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FINITE p-GROUPS WITH EXACTLY TWO NONLINEAR NON-FAITHFUL IRREDUCIBLE CHARACTERS

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Abstract. Let G be a finite group with exactly two nonlinear non-faithful irreducible characters. We discuss the properties of G and classify finite p-groups with exactly two nonlinear non-faithful irreducible characters.

Keywords: p-group; nonlinear irreducible character; non-faithful character

MSC 2010: 20C15

1. Introduction

Iranmanesh and Saeidi [4] studied finite groups with exactly one nonlinear non-faithful irreducible character. And Saeidi [6] classified solvable groups with a unique nonlinear non-faithful irreducible character. We consider the following case in this note.

Hypothesis $(*)$:
A finite group has exactly two nonlinear non-faithful irreducible characters.

Let G be a finite group with exactly two nonlinear non-faithful irreducible characters χ_1, χ_2. Let $K_1 = \ker\chi_1$, $K_2 = \ker\chi_2$, and write $L = K_1 \cap K_2$.

In this note, we will show some properties of groups which satisfy Hypothesis $(*)$. Our main conclusion is the classification of finite p-groups with exactly two nonlinear non-faithful irreducible characters. In fact, we have the following result.

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Theorem 1.1. *A p-group G has exactly two nonlinear non-faithful irreducible characters if and only if one of the following assertions holds.*

(1) *G is a 2-group with nilpotence class 2, $G/\mathbf{Z}(G)$ is elementary abelian, $\mathbf{Z}(G) \cong C_2 \times C_2$ and $|G'| = 2$.*
(2) *G is a group of order 32 of nilpotence class 3 and $\mathbf{Z}(G) \cong C_2$ or $\mathbf{Z}(G) \cong C_4$.*
(3) *G is a group of order 81 of nilpotence class 3.*

All groups considered in this note are finite. The notation and terminology are standard, and one can refer to [5] and [3].

2. Preliminaries

We first discuss the intersection of the kernels of irreducible nonlinear characters of a finite group. There is a modular form of the intersection of the kernels of irreducible Brauer characters in [8]. Moreover, in this section, we give some properties of groups which satisfy Hypothesis $(*)$, and state facts which are important to prove the main theorem of this note.

Proposition 2.1. *Let $\mathrm{Irr}(G)$ and $\mathrm{Lin}(G)$ be the sets of irreducible characters and linear characters of a group G, respectively. If $\mathrm{Irr}(G) \supsetneq \mathrm{Lin}(G)$, then*

(i) $\bigcap\limits_{\varphi\in\mathrm{Irr}(G)-\mathrm{Lin}(G)} \ker\varphi = 1$. *In particular, if G has a unique nonlinear irreducible character χ, then χ is faithful; and if G has exactly two nonlinear irreducible characters χ, φ, then $\ker\chi \cap \ker\varphi = 1$.*
(ii) *if G satisfies Hypothesis $(*)$ and $L > 1$, then $\mathbf{Z}(G)$ is cyclic, where $\mathbf{Z}(G)$ is the center of G;*
(iii) *if G satisfies Hypothesis $(*)$, then every normal subgroup of G not containing G' is among K_1, K_2 and L;*
(iv) *if $\mathbf{Z}(G) \neq 1$ and G satisfies Hypothesis $(*)$ and $L > 1$, then L is a minimal normal subgroup of G of a prime order. Moreover, if $G' \cap L = 1$, then $G' \cap K_i = 1$ for $i = 1, 2$.*

Proof. Write $U = \bigcap\limits_{\varphi\in\mathrm{Irr}(G)-\mathrm{Lin}(G)} \ker\varphi$ and $h \in U$. Then $\varphi(h) = \varphi(1)$ for $\varphi \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$. For any $\lambda \in \mathrm{Lin}(G)$, λ can act on $\mathrm{Irr}(G)$ by multiplication. Then $\exists\,\theta \in \mathrm{Irr}(G) - \mathrm{Lin}(G)$ such that $\varphi = \lambda\theta$. Thus

$$\varphi(1) = \varphi(h) = \lambda(h)\theta(h) = \lambda(h)\theta(1) = \lambda(h)\varphi(1), \quad \lambda(h) = 1.$$

