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SUBSETS OF NONEMPTY JOINT SPECTRUM IN TOPOLOGICAL ALGEBRAS

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Abstract. We give a necessary and a sufficient condition for a subset S of a locally convex Waelbroeck algebra $\mathcal{A}$ to have a non-void left joint spectrum $\sigma_l(S)$. In particular, for a Lie subalgebra $L \subset \mathcal{A}$ we have $\sigma_l(L) \neq \emptyset$ if and only if $[L, L]$ generates in $\mathcal{A}$ a proper left ideal.

We also obtain a version of the spectral mapping formula for a modified left joint spectrum. Analogous theorems for the right joint spectrum and the Harte spectrum are also valid.

Keywords: joint spectrum; Waelbroeck algebra; commutator; spectral mapping formula

MSC 2010: 47A13, 47A60, 46H30

1. Introduction

Let $\mathcal{A}$ be a complex associative algebra with unit e. The left (right) joint spectrum of a k-tuple $\bar{a} = (a_1, \ldots, a_k)$ of elements of $\mathcal{A}$ is defined as the set of k-tuples of complex numbers $\bar{\lambda} = (\lambda_1, \ldots, \lambda_k)$ such that the left (right) ideal generated by the elements $a_1 - \lambda_1 e, \ldots, a_k - \lambda_k e$ is nontrivial.

It is known that in the case of a Banach algebra and even of a Waelbroeck algebra $\mathcal{A}$ the joint spectrum of an arbitrary commutative k-tuple $\bar{a} = (a_1, \ldots, a_k)$ of elements of $\mathcal{A}$ is nonempty. Moreover, if $P\colon \mathbb{C}^k \to \mathbb{C}^m$ is a polynomial mapping, then the spectral mapping formula

$$\sigma_l(P(a_1, \ldots, a_k)) = P(\sigma_l(a_1, \ldots, a_k)) \tag{1}$$

is valid as well as an analogous formula for the right joint spectrum, see [1], [7], [6].

The notion of the joint spectrum can be extended to infinite subsets of $\mathcal{A}$. For a given subset $S \subset \mathcal{A}$ denote by $I_l(S)$ (or $I_r(S)$) the left (right) ideal generated by

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the elements of the set S. Define the left (right) joint spectrum of the set S as the space of all functions $\sigma\colon S \to \mathbb{C}$ such that the left ideal $I_l(\{a-\sigma(a)e\colon a \in S\})$ (or $I_r(\{a-\sigma(a)e\colon a \in S\})$) generated by elements of the form $a-\sigma(a)e$, $a \in S$ is proper.

The condition of the commutativity of elements of S is not necessary for $\sigma_l(S)$ to be nonempty (see e.g. [2], [3], [5]).

The principal aim of this paper is to determine a necessary and a sufficient condition assuring $\sigma_l(S) \neq \emptyset$. It appears that the Lie algebra structure of $\mathcal{A}$ is very relevant in this study.

Let $[S,S]$ denote the commutator of a set $S \subset \mathcal{A}$:

$$[S,S] = \{[a,b] = ab - ba\colon\ a,b \in S\}.$$

By $B(S)$ we denote the unital subalgebra of $\mathcal{A}$ generated by the set S.

If the left joint spectrum of S is nonempty and $\sigma \in \sigma_l(S)$, the (proper) left ideal $I_l(\{s-\sigma(s)e\colon s \in S\})$ contains the set $[S,S]$ as well as the set $[S,S]B(S) = \{cb\colon c \in [S,S],\ b \in B(S)\}$. Consequently, $I_l([S,S]B(S))$ is a proper ideal.

We prove that for locally convex Waelbroeck algebras (in particular for Banach algebras) the latter condition is sufficient for $\sigma_l(S)$ to be nonempty.

The condition simplifies if the set S is a subalgebra of $\mathcal{A}$ or a Lie subalgebra of $(\mathcal{A},[\cdot,\cdot])$. It takes then the form $I_l([S,S]) \neq \mathcal{A}$.

