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LOCAL SUPERDERIVATIONS ON LIE SUPERALGEBRA $\mathfrak{q}(n)$

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Abstract. Let $\mathfrak{q}(n)$ be a simple strange Lie superalgebra over the complex field $\mathbb{C}$. In a paper by A. Ayupov, K. Kudaybergenov (2016), the authors studied the local derivations on semi-simple Lie algebras over $\mathbb{C}$ and showed the difference between the properties of local derivations on semi-simple and nilpotent Lie algebras. We know that Lie superalgebras are a generalization of Lie algebras and the properties of some Lie superalgebras are similar to those of semi-simple Lie algebras, but $\mathfrak{p}(n)$ is an exception. In this paper, we introduce the definition of the local superderivation on $\mathfrak{q}(n)$, give the structures and properties of the local superderivations of $\mathfrak{q}(n)$, and prove that every local superderivation on $\mathfrak{q}(n)$, $n > 3$, is a superderivation.

Keywords: simple Lie superalgebra; superderivation; local superderivation

MSC 2010: 16W55, 17B20, 17B40

1. Introduction

Many people have begun to research the structures and properties of local derivations since the concept of local derivation from a C^*-algebra A into a Banach A-bimodule was introduced by Kadison in 1990, see [6]. In [6] Kadison proved that every continuous local derivation from a von Neumann algebra M into a dual Banach M-bimodule is a derivation. Inspired by the results proved by Kadison, it is natural to pose the question whether every (continuous) local derivation on a certain associative or nonassociative algebra is a derivation? If it is not true, one needs to provide examples of algebras on which local derivations are not derivations.

Around this question, the local derivations on different algebras were investigated. In [2] and [9], the authors studied the local derivations of the full matrix algebras and subrings of matrix rings, respectively. In [2] it was proved that for $n \geqslant 3$, every local

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derivation $\varphi\colon M_n(R) \to M_n(\mathcal{M})$ is a derivation provided R is a commutative ring with unity element and $\mathcal{M}$ is a 2-torsion free unital R-bimodule such that $rm = mr$ for all $r \in R$ and $m \in \mathcal{M}$. For all finite incidence algebras, each local derivation is a derivation, see [9], where finite incidence algebras are algebras of square matrices (of finite size) with zero entries at places whose pairs of indices do not belong to a given reflexive transitive relation. Local derivations on a noncommutative Arens algebra, a von Neumann algebra and a nest subalgebra have been researched in [4], [1], [7] and [11] and obtained the affirmative answer. In particular, the authors considered the existence problem of local derivations which are not derivations on algebras of measurable operators in [1].

In [3], the authors investigated local derivations on finite-dimensional Lie algebras and proved that every local derivation on semisimple Lie algebra over the complex field $\mathbb{C}$ is a derivation. However, on finite-dimensional nilpotent Lie algebras L with $\dim L \geqslant 3$ there exist local derivations which are not derivations. This means a fundamental difference between semisimple and nilpotent Lie algebras as concerns their local derivations.

Motivated by [3], we introduce the definition of local superderivations on Lie superalgebras since a Lie superalgebra is a generalization of a Lie algebra to include a $\mathbb{Z}_2$-grading. It is natural ask whether every local superderivation on finite-dimensional classical Lie superalgebra is a superderivation. In 1977, Kac in [5] pointed out that finite dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero can be classified into classical Lie superalgebras and Cartan type Lie superalgebras. The classical Lie superalgebras over $\mathbb{C}$ consist of simple Lie algebras, basic classical Lie superalgebras and the strange Lie superalgebras. Now we have known the result that local superderivations are superderivations on simple Lie algebras, see [3], but the structures and properties of the strange Lie superalgebras are different from the simple Lie algebras. For example, the superderivations are not inner. The strange Lie superalgebras include two series: $\mathfrak{p}(n)$ and $\mathfrak{q}(n)$. In this paper, we will be devoted to investigating local superderivations on $\mathfrak{q}(n)$ with $n > 3$.

2. The strange Lie superalgebras $\mathfrak{q}(n)$

Let us recall the strange Lie superalgebra $\mathfrak{q}(n)$ over the complex field $\mathbb{C}$ in [8].