Since λ is arbitrary, it follows that $h \in \bigcap_{\lambda \in \mathrm{Lin}(G)} \ker \lambda = G'$, where G' is the derived subgroup of G, and then

$$h \in U \cap G' = \bigcap_{\mu \in \mathrm{Irr}(G)} \ker \mu = 1.$$

Therefore, we have that $\bigcap_{\varphi \in \mathrm{Irr}(G) - \mathrm{Lin}(G)} \ker \varphi = 1$. Immediately, if G has a unique nonlinear irreducible character χ, then $\ker \chi = 1$.

If G satisfies Hypothesis $(*)$ and $L > 1$, then G has at least one nonlinear faithful irreducible character and so $\mathbf{Z}(G)$ is cyclic.

Let G satisfy Hypothesis $(*)$ and let N be a normal subgroup of G not containing the derived subgroup G'. Then the number of nonlinear irreducible characters of G/N is no more than 2. Let $\mathrm{Irr}_1(G)$ denote the set of all the nonlinear irreducible characters of G. So we have the following two cases.

Case (a): When $|\mathrm{Irr}_1(G/N)| = 1$, it follows that $\mathrm{Irr}_1(G/N) = \{\widehat{\chi}_1\}$ or $\{\widehat{\chi}_2\}$, where $\widehat{\chi}_i(Ng) = \chi_i(g)$ for $g \in G$, $i = 1, 2$. If $\mathrm{Irr}_1(G/N)| = \{\widehat{\chi}_1\}$, we deduce that $\ker \widehat{\chi}_1 = \overline{1}$ by (i). Therefore $N = \ker \chi_1$ since $\ker \chi_1/N = \ker \widehat{\chi}_1$. If $\mathrm{Irr}_1(G/N)| = \{\widehat{\chi}_2\}$, for the same reason as above, we have $N = \ker \chi_2$.

Case (b): When $|\mathrm{Irr}_1(G/N)| = 2$, it follows that $\mathrm{Irr}_1(G/N) = \{\widehat{\chi}_1, \widehat{\chi}_2\}$. By (i), we have that $\ker \widehat{\chi}_1 \cap \ker \widehat{\chi}_2 = \overline{1}$. Since $(\ker \chi_1/N) \cap (\ker \chi_2/N) = \ker \widehat{\chi}_1 \cap \ker \widehat{\chi}_2$, it follows that $N = \ker \chi_1 \cap \ker \chi_2 = L$.

By the above proof, we have that N is among K_1, K_2 and L.

By (iii), it follows that $G' \cap L = 1$ or $G' \cap L = L$. In both cases, we have that L is a minimal normal subgroup of G. If $G' \subseteq \mathbf{Z}(G)$, then G' is cyclic and so $L \subseteq \mathbf{Z}(G)$ and then L is of a prime order. If $\mathbf{Z}(G)$ is not containing G', then it follows by (iii) that $\mathbf{Z}(G)$ is among K_1, K_2 and L. Thus $L \subseteq \mathbf{Z}(G)$ and so L is of a prime order. Also, if $G' \cap L = 1$, then $G' \cap K_i = 1$ for $i = 1, 2$. Otherwise, $G' \cap K_i = L$ by (iii) and we have that

$$L = G' \cap K_i = G' \cap K_i \cap L = (G' \cap L) \cap K_i = 1,$$

a contradiction. □

Seitz proved in [7] the following lemma which will be used many times in the next section.