If $\mathcal{A}$ is finitely generated and $S = \mathcal{A}$, we obtain exactly the statements proved in [3].

In Section 4 we obtain a formula analogous to the spectral mapping formula (1), but the spectrum used in this case is a modified left spectrum calculated modulo the ideal $I_l([S,S]B(S))$.

2. The commutator theorem

The necessary condition for $\sigma_l(S)$ to be nonempty is purely algebraic.

Proposition 2.1. *Let $\mathcal{A}$ be a complex, associative, unital algebra. Let $S \subset \mathcal{A}$. For arbitrary function $\sigma\colon S \to \mathbb{C}$ the ideal $J = I_l(\{a-\sigma(a)e\colon a \in S\})$ contains $[S,S]$ and satisfies $Js \subset J$ for every $s \in S$.*

Proof. For all $a,b \in S$

$$[a,b] = [a-\sigma(a)e, b-\sigma(b)e] \in J.$$

For every $a,s \in S$ we have

$$(a-\sigma(a)e)s = (a-\sigma(a)e)(s-\sigma(s)e) + \sigma(s)(a-\sigma(a)e) \in J,$$

which proves $Js \subset S$ for every $s \in S$. □

Corollary 2.2. *Let $S \subset \mathcal{A}$. If $\sigma_l(S) \neq \emptyset$ then the set $[S,S]B(S)$ is contained in a proper left ideal in $\mathcal{A}$.*

Proof. If $\sigma \in \sigma_l(S)$, then $J = I_l(\{a-\sigma(a)e\colon\ a \in S\})$ is a proper ideal containing $[S,S]$. It is right S-invariant, hence $[S,S]B(S) \subset J$. The statement follows. $\square$

The sufficient condition for $\sigma_l(S) \neq \emptyset$ will be proved under an additional topological assumption.

Recall that a topological algebra $\mathcal{A}$ is called Waelbroeck algebra if the set $G(\mathcal{A})$ of elements invertible in $\mathcal{A}$ is open and the inversion $G(\mathcal{A}) \ni a \to a^{-1} \in G(\mathcal{A})$ is continuous.

Many important properties of Banach algebras are valid for locally convex Waelbroeck algebras. Let $\mathcal{A}$ be a locally convex Waelbroeck algebra. All maximal ideals in $\mathcal{A}$ are closed. The spectrum of a single element $a \in \mathcal{A}$ is non-void and the joint spectrum of a k-tuple is compact.

If a subalgebra $\mathcal{A}_1 \subset \mathcal{A}$ is closed under the inverse, then $\mathcal{A}_1$ is a Waelbroeck algebra. For a closed two-sided ideal $J \subset \mathcal{A}$, the quotient algebra $\mathcal{A}/J$ is a Waelbroeck algebra as well.

Theorem 2.3. *Let $\mathcal{A}$ be a locally convex Waelbroeck algebra and let $S \subset \mathcal{A}$. Then for every proper left ideal $I \subset \mathcal{A}$ which contains $[S,S]$ and satisfies $IS \subset I$ there exists in $\mathcal{A}$ a maximal ideal M and $\sigma \in \sigma_l(S)$ such that $I + I_l(\{a-\sigma(a)e\colon\ a \in S\} \subset M)$.*

Proof. Let $\mathcal{J}$ be the family of all proper left ideals J in $\mathcal{A}$ such that $I \subset J$ and $Ju \subset J$ for all $u \in S$. The set $\mathcal{J}$ is ordered by inclusion. By Kuratowski-Zorn lemma there is a maximal element M in $\mathcal{J}$.

Define the action of the product $\mathcal{A} \times S$ on the space $X = \mathcal{A}/M = \pi(\mathcal{A})$ by the formula

$$T_{(a,s)}[x] = L_a R_s [x] = [axs]$$

for $a \in \mathcal{A}$, $s \in S$ and $[x] = x + M$, $x \in \mathcal{A}$.

