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ZEROS OF A CERTAIN CLASS OF GAUSS HYPERGEOMETRIC POLYNOMIALS

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Abstract. We prove that as $n \to \infty$, the zeros of the polynomial

$$
{}_2F_1\left[\begin{matrix} -n, \alpha n + 1 \\ \alpha n + 2 \end{matrix}; z\right]
$$

cluster on (a part of) a level curve of an explicit harmonic function. This generalizes previous results of Boggs, Driver, Duren et al. (1999–2001) to the case of a complex parameter α and partially proves a conjecture made by the authors in an earlier work.

Keywords: asymptotic zero-distribution; hypergeometric polynomial; saddle point method

MSC 2010: 33C05, 30C15

1. Introduction

The generalized hypergeometric function ${}_AF_B$ with A numerator and B denominator parameters is defined by

$$
(1) \qquad {}_AF_B\left[\begin{matrix} (a)_A \\ (b)_B \end{matrix}; z\right] = {}_AF_B\left[\begin{matrix} a_1, a_2, \ldots, a_A \\ b_1, b_2, \ldots, b_B \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{j=A} (a_j)_k}{\prod_{j=1}^{j=B} (b_j)_k} \frac{z^k}{k!},
$$

where $a_i \in \mathbb{C}$, $b_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $1 \leqslant i \leqslant A$, $1 \leqslant j \leqslant B$ and $(\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1) = \Gamma(\alpha+k)/\Gamma(\alpha)$ is the Pochhammer symbol. If any of the numerator parameters is a negative integer, say $a_1 = -n$, $n \in \mathbb{N}$, the series terminates and reduces to

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a polynomial of degree n, called a *generalized hypergeometric polynomial.* In this note, we consider the asymptotic zero distribution of the hypergeometric polynomial

$$p_n(z) = {}_2F_1(-n, a_2(n); b_1(n); z) \tag{2}$$

with complex parameters depending linearly on n as $n \to \infty$.

In [1], based on our experimental evidence and results by previous authors, we made a conjecture on the asymptotic behaviour of zeros of a certain class of such hypergeometric polynomials. The purpose of the present note is to partially prove this conjecture.

In [7], Driver and Duren studied the zeros of the hypergeometric polynomial ${}_2F_1(-n, kn+1; kn+2; z)$ for integers $k, n > 0$. They used the Euler integral representation together with a general theorem of Borwein and Chen and proved the following result.

Theorem 1. *Given k and $n \in \mathbb{N}$, the zeros of the hypergeometric polynomial*

$$ {}_2F_1(-n, kn+1; kn+2; z)$$

cluster on the loop of the lemniscate

$$\left\{z\colon\ |z^k(z-1)| = \frac{k^k}{(k+1)^{k+1}};\ \mathrm{Re}(z) > \frac{k}{k+1}\right\}$$

as $n \to \infty$.

P r o o f. See [7], Theorem 1. □

In [4], Boggs and Duren gave the following extension of Theorem 1.

Theorem 2. *For arbitrary $k > 0$ and $l > 0$, the zeros of the hypergeometric polynomial*

$${}_2F_1(-n, kn+l+1; kn+l+2; z)$$

cluster when $n \to \infty$, on the loop of the lemniscate

$$|z^k(z-1)| = \frac{k^k}{(k+1)^{k+1}} \quad \text{with } \mathrm{Re}(z) > \frac{k}{k+1}.$$

P r o o f. See [4], Theorem 2. □

Below, we study what happens when the parameters are allowed to be complex. Our main result is as follows.

Theorem 3. *Let $\alpha = \eta + \mathrm{i}\zeta$ with $\eta > 0$ and $\zeta \neq 0$. The zeros of the hypergeometric polynomials*

$$p_n(z) = {}_2F_1\left[\begin{matrix}-n, \alpha n + 1\\ \alpha n + 2\end{matrix}; z\right]$$

cluster when $n \to \infty$, on the level curve

$$|z^\alpha(1-z)| = \left|\left(\frac{\alpha}{\alpha+1}\right)^\alpha (\alpha+1)^{-1}\right|.$$

In [1] it was proved (using an indirect method based on [5]) that a convergent sequence of zeros clusters along a level curve of the function $|z^\alpha(1-z)|$. The above theorem identifies this curve as the unique level curve passing through the saddle point of the function. In line with the earlier results of [1], [4] and [7], the zeros will not cluster on the whole level curve. In Lemma 3 and Proposition 2, we describe a zero-free region, which includes the left half plane. Following the above references, our proof uses the Euler integral representation of hypergeometric polynomials and the saddle point method.