Let $\mathbb{Z}$ be the set of integers and $\mathbb{Z}_2$ the residue class modulo 2. The two elements of $\mathbb{Z}_2$ will be denote by 0 and 1. Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded vector space and let $\tau\colon V \to V$ be an odd linear map such that $-\tau^2$ is the identity. The centralizer $L(\tau)$ of τ in $\mathfrak{gl}(V)$ is a subalgebra. Let $e_1, \ldots, e_n$ be a basis for V_0 and set $e_{n+i} = -\tau(e_i)$

for $i = 1, \dots, n$. Then the matrix of τ with respect to the basis $e_1, \dots, e_{2n}$ is

$$J_n = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Now let $\widehat{\mathfrak{q}}(n) = L(\tau)$ be the centralizer of J_n in $\mathfrak{gl}(n, n)$. Therefore,

$$\widehat{\mathfrak{q}}(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} :\ A, B \in \mathfrak{gl}(n) \right\}.$$

Let $\widetilde{\mathfrak{q}}(n)$ be the subalgebra of $\widehat{\mathfrak{q}}(n)$:

$$\widetilde{\mathfrak{q}}(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} :\ A \in \mathfrak{gl}(n),\ B \in \mathfrak{sl}(n) \right\}.$$

Then $\widetilde{\mathfrak{q}}(n)$ has a one-dimensional center $\mathbb{C}I_{2n}$. Let

$$\mathfrak{q}(n) = \widetilde{\mathfrak{q}}(n)/\mathbb{C}I_{2n}.$$

By abuse of notation, we denote the image in $\mathfrak{q}(n)$ of a matrix $X \in \widetilde{\mathfrak{q}}(n)$ again by X. When $n > 3$, $\mathfrak{q}(n)$ is a simple Lie superalgebra, and $\mathfrak{q}(n)$ is a $\mathbb{Z}_2$-graded vector space, i.e., $\mathfrak{q}(n) = \mathfrak{q}(n)_0 \oplus \mathfrak{q}(n)_1$, where $\mathfrak{q}(n)_0$ is a Lie algebra and $\mathfrak{q}(n)_1$ is a $\mathfrak{q}(n)_0$-module. Let H_0 be a Cartan subalgebra of $\mathfrak{q}(n)_0$. To describe the roots of $\mathfrak{q}(n)$, define $\varepsilon_i \in H_0^*$ by

$$\varepsilon_i \left(\begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \right) = a_i,$$

where h is the diagonal matrix with entries $(a_1, \dots, a_n)$. Also denote by $E_{ij} \in \mathfrak{gl}(n, n)$ the matrix with 1 in row i and column j and 0 elsewhere. When $1 \leqslant i \neq j \leqslant n$, let

$$A_{ij} = E_{ij} + E_{n+i,n+j}, \quad B_{ij} = E_{i,n+j} + E_{n+i,j}.$$

Let $h_i = E_{ii} + E_{n+i,n+i} - E_{i+1,i+1} - E_{n+i+1,n+i+1}$, $1 \leqslant i \leqslant n-1$ be a basis of H_0. Let $h_i' = E_{n+i,i} + E_{i,n+i} - E_{n+i+1,i+1} - E_{i+1,n+i+1}$, $1 \leqslant i \leqslant n-1$ be a basis of H_1, where $H_1 = \left\{ \left(\begin{smallmatrix} 0 & h' \\ h' & 0 \end{smallmatrix}\right) :\ h' \in \mathfrak{sl}(n) \text{ is a diagonal matrix} \right\}$. The roots of $\mathfrak{q}(n)$ are given by $\Delta = \Delta_0 = \Delta_1 = \{\varepsilon_i - \varepsilon_j :\ i \neq j\}$. Take $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \in H_0^* :\ 1 \leqslant i \leqslant n-1\}$, then Π is a basis of Δ, and every root $\alpha \in \Delta$ is a $\mathbb{Z}$-linear combination of α_i, $1 \leqslant i \leqslant n-1$. For $\alpha, \beta \in \Delta$, $\beta \neq -\alpha$ there exists a unique pair of integers p and q such that all $\beta - p\alpha, \dots, \beta, \dots, \beta + q\alpha$ are roots, and this finite sequence is said to be the α-string through β. Obviously, the α-string through β on $\mathfrak{q}(n)$ contains at most two roots. The subsets Δ^+ and Δ_γ^+ of Δ and Δ_γ, respectively, are defined by

$$\Delta^+ = \Delta_\gamma^+ = \{\varepsilon_i - \varepsilon_j :\ 1 \leqslant i < j \leqslant n\}, \quad \gamma = 0, 1.$$