We claim that the only subspaces in X invariant under all operators $T_{(a,s)}$ are 0 and X. Effectively, if $V \subset X$ is an invariant vector space and $J = \pi^{-1}(V)$, then J is a left ideal in $\mathcal{A}$ which contains M. Moreover, it satisfies $Ju \subset J$ by the invariance of V under the operators R_s, $s \in S$. By the maximality of M we obtain $J = M$ or $J = \mathcal{A}$.

By the Schur lemma argument, the commutant D of the family of operators $\mathcal{T} = \{T_{(a,s)}\colon\ (a,s) \in \mathcal{A} \times S\}$ is a division algebra.

The commutant D can be described in terms of elements of the algebra $\mathcal{A}$. Let

$$E = \{a \in \mathcal{A}\colon\ Ma \subset M \text{ and } as - sa \in M,\ s \in S\}.$$

The space E is a subalgebra in $\mathcal{A}$, it contains the unit and the ideal M, which forms in E a two-sided ideal. For every $a \in E$ the operator $R_a[x] = [xa]$ defines an element of D. We claim that all elements of D are of this form. Let $R \in D$. Then $R[x] = RL_x[e] = L_x R[e]$. If $R[e] = [a]$, then $R[x] = [xa] = R_a[x]$.

The mapping $E \ni a \to R_a \in D$ is onto. Its kernel is just the ideal M hence $D \cong \mathcal{A}/M$. All elements of the commutant D are continuous operators on X as projections on X of operators of multiplication in $\mathcal{A}$.

Note that the subalgebra E is closed in $\mathcal{A}$ and it is closed with respect to the operations of inverse in $\mathcal{A}$. Namely, if $a \in \mathcal{A}$ has an inverse $a^{-1} \in \mathcal{A}$, then the space $M' = M + Ma^{-1}$ is a left ideal in $\mathcal{A}$ which contains M. It is proper, because $M'a = Ma + M \subset M$.

If $b \in S$, then $ba = ab + m$, where $m \in I_l([S,S]) \subset M$. We obtain

$$\begin{aligned} M'b &= (M + Ma^{-1})b \subset M + Ma^{-1}baa^{-1} = M + Ma^{-1}(ab+m)a^{-1} \\ &= M + Mb + Ma^{-1}ma^{-1} \subset M + Ma^{-1} = M'. \end{aligned}$$

The ideal M' is of the class $\mathcal{J}$, hence by the maximality of M it follows that $M' = M$ and $Ma^{-1} \subset M$. It holds $a^{-1}b - ba^{-1} = a^{-1}(ba - ab)a^{-1} = a^{-1}ma^{-1} \in M$, hence effectively $a^{-1} \in E$.

As an inverse-closed subalgebra of a locally convex Waelbroeck algebra, E is of the same class, as well as the algebra $D = E/M$.

By the Gelfand-Mazur theorem for locally convex Waelbroeck algebras it follows that $D \cong \mathbb{C}$. The condition $[S,S] \subset I_l([S,S]) \subset M$ implies that $S \subset E$. For every $s \in S$ there is $\sigma(s) \in \mathbb{C}$ such that $R_s[x] = \sigma(s)[x]$ for every $[x] \in X$. In particular $s - \sigma(s)e \in M$.

The action of the operators $L_a R_x$, $a \in \mathcal{A}$, $x \in S$ on $X = \mathcal{A}/M$ is irreducible. However, the operators R_x are scalars, hence the representation L of $\mathcal{A}$ on X is irreducible. It means that M is a maximal ideal in $\mathcal{A}$ (not only as an element of the family $\mathcal{J}$). The statement follows. □

By Proposition 2.1 and Theorem 2.3 we obtain the following.

Theorem 2.4. *Let $\mathcal{A}$ be a locally convex Waelbroeck algebra and let $S \subset \mathcal{A}$. Then $\sigma_l(S) \neq \emptyset$ if and only if $I_l([S,S]B(S)) \neq \mathcal{A}$.*

If S is finite and satisfies $[S,S] = 0$, we obtain the theorems mentioned in Introduction about the joint spectrum for k-tuples of commuting elements.