2. Integral representation of the hypergeometric function and the saddle point method

2.1. Integral representation of the hypergeometric function. The Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ has the following integral representation due to Euler (see [2], Theorem 2.2.1). If $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, then

$$ {}_2F_1\left[\begin{matrix}a, b\\ c\end{matrix}; z\right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,\mathrm{d}t, \tag{3}$$

in the z plane cut from 1 to ∞ along the real axis. Here it is understood that $\arg t = \arg(1-t) = 0$ and that $(1-zt)^{-a}$ takes its principal value.

Choosing $a = -n$, $b = \alpha n + 1$ and $c = b + 1$, where $n \in \mathbb{N}$ and α is a complex number, $\mathrm{Re}(\alpha) > 0$, we have

$${}_2F_1\left[\begin{matrix}-n, \alpha n + 1\\ \alpha n + 2\end{matrix}; z\right] = (\alpha n + 1)\int_0^1 t^{\alpha n}(1-tz)^n\,\mathrm{d}t.$$

This identity is valid for z arbitrary, since the right-hand side is an entire function of z when $\mathrm{Re}(\alpha) > 0$. We reiterate that the integrand uses the principal branch of the logarithm to define t^α. Recall that this branch is defined in the complement to the negative real axis, and coincides with the real logarithm on the positive real axis.

2.2. The saddle point method. Let γ be a contour in the complex plane and suppose that the function φ is holomorphic in a neighborhood of this contour. We study the asymptotics as $n \to \infty$ of an integral:

$$I_n(z) = \int_\gamma \mathrm{e}^{n\varphi(z)}\,\mathrm{d}z. \tag{4}$$

The idea of the saddle point method is to deform the contour to a contour following the paths of steepest ascent/descent of $\varphi(z) = u(x,y) + \mathrm{i}v(x,y)$. It is an immediate consequence of the Cauchy-Riemann equations that the gradient of u is given (as a complex vector) by $\overline{\varphi}'(z)$. It is orthogonal to the gradient of v, and hence if $\varphi'(z_0) \neq 0$, there is a unique curve through z_0 characterized by both the property that $\operatorname{Im}\varphi(z) = \operatorname{Im}\varphi(z_0)$ and that along this curve $\operatorname{Re}\varphi(z)$ grows fastest. This curve is called a curve of *steepest ascent* (or *descent*, if we reverse the direction). A simple saddle point is a point t_0 where $\varphi'(t_0) = 0$ and $\varphi''(t_0) \neq 0$. In a small neighborhood of a simple saddle point t_0, the level curve $\operatorname{Im}\varphi(z) = \operatorname{Im}\varphi(t_0)$ consists of two analytic curves that intersect at t_0 and separate the neighborhood of t_0 into four sectors. Of the four curve segments starting at t_0, two, say γ_1 and γ_2, will satisfy the assumption that $\operatorname{Re}\varphi(z) < \operatorname{Re}\varphi(z_0)$, if $z_0 \neq z \in \gamma_i$, $i = 1, 2$. These curves are called the curves of *steepest descent* (or *ascent*, if we reverse the direction).

Proposition 1. *Let t_0 be the unique simple saddle point of φ. Suppose that for two points $a \in \gamma_1$, $b \in \gamma_2$, the curve segments from a to t_0 along γ_1 and from t_0 to b along γ_2 are finite. If γ is the curve from a to b along γ_1 and γ_2, then*

$$I_n(z) = \sqrt{\frac{2\pi}{-\varphi''(t_0)}}\, n^{-1/2} \mathrm{e}^{n\varphi(t_0)} (1 + O(n^{-1})).$$

Proof. See [3], 7.3.11, or [6]. $\square$

3. Main result

We apply the saddle point method to evaluate the asymptotic expansion of the Euler integral

$$ {}_2F_1\left[\begin{matrix} -n, \alpha n+1 \\ \alpha n+2 \end{matrix}; z\right] = (\alpha n+1)\int_0^1 t^{\alpha n}(1-zt)^n \,\mathrm{d}t, $$

where $\alpha = \eta + \mathrm{i}\zeta$, $\eta > 0$ and $\zeta \neq 0$. Set

$$ p_n = \int_0^1 [g(t)]^n \,\mathrm{d}t, $$

where $g(t) = t^\alpha(1-tz)$ (using the principal branch of the logarithm to define t^α). The function $g(t)$ vanishes at 0 and $1/z$, and has $t = 0$ as its only branch point.