Moreover, if $\alpha = \varepsilon_i - \varepsilon_j$, let

$$\mathfrak{q}(n)^\alpha = \{x \in \mathfrak{q}(n)^\alpha \colon\ [h, x] = \alpha(h)x \ \forall\, h \in H_0\};$$

then $\mathfrak{q}(n)^\alpha = \mathfrak{q}(n)^\alpha_{\bar 0} \oplus \mathfrak{q}(n)^\alpha_{\bar 1}$, where $\mathfrak{q}(n)^\alpha_{\bar 0} = \mathbb{C}A_{ij}$, $\mathfrak{q}(n)^\alpha_{\bar 1} = \mathbb{C}B_{ij}$. With these definitions we have a root space decomposition

$$\mathfrak{q}(n) = H \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{q}(n)^\alpha,$$

where $H = \mathfrak{q}(n)^0 = H_0 \oplus H_1$ is the centralizer of H_0 in $\mathfrak{q}(n)$. If $\alpha = \varepsilon_i - \varepsilon_j \in \Delta$, $1 \leqslant i \neq j \leqslant n$, take $x_\alpha = A_{ij} \in \mathfrak{q}(n)^\alpha_{\bar 0}$, $y_\alpha = B_{ij} \in \mathfrak{q}(n)^\alpha_{\bar 1}$. The set

$$\{h_k, h'_k, x_\alpha, y_\alpha \colon\ 1 \leqslant k \leqslant n-1, \quad \alpha \in \Delta\}$$

forms a basis of $\mathfrak{q}(n)$. Its Lie operations are as follows: suppose that $\alpha = \varepsilon_i - \varepsilon_j = \sum_{k=1}^{n-1} a_k\alpha_k$, $\beta \in \Delta$, $a_k \in \mathbb{Z}$,

$$[h_k, x_\alpha] = \alpha(h_k)x_\alpha, \quad [h_k, y_\alpha] = \alpha(h_k)y_\alpha, \quad [h_k, h_l] = 0, \quad [h_k, h'_l] = 0,$$

$$[x_\alpha, x_\beta] = \begin{cases} N_{\alpha,\beta}x_{\alpha+\beta} & \text{if } \alpha+\beta \in \Delta, \\ 0 & \text{if } \alpha+\beta \notin \Delta, \end{cases} \qquad [y_\alpha, y_\beta] = \begin{cases} x_{\alpha+\beta} & \text{if } \alpha+\beta \in \Delta, \\ 0 & \text{if } \alpha+\beta \notin \Delta, \end{cases}$$

$$[x_\alpha, y_\beta] = \begin{cases} N_{\alpha,\beta}y_{\alpha+\beta} & \text{if } \alpha+\beta \in \Delta, \\ 0 & \text{if } \alpha+\beta \notin \Delta, \end{cases} \qquad N_{\alpha,\beta} = \pm 1,$$

$$[x_\alpha, y_{-\alpha}] = E_{i,n+i} + E_{n+i,n} - E_{j,n+j} - E_{n+j,j},$$

$$[x_\alpha, x_{-\alpha}] = h_\alpha = \sum_{k=1}^{n-1} a_k h_k, \qquad [y_\alpha, y_{-\alpha}] = E_{ii} + E_{n+i,n+i} + E_{jj} + E_{n+j,n+j},$$

$$[h'_k, x_\alpha] = \alpha(h_k)y_\alpha, \quad [h'_k, y_\alpha] = N_{\alpha,k}x_\alpha, \quad N_{\alpha,k} = \delta_{ki} - \delta_{k+1,i} + \delta_{kj} - \delta_{k+1,j} \in \mathbb{Z},$$