Note that the condition $I_l([S,S]B(S)) \neq \mathcal{A}$ is satisfied if $[S,S]B(S)$ consists of topological zero divisors or the operators of multiplication by $a \in [S,S]$ are compact. The theory of Toeplitz operators provides examples of the latter situation.

3. Joint spectrum in subalgebras

The condition $I_l([S,S]B(S)) \neq \mathcal{A}$ simplifies substatially if we suppose that $[S,S] \subset S$.

Proposition 3.1. *Let $\mathcal{A}$ be a complex associative algebra and let $S \subset \mathcal{A}$ satisfy $[S,S] \subset S$. Then the ideal $I_l([S,S])$ coincides with $I_l([S,S]B(S))$.*

Proof. It suffices to prove that $I_l([S,S])s \subset I_l([S,S])$ for every $s \in S$. Let $u, v, s \in S$. Then

$$[u,v]s = s[u,v] + [[u,v],s] \in I_l([S,S]).$$

The statement follows. □

The condition $[S,S] \subset S$ means that the linear span $L(S)$ of S is a Lie subalgebra of $\mathcal{A}$.

Corollary 3.2. *Let $\mathcal{A}$ be a locally convex Waelbroeck algebra. If L is a Lie subalgebra of $\mathcal{A}$, then $\sigma_l(L) \neq \emptyset$ if and only if $I_l([L,L]) \neq \mathcal{A}$.*

Obviously, the result is valid in the particular case $L = \mathcal{A}$.

Note that in this case $I_l([\mathcal{A},\mathcal{A}]) = I_r([\mathcal{A},\mathcal{A}])$ by Proposition 3.1 and that this ideal coincides with the two-sided ideal generated by $[S,S]$ in $\mathcal{A}$. It was observed in [3] in the case of Banach finitely generated algebra $\mathcal{A}$.

For an algebra B denote by $\widehat{B}$ the space of multiplicative functionals on B. The commutator $[B,B]$ generates in B a two-sided ideal contained in the kernel of every multiplicative functional on B. Let B be a subalgebra in $\mathcal{A}$. If $\varphi \in \widehat{B}$ and the kernel of φ generates in $\mathcal{A}$ a proper left ideal, then $\varphi \in \sigma_l(\mathcal{A})$. The following result, which is strictly algebraic, shows that under the condition $I_l([B,B]) \neq \mathcal{A}$ all elements of $\sigma_l(\mathcal{A})$ are of this form.

Proposition 3.3. *Let $\mathcal{A}$ be an associative unital algebra and let B be a subalgebra of $\mathcal{A}$ which contains the unit. Then $\sigma_l(B) \subset \widehat{B}$.*

Proof. Let $\sigma \in \sigma_l(B)$. We must show that σ is a linear multiplicative function. By definition of the spectrum, the ideal $J = I_l(\{a - \sigma(a)e\colon\ a \in B\})$ is proper. Since $a - \sigma(a)e \in J \cap B$ for all $a \in B$, it follows that $J \cap B$ is at most of codimension 1 in B. Since $[B,B] \subset B$, the ideal J satisfies $JB \subset J$, hence $J \cap B$ is a two-sided ideal in B. There is a unique multiplicative functional φ on B such that $J \cap B = \ker \varphi$, so that for all $a \in B$ it holds $a - \varphi(a)e \in J$. It proves that $\sigma = \varphi$. □

Corollary 3.4. *There is a bijection between the space $\sigma_l(B)$ and the set $\widehat{B}_{\mathcal{A}}$ of the multiplicative functionals on B such that* $\ker \varphi$ *generates a proper left ideal in $\mathcal{A}$.*

Now, suppose that $B = B(S)$ for some $S \subset \mathcal{A}$.

Obviously, if $\sigma \in \sigma_l(B(S))$, then the restriction of σ to S belongs to $\sigma_l(S)$. In this sense $\sigma_l(B(S)) \subset \sigma_l(S)$. Under the assumption $[S,S] \subset S$ we can prove that every element of $\sigma_l(S)$ has a unique extension which is an element of $\sigma_l(B(S))$.