3.1. The saddle point. Consider $\varphi(t) = \alpha \log t + \log(1-zt)$, where possibly different branches of the logarithm may be used for the two terms, depending on the simply-connected domain U where φ is defined. For each fixed $z \neq 0$, the underlying multi-valued function φ has two branch points, at $t = 0$ and $t = 1/z$. Since

$$ \varphi'(t) = \frac{\alpha - zt(\alpha+1)}{t(1-zt)}, \tag{5} $$

we see that $\varphi'(t) = 0 \iff t = t_0 := \alpha/(\alpha+1)z$. Since

$$ \varphi''(t) = -\frac{(1+\alpha)^3}{\alpha}z^2, $$

t_0 is a simple saddle point under the assumption $\alpha \neq -1$ and $z \neq 0$. Note that the derivative φ' is defined (as a meromorphic function) in the whole $\mathbb{C}$ by (5). This implies that paths of steepest ascent/descent are unambiguously defined through all points (except the poles 0, $1/z$ and the saddle point t_0) independently of posssible choices of a branch of logarithm in the definition of $\varphi(t)$. Namely, by the Cauchy-Riemann equations, the gradient (considered as a complex number) at t of $\operatorname{Re}\varphi$ is given by $\overline{\varphi'(t)}$. We say that the descent picture of $\varphi(t)$ is *branch-independent.*

Lemma 1. *Let $t_0 = \alpha/(\alpha+1)z$ be the saddle point of $\varphi(t)$, where $\alpha \neq -1$ and $z \neq 0$.*

(i) *There are exactly two paths of steepest descent, along which $\operatorname{Re}\varphi(t)$ decreases. Starting at the saddle point t_0, one, say γ_1, goes to 0 and the other, say γ_2, to $1/z$.*

(ii) *There are exactly two paths δ_1 and δ_2 of steepest ascent, along which* $\operatorname{Re}\varphi(t)$ *increases. Starting at the saddle point t_0, both go to infinity. Together they split the plane into two (closed) simply-connected regions D_0 and $D_{1/z}$, such that if $t \in D_0$, there is a path of steepest descent from t to 0, and correspondingly for $D_{1/z}$.*

Proof. (i) Consider the local behavior of $\varphi(t)$ around each of the points t_0, 0, $1/z$. At t_0, $\varphi(t)$ is approximated by $B(t-t_0)^2$, where $B = \frac{1}{2}\varphi''(t_0) = -\frac{1}{2}z^2(\alpha+1)^3/\alpha \neq 0$. Hence, there are two paths with $\operatorname{Im}\varphi(t) = \operatorname{Im}\varphi(t_0)$, starting at $t = t_0$, along which $\operatorname{Re}\varphi(t)$ decreases. They cannot escape to infinity, since we assumed that $\alpha = \eta + \zeta\mathrm{i}$, with $\eta > 0$, so $\operatorname{Re}\varphi(t) = (\eta+1)\log|t| + O(1)$ for $t \gg 0$ (for any choice of U and a branch of φ). Hence, the curves of steepest *ascent*, in a neighbourhood of infinity, approximate radial rays from the origin, and thus a curve of steepest *descent* cannot approach infinity.

There is only one saddle point, so the paths cannot intersect, and have to end up at either 0 or $1/z$, which are the only points where $\operatorname{Re}\varphi(t) = -\infty$ (again for any choice of U and branch of φ). It remains to show that the paths do not end up at the same point, and this follows easily by checking that locally there is at most one path *starting* at either 0 or $1/z$ with constant value of $\operatorname{Im}\varphi(t)$. Around $t = 1/z$ the function behaves as $\log(t-1/z)+\delta$ for some $\delta \in \mathbb{C}$, and so there will be at most one ray corresponding to constant imaginary value. This finishes the proof.