Lemma 2.1. *Let G be a finite group. Then G has exactly one nonlinear irreducible character if and only if one of the following conditions holds.*

(i) *G is an extraspecial 2-group.*

(ii) *G is a Frobenius group with elementary abelian Frobenius kernel G' and a cyclic Frobenius complement H, where $|G'| - 1 = |H|$.*

Zhang classified in [9] the groups with exactly two nonlinear irreducible characters.

Lemma 2.2. *Let G be a finite group with exactly two nonlinear irreducible characters. Then one of the following assertions holds.*

(1) *G is an extraspecial 3-group.*
(2) *G is a Frobenius group with abelian Frobenius complement H and elementary abelian Frobenius kernel N, and $2|H| = |N| - 1$.*
(3) *$G = (C_3 \times C_3) \rtimes Q_8$ is a Frobenius group with Frobenius complement Q_8.*
(4) *G is a 2-group with nilpotence class 3 with a normal series $G \rhd G' \rhd \mathbf{Z}(G) \rhd 1$, and $G/\mathbf{Z}(G)$ is an extraspecial 2-group, $|G'| = 4$, $|\mathbf{Z}(G)| = 2$.*
(5) *G is a 2-group with nilpotence class 2 with a normal series $G \rhd \mathbf{Z}(G) \rhd G' \rhd 1$, and $G/\mathbf{Z}(G)$ is elementary abelian, $|\mathbf{Z}(G)| = 4$, $|G'| = 2$. Furthermore, $|G| = 2^{2m}$, $m \in \mathbb{Z}$. (See [1], Theorem 6, page 281).*

3. p-groups

Suppose G satisfies Hypothesis $(*)$, and $L < K_1$, $L < K_2$. The following lemma indicates that we only need to consider 2-groups if we study p-groups satisfying those conditions.

Lemma 3.1. *Let a p-group G satisfy Hypothesis $(*)$. Let $L = K_1 \cap K_2$ and $L \lneq K_1$, $L \lneq K_2$. Then $p = 2$ and $|K_1| = |K_2| = 4$ if $L > 1$, $|K_1| = |K_2| = 2$ if $L = 1$.*

Proof. Notice that G/K_1 has only one nonlinear irreducible character. Also, since G is a finite p-group, we have that $p = 2$ by Lemma 2.1.

Assume that K_1/M is a chief factor of G. Then $G' \nleqslant M$. Otherwise, $G' \leqslant K_1$ and $\chi_1 \in \mathrm{Irr}(G/K_1)$, a contradiction. Also, since $L \lneq K_2$, we have $M \neq K_2$. Therefore, $M = L$ by Proposition 2.1 (iii) and then K_1/L is a chief factor of G. Similarly, K_2/L is also a chief factor of G.

Since every chief factor of a p-group has order p, it follows that $|K_1| = |K_2| = 2$ if $L = 1$. If $L > 1$, then it follows by Proposition 2.1 (iv) that $|L| = 2$ and so $|K_1| = |K_2| = 2^2$. □

A p-group G is said to satisfy the *strong condition* on normal subgroups provided that for any $N \trianglelefteq G$ either $G' \leqslant N$ or $N \leqslant \mathbf{Z}(G)$. If for any $N \trianglelefteq G$, either $G' \leqslant N$ or $|N\mathbf{Z}(G) : \mathbf{Z}(G)| \leqslant p$, then we say G satisfies the *weak condition* on normal subgroups. Fernández-Alcober and Moretó in [2] gave some results about finite groups satisfying the strong condition or the weak condition on normal subgroups.

