of R is either surjective or nilpotent. A secondary representation of the R-module M is an expression for M as a finite sum of secondary modules. If such a representation exists, we will say that M is representable. A prime ideal $\mathfrak{p}$ of R is said to be an attached prime of M if $\mathfrak{p} = (N :_R M)$ for some submodule N of M. If M admits a reduced secondary representation $M = S_1 + S_2 + \ldots + S_n$, then the set of attached primes $\mathrm{Att}_R(M)$ of M is equal to $\{\sqrt{0 :_R S_i}$ for $i = 1, \ldots, n\}$. Note that every Artinian R-module M is representable and the minimal elements of the set $\mathrm{V}(\mathrm{Ann}(M))$, the set of prime ideals of R containing the ideal $\mathrm{Ann}(M)$, belong to $\mathrm{Att}(M)$. It is well known that if N is a submodule of an Artinian R-module M, then $\mathrm{Att}(M/N) \subseteq \mathrm{Att}(M) \subseteq \mathrm{Att}(N) \cup \mathrm{Att}(M/N)$.

We now recall the concept of Noetherian dimension $\mathrm{Ndim}_R(M)$ of an R-module M. For $M = 0$ we define $\mathrm{Ndim}_R(M) = -1$. Then by induction, for any integer $t \geqslant 0$, we define $\mathrm{Ndim}_R(M) = t$ when

(i) $\mathrm{Ndim}_R(M) < t$ is false, and

(ii) for every ascending chain $M_1 \subseteq M_2 \subseteq \ldots$ of submodules of M there exists an integer m_0 such that $\mathrm{Ndim}_R(M_{m+1}/M_m) < t$ for all $m \geqslant m_0$.

Thus M is nonzero and finitely generated if and only if $\mathrm{Ndim}_R(M) = 0$. If M is an Artinian module, then $\mathrm{Ndim}_R(M) < \infty$. (For more details see [10] and [14].)

Recall that, for any R-module M, the cohomological dimension of M with respect to $\mathfrak{a}$ is defined as

$$\mathrm{cd}(\mathfrak{a}, M) = \sup\{i \in \mathbb{Z} \colon \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\}.$$

Also, in [12] we defined the homological dimension of M with respect to $\mathfrak{a}$ by

$$\mathrm{hd}(\mathfrak{a}, M) = \sup\{i \in \mathbb{Z} \colon \mathrm{H}^{\mathfrak{a}}_i(M) \neq 0\}.$$

It is easy to see that, if M is an Artinian R-module, then $\mathrm{hd}(\mathfrak{a}, M) \leqslant \mathrm{Ndim}_R(M)$ and $\mathrm{hd}(\mathfrak{m}, M) = \mathrm{Ndim}_R(M)$ by [5], Proposition 4.8, and [5], Proposition 4.10.

Throughout the paper, for an R-module M, $\mathrm{E}(R/\mathfrak{m})$ denotes the injective envelope of $R/\mathfrak{m}$ and $\mathrm{D}(\cdot)$ denotes the Matlis duality functor $\mathrm{Hom}_R(\cdot, \mathrm{E}(R/\mathfrak{m}))$. It is well known that $\mathrm{Ann}_R \mathrm{D}(M) = \mathrm{Ann}_R M$ and $\dim \mathrm{D}(M) = \dim M$. Also, if M is an Artinian R-module then $M \simeq \mathrm{DD}(M)$, and $\mathrm{D}(M)$ is a Noetherian $\widehat{R}$-module. (See [11], Theorem 1.6, and [4], Theorem 10.2.19.) Note that if M is an Artinian R-module, then $\mathrm{H}^{\mathfrak{a}}_i(M) \simeq \mathrm{D}(\mathrm{H}^i_{\mathfrak{a}}(\mathrm{D}(M)))$ for all i (see [5], Proposition 3.3), and therefore $\mathrm{hd}(\mathfrak{a}, M) = \mathrm{cd}(\mathfrak{a}, \mathrm{D}(M))$. Thus $\mathrm{hd}(\mathfrak{a}, M) \leqslant \dim \mathrm{D}(M) = \dim M$.

2. The results

There are many results about annihilators of local cohomology modules. For example, the following theorem is a main result of [1] on the annihilators of top local cohomology modules.