(ii) The second part follows similarly. $\square$

Given $z \in \mathbb{C}$, one of two cases may occur: either $1 \in D_0$ or $1 \in D_{1/z}$. Define

$$E := \{z\colon\ 1 \in D_{1/z}\}.$$

If $\alpha \in \mathbb{R}$,

$$z \in E \iff \operatorname{Re} z \geqslant \frac{\alpha}{\alpha+1}$$

(see [8]); but for a general α, we do not have a similar precise description. Instead, we can use the descent picture of $\psi(w) := \log w^\alpha(1-w)$. This function has a unique simple saddle point at $\alpha/(\alpha+1)$, and (as in the lemma) the paths of steepest ascent split the complex plane into two regions $\tilde{D}_0$ and $\tilde{D}_1$, containing the branch points 0 and 1 of the logarithm, respectively. Its descent picture is again branch-independent.

Lemma 2. $E = \tilde{D}_1$.

Proof. We use the notation of the preceding lemma and the fact that the descent pictures of φ and ψ only depend on the respective meromorphic derivative. The intuition is that

$$z^\alpha g(t) = z^\alpha t^\alpha(1-tz) = w^\alpha(1-w)$$

with $w = zt$ (using compatible determinations of log). We claim that there is a path of steepest descent from $t = 1$ to $t = 0$ with respect to $\varphi(t) = \log g(t)$ if and only if there is a path of steepest descent from $w = z$ to $w = 0$, with respect to ψ. In detail, $z\psi'(zt) = \varphi'(t)$. So if $\gamma(s),\ s \in [0,1]$, is a path from $z = \gamma(1)$ to $\gamma(0) = 0$ such that the tangent $\gamma'(s) = -\overline{\psi'(\gamma(s))}$, then $\delta(s) = z^{-1}\gamma(s)$ is a path from 1 to 0 such that $\delta'(s) = -z^{-1}\overline{\psi'(z^{-1}\gamma(s))} = -|z|^{-2}\overline{\varphi'(\delta(s))}$. Clearly δ will be a parametrization of a path of steepest descent for $\varphi(t)$. Hence $1 \in D_0 \iff z \in \tilde{D}_0$, which proves the lemma. □

From the assumption that $\eta > 0$ it follows that E^{c} (the complement of E) contains the left half plane.

Lemma 3. *Suppose that $z = x + \mathrm{i}y$ with $x \leqslant 0$. Then $z \in E^{\mathrm{c}}$.*

P r o o f. It suffices to show that the segment $\gamma(s) = sz$, $s \in [0,1]$ is an ascending curve for the function $\psi(w)$. For γ to be ascending from 0 it is enough to show that $\mathrm{Re}(\overline{\psi'(\gamma(s))\gamma'(s)}) > 0$, which after some simplifications is equivalent to $E(s) := \eta - 2\eta xs + (-x + \eta(x^2+y^2))s^2 + (x^2+y^2)s^3 \geqslant 0$. This is a cubic curve in the plane and clearly $\lim\limits_{s\to\infty} E(s) = \infty$, and it will have at most two stationary points, and an inflection point between them. Now $E''(s) = 0$ has the solution

$$s = \frac{x - \eta(x^2+y^2)}{3(x^2+y^2)} \leqslant 0,$$

by assumption. Since furthermore $E'(0) = -2\eta x > 0$ and $E(0) = \eta > 0$, $E(s)$ will be strictly increasing with $s \geqslant 0$. □

3.2. Zero-free region. We sketch a proof of the fact that the complement to E is a zero-free region.

Proposition 2. *If $z \notin E$, then z is not the limiting point of zeros of $p_n(z)$.*