Lemma 3.2. *Let G be a p-group.*

(1) *If G satisfies the strong condition on normal subgroups, then it has nilpotence class $c(G) \leqslant 3$. Furthermore,*
   (i) *if $c(G) = 2$, then $\exp G/\mathbf{Z}(G) = \exp G' = p$;*
   (ii) *if $c(G) = 3$, then $|G : \mathbf{Z}(G)| = p^3$ and $|G| \leqslant p^5$. Moreover, $|G| = 2^4$ for $p = 2$.*

(2) *If G satisfies the weak condition on normal subgroups, then it has nilpotence class $c(G) \leqslant 4$. Furthermore,*
   (i) *if $c(G) = 2$, then $\exp G/\mathbf{Z}(G) = \exp G' = p$ or p^2. Moreover, in the latter case $G/\mathbf{Z}(G) \cong C_{p^2} \times C_{p^2}$ and $G' \cong C_{p^2}$;*
   (ii) *if $c(G) = 4$, then $|G : \mathbf{Z}(G)| = p^4$, whereas for $c(G) = 3$ we have $|G : \mathbf{Z}(G)| = p^3, p^4$ or p^6 for odd p, and $|G : \mathbf{Z}(G)| = 2^3, 2^4$ when $p = 2$;*
   (iii) *if $c(G) = 4$ and $p = 2$, then $|G| = 2^5$.*

P r o o f. See Theorem D, Theorem F and Theorem G of [2]. □

Let a p-group G satisfy Hypothesis $(*)$. Obviously, there are three cases for $K_1 \cap K_2$: case (i) $L = 1$; case (ii) $1 < L < K_i$, $i = 1, 2$; and case (iii) $K_1 \leqslant K_2$ (or $K_2 \leqslant K_1$, but without loss of generality, we may assume $K_1 \leqslant K_2$). Next, we respectively discuss the structure of G according to the above cases. First, we have the following theorem.

Theorem 3.1. *A p-group G satisfies Hypothesis $(*)$ with $L = 1$ if and only if G is a 2-group of nilpotence class 2, $G/\mathbf{Z}(G)$ is elementary abelian, $\mathbf{Z}(G) \cong C_2 \times C_2$ and $|G'| = 2$.*

P r o o f. When G is a 2-group of nilpotence class 2, $G/\mathbf{Z}(G)$ is elementary abelian, $\mathbf{Z}(G) \cong C_2 \times C_2$ and $|G'| = 2$, by Lemma 2.2 (5) and since $\mathbf{Z}(G)$ is not cyclic, we know that G has exactly two nonlinear irreducible characters and both of them are non-faithful. It follows that $L = 1$ from Proposition 2.1.

Now, we assume that a p-group G satisfies Hypothesis $(*)$ and $L = 1$. First we have $p = 2$ and $|K_1| = |K_2| = 2$ by Lemma 3.1. Hence both K_1, K_2 are minimal normal subgroups of G and $K_1, K_2 \leqslant \mathbf{Z}(G)$. Using the fact that the nontrivial normal subgroups of G not containing G' are K_1 and K_2, we have that G satisfies the strong condition and so $c(G) \leqslant 3$. If $c(G) = 3$, then $G' \nleqslant \mathbf{Z}(G)$ and hence $\mathbf{Z}(G) = K_1$ or $\mathbf{Z}(G) = K_2$. When $\mathbf{Z}(G) = K_1$, we get that $K_2 \leqslant K_1$, contradicting that $L = 1$. Thus $\mathbf{Z}(G) \neq K_1$. For the same reason, $\mathbf{Z}(G) \neq K_2$. Therefore, we obtain that $c(G) = 2$. Also we have that $G/\mathbf{Z}(G)$ is elementary abelian and $\exp G' = 2$ by Lemma 3.2. Furthermore, G/K_i is extra-special, we can deduce that $\mathbf{Z}(G)/K_i = \mathbf{Z}(G/K_i) \cong C_2$, and hence $|\mathbf{Z}(G)| = 4$. We claim that $|G'| = 2$, otherwise, we must have that $\mathbf{Z}(G) = G' \cong C_2 \times C_2$. Thus we can obtain three normal subgroups of G which are different from one another and do not contain G'. That is impossible as the nontrivial normal subgroups of G not containing G' are K_1 and K_2. Then the claim follows. Therefore by Lemma 2.2 (5) we have that G has exactly two nonlinear irreducible characters. Thus G has no faithful irreducible characters, and so $\mathbf{Z}(G)$ is not cyclic, which implies that $\mathbf{Z}(G) \cong C_2 \times C_2$. □

In the next Lemma, we give some properties of p-groups which satisfy Hypothesis $(*)$ and $L > 1$.