$$[h'_k, h'_i] = 2(\delta_{ki} - \delta_{k+1,i})(E_{ii} + E_{n+i,n+i}) + 2(\delta_{ki} - \delta_{k,i+1})(E_{i+1,i+1} + E_{n+i+1,n+i+1}),$$

where

$$\delta_{ki} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases} \qquad \alpha(h_k) \in \mathbb{Z}$$

since

$$\alpha_i(h_k) = \begin{cases} 2, & k = i, \\ -1, & i = k \pm 1, \\ 0, & \text{otherwise} \end{cases}$$

is the integer and $\alpha \in \Delta$ is a $\mathbb{Z}$-linear combination of α_i, $1 \leqslant i \leqslant n-1$.

A linear mapping $\varphi\colon \mathfrak{q}(n) \to \mathfrak{q}(n)$ is called homogeneous of degree γ, $\gamma \in \mathbb{Z}_2$ if $\varphi(\mathfrak{q}(n)_\beta) \subseteq \mathfrak{q}(n)_{\gamma+\beta}$ for all $\beta \in \mathbb{Z}_2$. Let $\mathrm{D}_\gamma(\mathfrak{q}(n))$, $\gamma \in \mathbb{Z}_2$, be the subspace of all homogeneous of degree γ linear maps δ of $\mathfrak{q}(n)$ such that

$$\delta([x,y]) = [\delta(x),y] + (-1)^{\gamma\beta}[x,\delta(y)] \quad \forall\, x \in \mathfrak{q}(n)_\beta,\ \forall\, y \in \mathfrak{q}(n),\ \beta \in \mathbb{Z}_2.$$

Define $\mathrm{D}(\mathfrak{q}(n)) = \mathrm{D}_0(\mathfrak{q}(n)) \oplus \mathrm{D}_1(\mathfrak{q}(n))$. The elements of $\mathrm{D}(\mathfrak{q}(n))$ are called superderivations of $\mathfrak{q}(n)$. $\mathrm{D}(\mathfrak{q}(n))$ is called the Lie superalgebra of superderivations of $\mathfrak{q}(n)$. For $a \in \mathfrak{q}(n)$, a linear mapping $\operatorname{ad} a\colon \mathfrak{q}(n) \to \mathfrak{q}(n)$ such that $\operatorname{ad} a(b) = [a,b]$ for all $b \in \mathfrak{q}(n)$ is a superderivation which is called inner, all others are called outer.

Definition 2.1. A homogeneous linear mapping $\varphi\colon \mathfrak{q}(n) \to \mathfrak{q}(n)$ of degree γ is called a local homogeneous superderivation of degree γ if for any element $x \in \mathfrak{q}(n)$ there exists a superderivation $\delta_x\colon \mathfrak{q}(n) \to \mathfrak{q}(n)$(depending on x) such that $\varphi(x) = \delta_x(x)$. If $\gamma = 0$, we call φ an even local superderivation; if $\gamma = 1$, we call φ an odd local superderivation. Let LD_γ be the set of all local homogeneous superderivations of degree γ. The elements of $\mathrm{LD} = \mathrm{LD}_0 + \mathrm{LD}_1$ are called local superderivations on $\mathfrak{q}(n)$.

Obviously, every superderivation is a local superderivation on $\mathfrak{q}(n)$. The sum of two local superderivations is also a local superderivations on $\mathfrak{q}(n)$.

3. The structure of the superderivation algebra of $\mathfrak{q}(n)$

We first consider the superderivation algebra of $\mathfrak{q}(n)$. In [10] we can find the result of the superderivation algebra.

Proposition 3.1 ([10]). *The superderivation algebra of simple Lie superalgebra $\mathfrak{q}(n)$ is isomorphic to $\widehat{q}(n)/\mathbb{C}I_{2n}$, i.e., $\mathrm{D}(\mathfrak{q}(n)) \cong \widehat{q}(n)/\mathbb{C}I_{2n}$.*

Since $\mathfrak{q}(n)$, $n > 3$, is a simple Lie superalgebra, we have $\operatorname{ad}(\mathfrak{q}(n)) = \mathfrak{q}(n)$. Comparing $\operatorname{ad}(\mathfrak{q}(n))$ with $\mathrm{D}(\mathfrak{q}(n))$, it is easy to see that there exists an odd outer superderivation such that $\mathrm{D}(\mathfrak{q}(n)) = \operatorname{ad}(\mathfrak{q}(n)) \oplus \mathbb{C}d_1$, where d_1 is an odd outer superderivation. We will find this outer superderivation d_1.