Lemma 3.5. *Let $S \subset \mathcal{A}$ satisfy $[S,S] \subset S$. Denote by I the two-sided ideal generated in $B(S)$ by $[S,S]$. Let $(s_1,\dots,s_n)$ be an n-tuple of elements of S. If π is a permutation of n elements, then*

$$s_1 s_2 \dots s_n - s_{\pi(1)} \dots s_{\pi(n)} \in I.$$

If P, Q are polynomials of k and m variables, respectively, then for arbitrary $\bar{s} = (s_1,\dots,s_k)$ and $\bar{t} = (t_1,\dots,t_m)$ of elements of S it holds that

$$P(\bar{s})Q(\bar{t}) - Q(\bar{t})P(\bar{s}) \in I. \tag{2}$$

Proof. The group of permutations of n elements is generated by the elemental transpositions, hence it is sufficient to prove the statement for a transposition. We have

$$s_1 \dots s_k s_{k+1} \dots s_n - s_1 \dots s_{k+1} s_k \dots s_n = s_1 \dots s_{k-1}[s_k, s_{k+1}]s_{k+2} \dots s_n \in I.$$

Formula (2) is valid because every term of the product $P(\bar{s})Q(\bar{t})$ is a permutation of a coresponding term of $Q(\bar{t})P(\bar{s})$. The proof is complete. $\square$

One of the consequences of Lemma 3.5 is that for an arbitrary k-tuple $\bar{s} = (s_1,\dots,s_k)$ of elements of S and for every polynomial P of k variables the value $P(\bar{s}) = P(s_1,\dots,s_k)$ is uniquely defined as an element in the quotient algebra $B(S)/I$.

Theorem 3.6. *Let $\mathcal{A}$ be a locally convex Waelbroeck algebra and let $S \subset \mathcal{A}$ satisfy $[S,S] \subset S$. If $\sigma \in \sigma_l(S)$, then there exists $\varphi \in \widehat{B(S)}$ such that for every $b \in S$ it holds that $\sigma(b) = \varphi(b)$.*

Proof. We know by Proposition 3.1 that $I_l([S,S])B(S) = I_l([S,S])$. The ideal $I_l([S,S])$ contains $[S,S]$ and is two-sided $B(S)$-invariant, hence $I_l(I) = I_l([S,S])$. Equation (2) implies that $[B(S),B(S)] \subset I$.

We obtain $I_l([B(S),B(S)]) \subset I_l(I) = I_l([S,S])$, which proves the equality of these ideals. If $\sigma \in \sigma_l(S)$, then $I_l(\{(b - \sigma(b)e) \colon b \in S\})$ is a proper ideal containing $I_l([B(S),B(S)])$. This ideal is right $B(S)$-invariant.

By Theorem 2.3 applied to the algebra $B(S)$ in place of S and by Corollary 3.4 there exists $\varphi \in \widehat{B(S)}$ such that $\sigma = \varphi|S$. $\square$

In the case of a finite set S and for $\mathcal{A} = B(S)$ this result was proved in [3].

4. The spectral mapping theorem

As observed in the previous section, for a polynomial P of k variables and for $a_1, \dots a_k \in S$ the value $P(a_1, \dots, a_k)$ is defined as an element of $B(S)/I$ where I is the two-sided ideal in $B(S)$ generated by the set $[S, S]$.

In order to obtain an analogue of the spectral mapping theorem for elements of $B(S)$ we apply the concept of a joint spectrum associated to a left ideal $J \subset \mathcal{A}$. This notion was introduced and studied in [4].

Let $C \subset \mathcal{A}$. Let

$$\sigma_l^J(C) = \{\lambda \in \mathbb{C}^C \colon \; I_l(\{c - \lambda(c)e\}_{c \in C} + J) \neq \mathcal{A}\}.$$

Obviously, $\sigma_l^J(C) \subset \sigma_l(C)$ for arbitrary $C \subset \mathcal{A}$. By Theorem 2.3 the spectrum $\sigma_l^J(C)$ is nonempty if $[C, C] \subset J$ and $JC \subset J$.