Lemma 3.3. *Let a finite p-group G satisfy Hypothesis $(*)$ and let $L > 1$. Then $L < G'$ and G satisfies the weak condition on normal subgroups.*

Proof. By Proposition 2.1 (iv), we have that $G' \cap L = 1$ or $L < G'$. If $G' \cap L = 1$, then $G' \cap K_i = 1$, $i = 1, 2$ by Proposition 2.1. Thus G' and L are all minimal normal subgroups of G and since G is a p-group, we have $L, G' \subseteq \mathbf{Z}(G)$. Since $\mathbf{Z}(G)$ is cyclic, we have that $L = G'$, a contradiction. So $L < G'$.

Next, we prove that G satisfies the weak condition. First, $L \leqslant \mathbf{Z}(G)$. And by Proposition 2.1 (iii), we only need to prove that $|K_i\mathbf{Z}(G) : \mathbf{Z}(G)| = |K_i/(K_i \cap \mathbf{Z}(G))| \leqslant p$, $i = 1, 2$. Since $L \leqslant K_i \cap \mathbf{Z}(G) \leqslant K_i$, and K_i/L are chief factors of G or $K_i/L = 1$, $i = 1, 2$, the proof follows. □

In the rest of this paper, we consider the cases (ii) and (iii) stated above.

Theorem 3.2. *A p-group G satisfies Hypothesis $(*)$ with $1 < L < K_1$ and $1 < L < K_2$ if and only if G is a group of nilpotence class 3, order 32, and $\mathbf{Z}(G) \cong C_2$ or C_4.*

Proof. A computation in GAP using the GAP libraries shows that the groups G of order 32 and nilpotence class 3 are $((C_4 \times C_2) \rtimes C_2) \rtimes C_2$, $(C_8 \rtimes C_2) \rtimes C_2$, $C_2((C_4 \times C_2) \rtimes C_2) = (C_2 \times C_2)(C_4 \times C_2)$, $(C_2 \times D_8) \rtimes C_2$, $(C_2 \times Q_8) \rtimes C_2$ with $\mathbf{Z}(G) \cong C_2$ and $(C_4 \times C_4) \rtimes C_2$, $C_4D_8 = C_4(C_4 \times C_2)$, $(C_8 \times C_2) \rtimes C_2$ with $\mathbf{Z}(G) \cong C_4$. Using GAP's program to compute character tables we verified that in each case Hypothesis $(*)$ holds for G with $1 < L < K_1$ and $1 < L < K_2$.

Now we assume that G satisfies Hypothesis $(*)$, and $1 < L < K_1$, $1 < L < K_2$. First, $p = 2$ by Lemma 3.1. Notice that $L < G'$ by Lemma 3.3. Then L is the unique minimal normal subgroup of G. And we have that the nilpotence class satisfies $c(G) \leqslant 4$ by Lemma 3.2.

If $c(G) = 2$, then $G' \leqslant \mathbf{Z}(G)$ and so G' is cyclic. By Lemma 3.2, it follows that $\exp G' = 2$ or 2^2. We must have $\exp G' = 2^2$, otherwise, $|L| = |G'| = 2$ and since $L, G' \subseteq \mathbf{Z}(G)$, we get that $L = G'$, a contradiction. Thus $G' \cong C_4$ and $G/\mathbf{Z}(G) \cong C_4 \times C_4$. Note that G/L has exactly two nonlinear irreducible characters.