Lemma 3.2. *The linear mapping d_1 on $\mathfrak{q}(n)$ defined by*

$$d_1(\mathfrak{q}(n))_0) = 0, \quad d_1(h'_k) = h_k, \quad 1 \leqslant k \leqslant n-1, \qquad d_1(y_\alpha) = x_\alpha \quad \forall\, \alpha \in \Delta_1,$$

is an outer superderivation of $\mathfrak{q}(n)$.

P r o o f. By the definition of the superderivation, it is easy to prove that d_1 is an odd superderivation. Next, we will show that it is not an odd inner superderivation. Suppose that it is an inner superderivation, then there exists an element $u = \sum\limits_{k=1}^{n-1} u_k h'_k + \sum\limits_{\alpha\in\Delta} u_\alpha y_\alpha$ such that $d_1(x) = [u, x]$, $x \in \mathfrak{q}(n)_\gamma$, $\gamma = 0, 1$. Moreover, taking $x = h_i$, $i = 1, \ldots, n-1$, we have $d_1(h_i) = [u, h_i] = -\sum\limits_{\alpha\in\Delta} u_\alpha \alpha(h_i) y_\alpha = 0$. They implies $u_\alpha = 0$ for all $\alpha \in \Delta_1$. Using

$$d_1(x_{\alpha_i}) = [u, x_{\alpha_i}] = \sum_{k=1}^{n-1} u_k \alpha_k(h_i) y_{\alpha_i} = 0, \quad i = 1, \ldots, n-1,$$

we have a system of equations for u_k:

$$\sum_{k=1}^{n-1} u_k \alpha_k(h_i) = 0, \quad i = 1, \ldots, n-1.$$

We can put $u_k = 0$, $k = 1, \ldots, n-1$ since the coefficient matrix is nondegenerate. Then $d_1 = 0$, which is a contradiction. Therefore, d_1 is not an inner superderivation. □

4. Local superderivations on Lie superalgebra $\mathfrak{q}(n)$

The main result of this section is given as follows.

Theorem 4.1. *For $n > 3$, all local superderivations on $\mathfrak{q}(n)$ are superderivations.*

Recall that a homogeneous linear mapping $\varphi\colon \mathfrak{q}(n) \to \mathfrak{q}(n)$ of degree γ is called a local homogeneous superderivation of degree γ if for any element $x \in \mathfrak{q}(n)$, there exists a superderivation $\delta_x\colon \mathfrak{q}(n) \to \mathfrak{q}(n)$ (depending on x) such that $\varphi(x) = \delta_x(x)$.

Since any superderivation of $\mathfrak{q}(n)$ is $\mathrm{ad}(\mathfrak{q}(n)) \oplus \mathbb{C}d_1$, it follows that every even superderivation is an inner superderivation and every odd superderivation is a linear combination of an inner superderivation and d_1. For this algebra the above definition of a local homogeneous superderivation is reformulated as follows. A homogeneous linear mapping $\varphi\colon \mathfrak{q}(n) \to \mathfrak{q}(n)$ of degree γ is called a local homogeneous superderivation of degree γ if for any element $x \in \mathfrak{q}(n)$ there exists an element $u_x \in \mathfrak{q}(n), a_x \in \mathbb{C}$ such that $\varphi(x) = [u_x, x] + a_x d_1(x)$. More specifically, if φ is an even local superderivation on $\mathfrak{q}(n)$, then for any homogeneous element $x \in \mathfrak{q}(n)_\gamma$ there exists an element $u_x \in \mathfrak{q}(n)_0$ such that $\varphi(x) = [u_x, x]$. If y is any homogeneous even element of $\mathfrak{q}(n)$, then we can find an element v_y such that $\varphi(y) = [v_y, y]$ since $d_1(y) = 0$.