We are interested in the case of $J = I_l(I)$, where I denotes as before the two-sided ideal generated in $B(S)$ by the set $[S, S]$. In this case and for $C \subset B(S)$ we denote

$$\sigma_l^S(C) := \sigma_l^J(C).$$

Under the condition $[S, S] \subset S$ it holds that $\sigma_l^S(S) = \sigma_l(S)$ by Proposition 2.1.

Note that $\sigma_l^S(C + I) = \sigma_l^S(C)$, hence σ_l^S can be defined for subsets $\widetilde{C}$ in the quotient algebra $B(S)/I$.

In particular, $\sigma_l^S(P(a_1, \dots, a_k))$ makes sense for every polynomial mapping $P = (p_1, \dots, p_k)$.

Theorem 4.1. *Let $\mathcal{A}$ be a locally convex Waelbroeck algebra. Let $Z \subset \mathcal{A}$ satisfy $[Z, Z] \subset Z$, $I_l([Z, Z]) \neq \mathcal{A}$. Then $\sigma_l^Z(P(a_1, \dots, a_k)) = P(\sigma_l^Z(a_1, \dots, a_k))$ for every polynomial mapping of k variables $P = (p_1, \dots, p_k)$ and $\bar{a} = (a_1, \dots, a_k) \in Z^k$.*

Proof. For an arbitrary polynomial p_j of k variables the remainder formula states:

$$p_j(x_1, \dots, x_k) - p_j(\lambda_1, \dots, \lambda_k) = \sum_{i=1}^{k} q_{ij}(x_i - \lambda_i),$$

where q_{ij}, $i, j = 1, \dots, k$ are polynomials of k variables.

Let $\lambda \in \sigma_l^Z(a_1, \dots, a_k)$. Denote $\lambda_i = \lambda(a_i)$. Then there exists $h_j \in I$ such that

$$p_j(\bar{a}) - p_j(\lambda_1, \dots, \lambda_k) = \sum_{i=1}^{k} q_{ij}(\bar{a})(a_i - \lambda_i) + h_j.$$

The left ideal generated by the elements on the right-hand side of the equations is proper by definition, hence the statement

$$P(\sigma_l^Z(a_1,\dots,a_k)) \subset \sigma_l^Z(P(a_1,\dots,a_k))$$

is proved.

Now suppose that $\bar{\mu} = (\mu_1,\dots,\mu_k) \in \sigma_l^Z(p_1(\bar{a}),\dots,p_k(\bar{a}))$. This means that

$$J = I_l(\{p_j(\bar{a}) - \mu_j e\}_{1\leqslant j\leqslant k} + I) \neq \mathcal{A}.$$

This ideal satisfies the assumptions of Theorem 2.3 for $S = B(Z)$.

There is $\sigma \in \sigma_l(B(Z))$ such that $J + I_l(\{p - \sigma(p)e\}_{p\in B(Z)})$ is proper. Moreover, by Theorem 3.4 the function σ is a multiplicative functional on $B(Z)$, hence $\pi_j := \sigma(p_j(\bar{a})) = p_j(\sigma(a_1),\dots,\sigma(a_k))$.

There exists a proper ideal J_1 in $\mathcal{A}$ such that for arbitrary $g_1,\dots,g_k,h_1,\dots,h_k \in \mathcal{A}$ it holds that

$$\sum_{j=1}^{k} g_j(p_j(\bar{a}) - \mu_j e) + \sum_{j=1}^{k} h_j(p_j(\bar{a}) - \pi_j e) + I \subset J_1.$$

In particular for arbitrary g_i and $h_i = -g_i$ we have

$$\sum_{j=1}^{k} g_j(\mu_j - \pi_j) + I \subset J_1.$$

If $\mu_j - \pi_j \neq 0$ for some j, we obtain a contradiction $\mathcal{A} = J_1$. This proves $\mu_i = \pi_i = p_j(\sigma(a_1),\dots,\sigma(a_k))$ for $i = 1,\dots,k$. The proof is complete. $\square$

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