By Lemma 2.2 and $c(G/L) = 2$, it follows that $|\mathbf{Z}(G/L)| = 4$ and then $|\mathbf{Z}(G)/L| \leqslant 4$, $|\mathbf{Z}(G)| \leqslant 8$. Also, by [1], Theorem 6, page 281 we have that $|G/L| = 2^{2m}$, $m \in \mathbb{Z}$. Since $G' \leqslant \mathbf{Z}(G)$, it follows that $|\mathbf{Z}(G)| = 8$ or 4. If $|\mathbf{Z}(G)| = 4$, then $|G| = 2^6$ and $|G/L| = 2^5$, a contradiction. Thus $\mathbf{Z}(G) \cong C_8$. Also, note that G/K_1 has only one nonlinear irreducible character. By Lemma 2.1 (Seitz's Theorem), it follows that

$$2 = |\mathbf{Z}(G/K_1)| = |\mathbf{Z}(\chi_1) : K_1|.$$

Since $|K_1| = 4$, it follows that $|\mathbf{Z}(\chi_1)| = 8$. Therefore $\mathbf{Z}(G) = \mathbf{Z}(\chi_1) \supset K_1$. Since $K_1 \subseteq \mathbf{Z}(G)$ and $G' \subseteq \mathbf{Z}(G)$, we have that $K_1 = G'$ as they have the same order, a contradiction.

If $c(G) = 4$, by Lemma 3.2 it follows that $|G| = 2^5$, that is, G has maximal class. And then G' has index 4 in G and so $|G'| = 8$. Since G/L has exactly two nonlinear irreducible characters $\widehat{\chi}_1$, $\widehat{\chi}_2$ and $\ker \widehat{\chi}_i = K_i/L \neq \overline{1}$ for $i = 1, 2$, it follows that $\mathbf{Z}(G/L)$ is not cyclic and hence $|(G/L)'| = |G'/L| = 2$ by Lemma 2.2. So $|G'| = 4$. We arrive at a contradiction.

The remaining case is $c(G) = 3$. By Lemma 3.2 it follows that $|G : \mathbf{Z}(G)| = 2^3$ or 2^4. Note that $G' \nleq \mathbf{Z}(G)$. So $\mathbf{Z}(G) \in \{L, K_1, K_2\}$ by Proposition 2.1 (iii). Thus $|\mathbf{Z}(G)| = 2$ or 2^2. Again by $|G/L| = 2^{2m}$, we have that $|G| = 2^5$. □

Theorem 3.3. *A p-group G satisfies Hypothesis $(*)$ and $K_1 \leqslant K_2$ if and only if G is a group of nilpotence class 3 and order 3^4.*

Proof. First assume that G satisfies Hypothesis $(*)$ and $K_1 \leqslant K_2$. Then $L = K_1$ and so by Proposition 2.1 we have that $\mathbf{Z}(G)$ cyclic and K_1 is a minimal normal subgroup of G with $K_1 \leqslant \mathbf{Z}(G)$. Hence $|K_1| = p$. Moreover, Lemma 3.3 and Lemma 3.2 show that $K_1 < G'$ and $c(G) \leqslant 4$.

Assume that $K_1 \lneq K_2$. Then $\mathrm{Irr}_1(G/K_2) = \{\widehat{\chi}_2\}$, where $\mathrm{Irr}_1(G/K_2)$ denotes the set of nonlinear irreducible characters of G/K_2. So $p = 2$ by Lemma 2.1. Moreover, we have $\mathrm{Irr}_1(G/K_1) = \{\widehat{\chi}_1, \widehat{\chi}_2\}$. It follows that $c(G/K_1) = 2$ or 3 by Lemma 2.2.

Case (1), when $c(G/K_1) = 3$.

If $c(G) = 4$, then $|G| = 2^5$ by Lemma 3.2, and so $|G/K_1| = 2^4$. But groups with order 2^4 and nilpotence class 3 have three nonlinear irreducible characters, which is a contradiction.