Theorem 2.1 ([1], Theorem 2.3). *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a nonzero finitely generated R-module such that* $\operatorname{cd}(\mathfrak{a}, M) = \dim M$. *Then* $\operatorname{Ann}_R \mathrm{H}_{\mathfrak{a}}^{\dim M}(M) = \operatorname{Ann}_R(M/T_R(\mathfrak{a}, M))$, *where*

$$T_R(\mathfrak{a}, M) := \bigcup\{N\colon\ N \leqslant M \text{ and } \operatorname{cd}(\mathfrak{a}, N) < \operatorname{cd}(\mathfrak{a}, M)\}.$$

Here, as the dual case of the above result, we obtain some results about the annihilator of top local homology modules. At first, we define the following notation $N_R(\mathfrak{a}, M)$.

Definition 2.2. Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R and M a nonzero Artinian R-module with $\operatorname{hd}(\mathfrak{a}, M) = n$. We denote by $N_R(\mathfrak{a}, M)$ the smallest element of the set

$$\Sigma := \{N\colon\ N \text{ is a submodule of } M \text{ and } \operatorname{hd}(\mathfrak{a}, M/N) < n\}.$$

To prove our main result, we need the following lemmas.

Lemma 2.3. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R, and $0 \to L \to M \to N \to 0$ an exact sequence of Artinian R-modules. Then* $\operatorname{hd}(\mathfrak{a}, M) = \max\{\operatorname{hd}(\mathfrak{a}, L), \operatorname{hd}(\mathfrak{a}, N)\}$.

P r o o f. See [12], Lemma 2.1. □

Lemma 2.4. *Let $(R, \mathfrak{m})$ be a complete local ring, $\mathfrak{a}$ an ideal of R and M a nonzero Artinian module. Then* $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) \leqslant \operatorname{hd}(\mathfrak{a}, M)$ *for all* $\mathfrak{p} \in \operatorname{Att}(M)$.

P r o o f. See [12], Lemma 2.2. □

Lemma 2.5. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R and M an Artinian R-module. Then* $\operatorname{hd}(\mathfrak{a}, M) \leqslant \operatorname{cd}(\mathfrak{a}, R/\operatorname{Ann} M)$.

P r o o f. See [12], Lemma 2.3. □

Lemma 2.6. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R and M a nonzero Artinian R-module with* $\operatorname{hd}(\mathfrak{a}, M) = n$. *Let* $N := N_R(\mathfrak{a}, M)$. *Then the module N has the following properties:*

(i) *If* $\dim M = n$ *then* $\operatorname{hd}(\mathfrak{a}, N) = \dim N = n$.
(ii) *If* $\operatorname{Ndim}_R M = n$ *then* $\operatorname{hd}(\mathfrak{a}, N) = \operatorname{Ndim}_R N = n$.
(iii) *N has no proper submodule L such that* $\operatorname{hd}(\mathfrak{a}, N/L) < n$.
(iv) $\mathrm{H}_n^{\mathfrak{a}}(N) \simeq \mathrm{H}_n^{\mathfrak{a}}(M)$.
(v) *If* $\dim M = n$ *and R is complete then*

$$\operatorname{Att}_R(N) = \{\mathfrak{p} \in \operatorname{Att}_R(M)\colon\ \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = n\} = \operatorname{Ass}_R(\mathrm{H}_n^{\mathfrak{a}}(M)).$$

P r o o f. See [12], Lemma 2.4 and Theorem 2.5. □

Theorem 2.7. *Let* $(R,\mathfrak{m})$ *be a complete local ring,* $\mathfrak{a}$ *an ideal of* R *and* M *a nonzero Artinian module of Noetherian dimension* n *with* $\operatorname{hd}(\mathfrak{a},M)=n$. *Then*

$$\operatorname{Ann}_R(\mathrm{H}_n^{\mathfrak{a}}(M)) = \operatorname{Ann}_R(N(\mathfrak{a},M)).$$