S k e t c h o f p r o o f. Assume that $z_n \to z$ is a sequence such that $p_n(z_n) = 0$ and $z \notin E$. Hence $z_n \notin E$ for large enough n, and so there will be paths γ_n of steepest ascent from 0 to 1, and this means, using standard Laplace-type techniques, that the integral

$$0 = \left(\left|\int_0^1 (t^\alpha(1 - tz_n))^n \,\mathrm{d}t\right|\right)^{1/n}$$

will have limit $|1 - z|$ (compare the proof of Lemma 6). A contradiction. □

3.3. The first integral. Now we start the actual proof of the main theorem. Assume that $z \in E$. Then there is a path of steepest ascent from $1/z$ to 1, and a path from 0 to $1/z$, first as a path of steepest ascent to the saddle point, and then as a path of steepest descent to $1/z$. We accordingly deform the contour of integration to get

$$\int_0^1 [g(t)]^n \,\mathrm{d}t = \int_0^{1/z} [g(t)]^n \,\mathrm{d}t + \int_{1/z}^1 [g(t))]^n \,\mathrm{d}t = I_1 + I_2. \tag{6}$$

We detail the path from 0 to $1/z$ in the first integral. Note that the spiral γ_1 around 0 will intersect the line L between 0 and $1/z$ in an infinite set of points clustering towards 0. Choose an arbitrary $\varepsilon > 0$. Start at 0, walk along L a distance at most ε, until one of the intersection points w_ε with γ_1 is reached. Call this part L_ε and continue along the part of γ_1 that remains to z_0 and then continue to $1/z$ along γ_2. Starting with the principal branch of the logarithm that is defined at $z_0 = \alpha/((\alpha+1)z)$ and at $1/z$ (by Lemma 3), we may analytically continue this branch along γ_1 to w_ε; we interpret $t^\alpha := \mathrm{e}^{\alpha \log t}$ in the integral in this sense. Now if ε is small enough, for $t \in L_\varepsilon$, $|1 - zt| < 2$. So there exists $M > 0$ such that $|t^{\alpha n}(1 - zt)^n| < Mt^\eta$ for $t \in L_\varepsilon$. As a consequence, using that $\alpha = \eta + \zeta\,\mathrm{i}$, with $\eta > 0$, the integral has an estimate

$$\left| \int_0^{w_\varepsilon} t^{\alpha n}(1 - zt)^n \,\mathrm{d}t \right| \leqslant \frac{M\varepsilon^{\eta+1}}{\eta+1}. \tag{7}$$

We estimate the remaining integral between w_ε and $1/z$ using the saddle point method. Note that the paths γ_ε and γ_2 are finite.

Lemma 4. *Let $z \neq 0$. For any $\varepsilon > 0$,*

$$\left| \int_0^{1/z} [g(t)]^n \,\mathrm{d}t \right| = g(z_0)^n \sqrt{\frac{2\pi}{n|\varphi''(t_0)|}} + O\Big(\frac{1}{n}\Big) + \varepsilon. \tag{8}$$

P r o o f. Follows from Lemma 1 and (7). □

3.4. The second integral. Now we turn to the second integral

$$I_2(z) = \int_{1/z}^1 [t^\alpha(1 - zt)]^n \,\mathrm{d}t.$$

Since $z \in E$, we have a path of steepest ascent from $1/z$ to 1. Heuristically, the value $(1 - z)^n$ at $t = 1$ asymptotically dominates the integral. We will now prove this. Start with the following clever change of variables, that we have taken from [7].

Define $t = t(s)$ implicitly, using the relation

$$t^\alpha(1 - zt) = s(1 - z).$$

Note that $s = 0 \Leftrightarrow t = 1/z$ and $s = 1 \Leftrightarrow t = 1$. We get a path $\Delta(s) = \{t(s)\colon 0 \leqslant s \leqslant 1\}$ from $1/z$ to 1.

Lemma 5. $\int_{1/z}^1 [g(t)]^n \, \mathrm{d}t = \int_\Delta [g(t)]^n \, \mathrm{d}t.$

Proof. Since $|g(t)| = O(|t|^\eta)$ and $\eta > 0$, the integral over a circle around the origin goes to 0 as the radius of the circle decreases. Hence, we can move the path of integration freely in the complex plane. □

We have that $t^{\alpha-1}(\alpha - (\alpha+1)zt) \, \mathrm{d}t = (1 - z) \, \mathrm{d}s$. Hence,

$$I_2 = (1 - z)^n \int_0^1 \frac{t(1 - zt)s^{n-1}}{\alpha - (\alpha + 1)zt} \, \mathrm{d}s = (1 - z)^n K(z),$$

$$|I_2|^{1/n} = |1 - z||K(z)|^{1/n}. \tag{9}$$

The calculation of the limit is completed by proving

$$\lim_{n\to\infty} |K(z)|^{1/n} = 1 \tag{10}$$

and

$$\liminf_{n\to\infty} |k(z)| \geqslant 1.$$

Indeed, set

$$f(s) = \frac{t(1 - zt)}{\alpha - (\alpha + 1)zt}.$$

Then (10) is a consequence of the following lemma.