If $c(G) = 3$, then $|G/\mathbf{Z}(G)| = 2^3$ or 2^4 by Lemma 3.2. Note that $G' \nleq \mathbf{Z}(G)$, hence $\mathbf{Z}(G) = K_1$ or K_2. If $\mathbf{Z}(G) = K_1$, then $|\mathbf{Z}(G)| = 2$ and so $|G| = 2^4$ or 2^5. When $|G| = 2^4$, since $c(G) = 3$, it follows that G has just one nonlinear non-faithful irreducible character. This is a contradiction. When $|G| = 2^5$, we would obtain a contradiction in the same way as in the situation of $c(G) = 4$. If $\mathbf{Z}(G) = K_2$, then G satisfies the strong condition on normal subgroups. By Lemma 3.2, we have

$|G| = 2^4$ which is not possible as above. Since $c(G/K_1) = 3$, it is not possible that $c(G) = 2$.

Case (2), when $c(G/K_1) = 2$.

By Lemma 2.2, we get $|G'/K_1| = 2$ and $|\mathbf{Z}(G/K_1)| = 4$. Since G/K_1 has a faithful nonlinear irreducible character $\widehat{\chi}_1$, it follows that $\mathbf{Z}(G/K_1) = C_4$. Therefore, we deduce that G'/K_1 is the unique minimal normal subgroup of G/K_1. Thus we get that all nonlinear irreducible characters of G/K_1 are faithful, contradicting the fact that $\widehat{\chi}_2$ is non-faithful for G/K_1.

Therefore we get that $K_1 = K_2$.

Write $K_1 = K_2 = K$. Then K is the only nontrivial normal subgroup of G which does not contain G'. Since $K \leqslant \mathbf{Z}(G)$, it follows that G satisfies the strong condition on normal subgroups and hence $c(G) \leqslant 3$. If $c(G) = 2$, then G' is cyclic as $G' \leqslant \mathbf{Z}(G)$. Therefore $|G'| = p$ as $\exp G' = p$. It follows that $K = G'$, a contradiction. So $c(G) = 3$. If $p = 2$, then $|G| = 2^4$ by Lemma 3.2. That is impossible as above. Hence $p \neq 2$. Note that G/K has exactly two nonlinear irreducible characters. Then G/K must be an extra-special 3-group by Lemma 2.2 and so $p = 3$. Moreover, by Lemma 3.2, we have $|G : \mathbf{Z}(G)| = 3^3$. Since $G' \nleqslant \mathbf{Z}(G)$ and by Proposition 2.1 (iii), we obtain that $K = \mathbf{Z}(G)$. It follows that $|\mathbf{Z}(G)| = 3$. Hence $|G| = 3^4$ and G has maximal class.

Conversely, assume that $|G| = 3^4$ and $c(G) = 3$. Then $|\mathbf{Z}(G)| = 3$ and $G/\mathbf{Z}(G)$ is an extra-special 3-group by the properties of groups of order 3^4 with maximal class. Hence $G/\mathbf{Z}(G)$ has exactly two nonlinear irreducible characters by Lemma 2.2 and they are faithful. It indicates that there are $\chi_1, \chi_2 \in \mathrm{Irr}_1(G)$ and $\ker \chi_1 = \ker \chi_2 = \mathbf{Z}(G)$. Since $\ker \chi_1 \cap \ker \chi_2 \neq 1$, it follows that G must have $\varphi \in \mathrm{Irr}_1(G)$ and $\varphi \neq \chi_1, \chi_2$. Suppose $\ker \varphi \neq 1$. Since $\mathbf{Z}(G) \leqslant \ker \varphi$, it follows that $\varphi \in \mathrm{Irr}_1(G/\mathbf{Z}(G))$. So we find three nonlinear irreducible characters in $G/\mathbf{Z}(G)$, which is impossible. Consequently, φ must be faithful and so G has exactly two nonlinear non-faithful irreducible characters. □

Finally, the proof of Theorem 1.1 in Introduction is immediately available by Theorems 3.1, 3.2 and 3.3.

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