P r o o f. Let $N := N(\mathfrak{a},M)$. By Lemma 2.6 (iv) $\operatorname{Ann}_R(\mathrm{H}_n^{\mathfrak{a}}(M)) = \operatorname{Ann}_R(\mathrm{H}_n^{\mathfrak{a}}(N))$. By [5], Proposition 3.3, $\mathrm{H}_n^{\mathfrak{a}}(N) \simeq \mathrm{D}(\mathrm{H}_{\mathfrak{a}}^n(\mathrm{D}(N))$. Thus we get

$$\operatorname{Ann}_R(\mathrm{H}_n^{\mathfrak{a}}(N)) = \operatorname{Ann}_R(\mathrm{D}(\mathrm{H}_{\mathfrak{a}}^n(\mathrm{D}(N)))) = \operatorname{Ann}_R(\mathrm{H}_{\mathfrak{a}}^n(\mathrm{D}(N))).$$

Since $(R,\mathfrak{m})$ is a complete local ring, $\dim_R N = \operatorname{Ndim}_R N$ by [7], Corollary 2.5. But, by Lemma 2.6 (ii), $\operatorname{Ndim}_R(N) = n$ and so $\dim \mathrm{D}(N) = \dim(N) = n$. Now, by Theorem 2.1 we conlude that $\operatorname{Ann}_R(\mathrm{H}_{\mathfrak{a}}^n(\mathrm{D}(N))) = \operatorname{Ann}_R(\mathrm{D}(N)/T_R(\mathfrak{a},\mathrm{D}(N)))$. If we show that $T_R(\mathfrak{a},\mathrm{D}(N)) = 0$ then we have $\operatorname{Ann}_R(\mathrm{H}_n^{\mathfrak{a}}(M)) = \operatorname{Ann}_R(\mathrm{D}(N)) = \operatorname{Ann}_R(N)$ and the proof is complete.

By definition $T_R(\mathfrak{a},\mathrm{D}(N)) = \bigcup\{U\colon\ U \leqslant \mathrm{D}(N) \text{ and } \operatorname{cd}(\mathfrak{a},U) < \operatorname{cd}(\mathfrak{a},\mathrm{D}(N))\}$. Let $0 \neq U$ be a submodule of $\mathrm{D}(N)$ such that $\operatorname{cd}(\mathfrak{a},U) < n$. Then the exact sequence $0 \to U \to \mathrm{D}(N) \to \mathrm{D}(N)/U \to 0$ implies the following exact sequence:

$$0 \to \mathrm{D}(\mathrm{D}(N)/U) \to \mathrm{DD}(N) \to \mathrm{D}(U) \to 0.$$

But $\mathrm{DD}(N) \simeq N$ and so we conclude that there is a proper submodule L of N such that $N/L \simeq \mathrm{D}(U)$. On the other hand, $\operatorname{hd}(\mathfrak{a},\mathrm{D}(U)) = \operatorname{cd}(\mathfrak{a},\mathrm{DD}(U)) = \operatorname{cd}(\mathfrak{a},U) < n$. Hence $\operatorname{hd}(\mathfrak{a},N/L) < n$ which is a contradiction by Lemma 2.6 (ii). Therefore $T_R(\mathfrak{a},\mathrm{D}(N)) = 0$, which completes the proof. □

In the following result, we will show that for a complete local ring $(R,\mathfrak{m})$ all associated primes of $\mathrm{H}_n^{\mathfrak{a}}(M)$ are minimal.

Corollary 2.8. *Let* $(R,\mathfrak{m})$ *be a complete local ring,* $\mathfrak{a}$ *an ideal of* R *and* M *a nonzero Artinian* R*-module of Noetherian dimension* n *with* $\operatorname{hd}(\mathfrak{a},M)=n$. *Then*

$$\operatorname{Ass}_R(\mathrm{H}_n^{\mathfrak{a}}(M)) = \min \operatorname{Att}_R N(\mathfrak{a},M) = \min\, \operatorname{Ass}_R(\mathrm{H}_n^{\mathfrak{a}}(M)).$$

P r o o f. Since $\mathrm{H}_n^{\mathfrak{a}}(M) \simeq \mathrm{D}(\mathrm{H}_{\mathfrak{a}}^n(\mathrm{D}(M))$, we have

$$\operatorname{Ass}_R(\mathrm{H}_n^{\mathfrak{a}}(M)) = \operatorname{Ass}_R(\mathrm{D}(\mathrm{H}_{\mathfrak{a}}^n(\mathrm{D}(M))).$$