Lemma 6. *Assume that $f(z)$ is a nonzero function, analytic in a domain D, containing the unit interval* $[0, 1]$. *Then*

$$\lim_{n\to\infty} \left| \int_0^1 f(s)s^{n-1} \, \mathrm{d}s \right|^{1/n} = 1.$$

Proof. If $f(z)$ is constant, it is trivial. Obviously, both $\operatorname{Re} f(z)$ and $\operatorname{Im} f(z)$ cannot be identically zero on $[0, 1]$; assume that $\operatorname{Re} f(z) \not\equiv 0$. Let $0 \leqslant \delta < 1$ be minimal with respect to the property that $h(z) := \operatorname{Re} f(z)$ is of constant sign in $[\delta, 1]$, and assume that $h(s) \geqslant 0$ in $[\delta, 1]$. For each $C\colon\ 0 < C < \max\{h(s)\colon\ s \in [\delta, 1]\}$ there is an interval $[\delta_2, \delta_1] \subset [\delta, 1]$ with $\delta_1 \neq \delta_2$, such that $\delta_2 \leqslant s \leqslant \delta_1 \Longrightarrow h(s) \geqslant C$.

Set $I(n) := \int_0^1 f(s)s^{n-1}\,\mathrm{d}s$ and $M := \sup\{|h(s)|\colon\ s \in [0, 1]\}$. Then

$$\begin{aligned}\operatorname{Re} I(n) &\geqslant \int_0^\delta h(s)s^{n-1}\,\mathrm{d}s + \int_\delta^1 h(s)s^{n-1}\,\mathrm{d}s \geqslant \int_0^\delta -Ms^{n-1}\,\mathrm{d}s + \int_{\delta_2}^{\delta_1} Cs^{n-1}\,\mathrm{d}s \\ &= -M\frac{\delta^n}{n} + C\frac{\delta_1^n}{n} - C\frac{\delta_2^n}{n} = \frac{\delta_1^n}{n}\Big(-\Big(\frac{\delta}{\delta_1}\Big)^n M - \Big(\frac{\delta_2}{\delta_1}\Big)^n C + C\Big).\end{aligned}$$

Since $\delta/\delta_1 < 1$ and $\delta_2/\delta_1 < 1$, there exists n_0 such that for $n > n_0$

$$\operatorname{Re} I(n) \geqslant \frac{\delta_1^n}{n}\frac{C}{2} \Longrightarrow \liminf |I(n)|^{1/n} \geqslant \delta_1.$$

But clearly $\lim_{c \to 0} \delta_1 = 1$. On the other hand, $\limsup |I(n)|^{1/n} \leqslant 1$, since f is bounded on $[0, 1]$. This finishes the proof. □

3.5. Proof of the main theorem. By (6), a zero z_n of the polynomial $p_n(z)$ in (2) satisfies the equation $I_1 = -I_2$. If a sequence of zeros $z_n \to z \in E$, we get

$$\lim_{n\to\infty} |I_1|^{1/n} = \lim_{n\to\infty} |I_2|^{1/n} = |1 - z|,$$

by (10) and (9.) It follows from formula (3) that $z = 0$ is not a zero of p_n. Hence, Lemma 1 and Lemma 4 apply and we have that

$$\lim_{n\to\infty} |I_1|^{1/n} = |t_0^\alpha(1 - zt_0)|.$$

An easy calculation then shows that

$$\Big|\Big(\frac{\alpha}{\alpha+1}\Big)^\alpha (\alpha+1)^{-1}\Big| = |z^\alpha||1 - z|.$$

(The form of the left-hand side is chosen to avoid problems with changing the branch of the logarithm; note that $\alpha/(\alpha+1)$ is in the right half-plane, as well as z, by Lemma 3.) This finishes the proof. □

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