By [4], Theorem 7.1.6, $\mathrm{H}^n_{\mathfrak{a}}(\mathrm{D}(M))$ is an Artinian R-module and so

$$\mathrm{Ass}_R(\mathrm{D}(\mathrm{H}^n_{\mathfrak{a}}(\mathrm{D}(M)))) = \mathrm{Att}_R(\mathrm{H}^n_{\mathfrak{a}}(\mathrm{D}(M)))$$

by [15], Theorem 2.3. But by [13], Theorem 2.11,

$$\begin{aligned}\mathrm{Att}_R(\mathrm{H}^n_{\mathfrak{a}}(\mathrm{D}(M))) = \min \mathrm{V}(\mathrm{Ann}_R \mathrm{H}^n_{\mathfrak{a}}(\mathrm{D}(M))) &= \min \mathrm{V}(\mathrm{Ann}_R \mathrm{D}(\mathrm{H}^n_{\mathfrak{a}}(\mathrm{D}(M))))\\ &= \min \mathrm{V}(\mathrm{Ann}_R(\mathrm{H}^{\mathfrak{a}}_n(M))).\end{aligned}$$

On the other hand, by Theorem 2.7 and [11], Proposition 2.10, we have

$$\min \mathrm{V}(\mathrm{Ann}_R(\mathrm{H}^{\mathfrak{a}}_n(M))) = \min \mathrm{V}(\mathrm{Ann}_R N(\mathfrak{a}, M)) = \min \mathrm{Att}_R N(\mathfrak{a}, M).$$

Since $\mathrm{Att}_R(N(\mathfrak{a}, M)) = \mathrm{Ass}_R(\mathrm{H}^{\mathfrak{a}}_n(M))$ by Lemma 2.6 (v), we get $\min \mathrm{Att}_R N(\mathfrak{a}, M) = \min\ \mathrm{Ass}_R(\mathrm{H}^{\mathfrak{a}}_n(M))$. The proof is complete. □

Lemma 2.9. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R and M a nonzero Artinian R-module of Noetherian dimension n with $\mathrm{hd}(\mathfrak{a}, M) = n$. Then $N(\mathfrak{a}, M) = N(\mathfrak{a}\widehat{R}, M)$.*

Proof. By [16], Remark 2.6, for any submodule L of M and any integer i, $\mathrm{H}^{\mathfrak{a}}_i(M/L) \simeq \mathrm{H}^{\mathfrak{a}\widehat{R}}_i(M/L)$ as R-modules. Thus $\mathrm{hd}(\mathfrak{a}, M/L) = \mathrm{hd}(\mathfrak{a}\widehat{R}, M/L)$ and so $N(\mathfrak{a}, M) = N(\mathfrak{a}\widehat{R}, M)$. □

In the next result we provide a generalization of Theorem 2.7 by eliminating the complete hypothesis.

Theorem 2.10. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R and M a nonzero Artinian R-module of Noetherian dimension n with $\mathrm{hd}(\mathfrak{a}, M) = n$. Then*

$$\mathrm{Ann}_R(\mathrm{H}^{\mathfrak{a}}_n(M)) = \mathrm{Ann}_R(N(\mathfrak{a}, M)).$$

Proof. Let $N := N(\mathfrak{a}, M)$. Since $\mathrm{Ann}_R N \subseteq \mathrm{Ann}_R(\mathrm{H}^{\mathfrak{a}}_n(N))$ and by Lemma 2.6 (iv) $\mathrm{Ann}_R(\mathrm{H}^{\mathfrak{a}}_n(N)) = \mathrm{Ann}_R(\mathrm{H}^{\mathfrak{a}}_n(M))$ we have $\mathrm{Ann}_R N \subseteq \mathrm{Ann}_R(\mathrm{H}^{\mathfrak{a}}_n(M))$. Now we show that $\mathrm{Ann}_R(\mathrm{H}^{\mathfrak{a}}_n(M)) \subseteq \mathrm{Ann}_R N$.

Let $x \in \mathrm{Ann}_R(\mathrm{H}^{\mathfrak{a}}_n(M))$. By [16], Remark 2.6, $\mathrm{H}^{\mathfrak{a}}_n(M) \simeq \mathrm{H}^{\mathfrak{a}\widehat{R}}_n(M)$ as R-modules. Thus $x \in \mathrm{Ann}_R(\mathrm{H}^{\mathfrak{a}\widehat{R}}_n(M))$. Note that $\widehat{R}$-module $\mathrm{H}^{\mathfrak{a}\widehat{R}}_n(M)$ is an R-module by means of f, where $f\colon R \to \widehat{R}$ is natural ring homomorphism. Thus $f(x) \in \mathrm{Ann}_{\widehat{R}}(\mathrm{H}^{\mathfrak{a}\widehat{R}}_n(M))$. By [7], Remark 1 (ii), $\mathrm{Ndim}_{\widehat{R}} M = \mathrm{Ndim}_R M = n$. Thus by Theorem 2.7 $\mathrm{Ann}_{\widehat{R}}(\mathrm{H}^{\mathfrak{a}\widehat{R}}_n(M)) = \mathrm{Ann}_{\widehat{R}}(N(\mathfrak{a}\widehat{R}, M))$. From this we get that $f(x) \in \mathrm{Ann}_{\widehat{R}}(N(\mathfrak{a}\widehat{R}, M))$. By Lemma 2.9 $f(x) \in \mathrm{Ann}_{\widehat{R}}(N(\mathfrak{a}, M))$. Since $f(x) = (x+\mathfrak{m}^n)_{n\in\mathbb{N}}$ and $N(\mathfrak{a}, M)$ is an Artinian R-module we have $x \in \mathrm{Ann}_R(N(\mathfrak{a}, M))$ (see [4], Remark 10.2.9). This completes the proof. □

In the next result, we determine the annihilator of top local homology module $\mathrm{H}_n^{\mathfrak{a}}(M)$ in terms of a secondary representation of M.

Theorem 2.11. *Let $(R,\mathfrak{m})$ be a complete local ring, $\mathfrak{a}$ an ideal of R and M a nonzero Artinian module of Noetherian dimension n with $\mathrm{hd}(\mathfrak{a},M)=n$. Let $M=N_1+N_2+\ldots+N_t$ be a secondary representation of M where N_i is a $\mathfrak{p}_i$-secondary submodule of M. Then*

$$\mathrm{Ann}_R(\mathrm{H}_n^{\mathfrak{a}}(M))=\mathrm{Ann}_R\Bigg(\sum_{\mathrm{cd}(\mathfrak{a},R/\mathfrak{p}_j)=n}N_j\Bigg).$$

Proof. Let $U:=\sum\limits_{\mathrm{cd}(\mathfrak{a},R/\mathfrak{p}_j)=n}N_j$. By Theorem 2.10, it is sufficient to show that U is a smallest element of the set

$$\Sigma:=\{N'\colon\ N' \text{ is a submodule of } M \text{ and } \mathrm{hd}(\mathfrak{a},M/N')<n\}.$$

At first we show that $\mathrm{hd}(\mathfrak{a},M/U)<n$. Let $U':=\sum_{\mathrm{cd}(\mathfrak{a},R/\mathfrak{p}_j)<n}N_j$. By Lemma 2.3, $\mathrm{hd}(\mathfrak{a},U')=\max\{\mathrm{hd}(\mathfrak{a},N_j)\colon\ \mathrm{cd}(\mathfrak{a},R/\mathfrak{p}_j)<n\}$. But by Lemma 2.5 and [8], Theorem 1.2,

$$\mathrm{hd}(\mathfrak{a},N_j)\leqslant\mathrm{cd}(\mathfrak{a},R/\mathrm{Ann}(N_j))=\mathrm{cd}(\mathfrak{a},R/\sqrt{\mathrm{Ann}(N_j)})=\mathrm{cd}(\mathfrak{a},R/\mathfrak{p}_j)<n.$$

Thus $\mathrm{hd}(\mathfrak{a},U')<n$. We conclude that,

$$\mathrm{hd}(\mathfrak{a},M/U)=\mathrm{hd}(\mathfrak{a},(U+U')/U)=\mathrm{hd}(\mathfrak{a},U'/U\cap U')<\mathrm{hd}(\mathfrak{a},U')<n.$$

Thus $U\in\Sigma$.

Now let L be a proper submodule of U. Since $U/L\neq 0$, $\mathrm{Att}_R(U/L)\neq\varphi$. Take $\mathfrak{p}_0\in\mathrm{Att}_R(U/L)$. Thus $\mathfrak{p}_0\in\mathrm{Att}_RU$ and so $\mathrm{cd}(\mathfrak{a},R/\mathfrak{p}_0)=n$. Now Lemma 2.4 implies that $n\leqslant\mathrm{hd}(\mathfrak{a},U/L)$ and so by Lemma 2.3 $n\leqslant\mathrm{hd}(\mathfrak{a},U/L)\leqslant\mathrm{hd}(\mathfrak{a},M/L)$. Therefore U is a smallest element of the set Σ, as required. □

Corollary 2.12. *Let $(R,\mathfrak{m})$ be a complete local ring, $\mathfrak{a}$ an ideal of R and M a nonzero Artinian R-module of Noetherian dimension n with $\mathrm{hd}(\mathfrak{a},M)=n$. Let $\mathrm{Att}_RM\subseteq\{\mathfrak{p}\in\mathrm{Spec}R\colon\ \mathrm{cd}(\mathfrak{a},R/\mathfrak{p})=n\}$. Then $\mathrm{Ann}_R(\mathrm{H}_n^{\mathfrak{a}}(M))=\mathrm{Ann}_RM$.*

Proof. Let $M=N_1+N_2+\ldots+N_t$ be a secondary representation of M where N_i is a $\mathfrak{p}_i$-secondary submodule of M. By assumption $\mathrm{cd}(\mathfrak{a},R/\mathfrak{p}_j)=n$ for all $1\leqslant j\leqslant t$ and so we have $M=\sum_{\mathrm{cd}(\mathfrak{a},R/\mathfrak{p}_j)=n}N_j$. Now the result follows from Theorem 2.11. □

Corollary 2.13. *Let $(R, \mathfrak{m})$ be a complete local ring, $\mathfrak{a}$ an ideal of R and M a nonzero Artinian R-module of Noetherian dimension n with $\operatorname{hd}(\mathfrak{a}, M) = n$. Then*

$$\sqrt{\operatorname{Ann}_R(\mathrm{H}_n^{\mathfrak{a}}(M))} = \bigcap_{\substack{\mathfrak{p}\in \operatorname{Att}_R M \\ \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p})=n}} \mathfrak{p}.$$

P r o o f. Let $M = N_1 + N_2 + \ldots + N_t$ be a secondary representation of M where N_i is a $\mathfrak{p}_i$-secondary submodule of M. By Theorem 2.11

$$\begin{aligned}\sqrt{\operatorname{Ann}_R(\mathrm{H}_n^{\mathfrak{a}}(M))} &= \sqrt{\operatorname{Ann}_R\Big(\sum_{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j)=n} N_j\Big)} \\ &= \bigcap_{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j)=n} \sqrt{\operatorname{Ann}_R N_j} = \bigcap_{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j)=n} \mathfrak{p}_j.\end{aligned}$$

Since $\mathfrak{p}_j \in \operatorname{Att}_R M$ for all $j = 1, \ldots, t$, the proof is complete. □

Corollary 2.14. *Let $(R, \mathfrak{m})$ be a complete local ring, $\mathfrak{a}$ an ideal of R and M a nonzero Artinian R-module of Noetherian dimension n with $\operatorname{hd}(\mathfrak{a}, M) = n$. Let $M = N_1+N_2+\ldots+N_t$ be a secondary representation of M where N_i is a $\mathfrak{p}_i$-secondary submodule of M. Then*

$$\operatorname{Supp}_R(\mathrm{H}_n^{\mathfrak{a}}(M)) = \bigcup_{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j)=n} \mathrm{V}(\operatorname{Ann}_R N_j).$$

P r o o f. Set $\Lambda_{\mathfrak{a}}(R) := \varprojlim_t R/\mathfrak{a}^t$. By [6], Theorem 5.3, $\mathrm{H}_n^{\mathfrak{a}}(M)$ is a Noetherian $\Lambda_{\mathfrak{a}}(R)$-module. Since R is $\mathfrak{m}$-adically complete and $\mathfrak{a} \subseteq \mathfrak{m}$, it follows that R is $\mathfrak{a}$-adically complete and so $\Lambda_{\mathfrak{a}}(R) \simeq R$. Thus $\mathrm{H}_n^{\mathfrak{a}}(M)$ is a Noetherian R-module. Hence $\operatorname{Supp}_R(\mathrm{H}_n^{\mathfrak{a}}(M)) = \mathrm{V}(\operatorname{Ann}_R(\mathrm{H}_n^{\mathfrak{a}}(M))$. On the other hand, by Theorem 2.11 we have

$$\begin{aligned}\mathrm{V}(\operatorname{Ann}_R(\mathrm{H}_n^{\mathfrak{a}}(M)) &= \mathrm{V}\Big(\operatorname{Ann}_R \sum_{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j)=n} N_j\Big) \\ &= \mathrm{V}\Big(\bigcap_{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j)=n} \operatorname{Ann}_R N_j\Big) = \bigcup_{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}_j)=n} \mathrm{V}(\operatorname{Ann}_R N_j),\end{aligned}$$

as required. □

In the next main result we extend Theorem 2.11 to noncomplete local rings.

Theorem 2.15. *Let $(R,\mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal of R and M a nonzero Artinian module of Noetherian dimension n with $\operatorname{hd}(\mathfrak{a},M)=n$. Let $M=N_1+N_2+\ldots+N_t$ be a secondary representation of M as an $\widehat{R}$-module where N_j is a $\mathfrak{P}_j$-secondary submodule of M. Then*

$$\operatorname{Ann}_R(\mathrm{H}_n^{\mathfrak{a}}(M))=\operatorname{Ann}_R\biggl(\sum_{\operatorname{cd}(\mathfrak{a}\widehat{R},\widehat{R}/\mathfrak{P}_j)=n}N_j\biggr).$$

Proof. Set $U:=\sum_{\operatorname{cd}(\mathfrak{a}\widehat{R},\widehat{R}/\mathfrak{P}_j)=n}N_j$, and let $f\colon R\to\widehat{R}$ be the natural ring homomorphism.

Now, let $x\in\operatorname{Ann}_R(U)$. Since U is an R-module by means of f, $f(x)\in\operatorname{Ann}_{\widehat{R}}U$. By Theorems 2.11 and 2.7 it follows that $f(x)\in\operatorname{Ann}_{\widehat{R}}N(\mathfrak{a}\widehat{R},M)$ and by Lemma 2.9 $f(x)\in\operatorname{Ann}_{\widehat{R}}N(\mathfrak{a},M)$. Since $N(\mathfrak{a},M)$ is an Artinian R-module we conclude that $x\in\operatorname{Ann}_RN(\mathfrak{a},M)$. Now by Theorem 2.10 we conclude that $x\in\operatorname{Ann}_R(\mathrm{H}_n^{\mathfrak{a}}(M))$.

Conversly, let $x\in\operatorname{Ann}_R(\mathrm{H}_n^{\mathfrak{a}}(M))$. Since $\mathrm{H}_n^{\mathfrak{a}}(M)\simeq\mathrm{H}_n^{\mathfrak{a}\widehat{R}}(M))$ as R-modules by [16], Remark 2.6, we have $x\in\operatorname{Ann}_R(\mathrm{H}_n^{\mathfrak{a}\widehat{R}}(M))$. Thus $f(x)\in\operatorname{Ann}_{\widehat{R}}(\mathrm{H}_n^{\mathfrak{a}\widehat{R}}(M))$ and by Theorem 2.11 $f(x)\in\operatorname{Ann}_{\widehat{R}}(U)$. Therefore $x\in\operatorname{Ann}_R(U)$. This completes the proof. □

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Author's address: Shahram Rezaei, Department of Mathematics, Faculty of Science, Payame Noor University (PNU), P.O. Box: 19395-4697, Tehran, Iran, e-mail: `Sha.rezaei@gmail.com`.