Suppose that φ is any local superderivation of $\mathfrak{q}(n)$. Fix a basis of $\mathfrak{q}(n)$:

$$h_k,\ h'_k,\ x_\alpha,\ y_\alpha,\quad 1 \leqslant k \leqslant n-1,\ \alpha \in \Delta,$$

which has been described in Section 2. Then φ on $\mathfrak{q}(n)$ can be represented as a 4×4 matrix of the following form:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}.$$

Since every local superderivation can be written as the sum of an even and an odd local superderivation, we will consider separately the even and the odd local superderivations.

We first consider the even local superderivations on $\mathfrak{q}(n)$. Suppose that φ_0 is an even local superderivation on $\mathfrak{q}(n)$. For elements $h_i \in \mathfrak{q}(n)_0$, $h'_i \in \mathfrak{q}(n)_1$, $i = 1, \ldots, n-1$, there exist elements

$$u = \sum_{k=1}^{n-1} u_k^i h_k + \sum_{\alpha\in\Delta_0} u_\alpha^i x_\alpha, \qquad u' = \sum_{k=1}^{n-1} v_k^i h_k + \sum_{\alpha\in\Delta_0} v_\alpha^i x_\alpha \in \mathfrak{q}(n)_0$$

respectively such that

$$\varphi_0(h_i) = [u, h_i] = -\sum_{\alpha\in\Delta_0} u_\alpha^i \alpha(h_i) x_\alpha,$$
$$\varphi_0(h'_i) = [u', h'_i] = -\sum_{\alpha\in\Delta_1} v_\alpha^i \alpha(h_i) y_\alpha.$$

It is easy to see that $A_{11} = A_{12} = A_{21} = A_{22} = 0$, $A_{14} = A_{23} = 0$. Thus, suppose that

$$\varphi_0(h_i) = \sum_{\alpha\in\Delta_0} a_{\alpha,i} x_\alpha, \quad \varphi_0(h'_i) = \sum_{\alpha\in\Delta_1} b_{\alpha,i} y_\alpha, \qquad i = 1, \ldots, n-1,$$

i.e. $A_{13} = (a_{\alpha,i})$, $A_{24} = (b_{\alpha,i})$.

Lemma 4.2. *Suppose that φ_0 is any even superderivation on $\mathfrak{q}(n)$. If $\varphi_0(\bar{h}) = 0$, where $\bar{h} = \sum\limits_{k=1}^{n-1} t^k h_k + \sum\limits_{k=1}^{n-1} t^k h'_k$ (t is an algebraic number of degree more than $n-1$), then $\varphi_0(H) = 0$.*

P r o o f. Let $\beta_0 \in \Delta$ be a given root. Since $h_1, \ldots, h_{n-1}$ is a basis of H_0, there exists an integer s, $1 \leqslant s \leqslant n-1$, such that $\beta_0(h_s) \neq 0$. Let

$$f_{\beta_0} = \frac{a_{\beta_0,s}}{\beta_0(h_s)}, \qquad f'_{\beta_0} = \frac{b_{\beta_0,s}}{\beta_0(h_s)}.$$

For $i \neq s$, we take $h = \beta_0(h_s)h_i - \beta_0(h_i)h_s$. On the one hand, we can find an element $u = \sum\limits_{k=1}^{n-1} u_k h_k + \sum\limits_{\alpha\in\Delta_0} u_\alpha x_\alpha$ such that $\varphi_0(h) = [u, h]$. The coefficient of x_{β_0} in this equation is

$$u_\beta(-\beta_0(h_s)\beta_0(h_i) + \beta_0(h_i)\beta_0(h_s)) = 0.$$

On the other hand, by the matrix A_{13}, we obtain that the coefficient of x_{β_0} is

$$\beta_0(h_s)a_{\beta_0,i} - \beta_0(h_i)a_{\beta_0,s} = \beta_0(h_s)(a_{\beta_0,i} - \beta_0(h_i)f_{\beta_0}) = 0.$$

Therefore, $a_{\beta_0,i} = \beta_0(h_i)f_{\beta_0}$ for all $i = 1, \ldots, n-1$. Similarly, using the coefficient of y_{β_0}, we can get $b_{\beta_0,i} = \beta_0(h_i)f'_{\beta_0}$.

It is sufficient to show that $A_{13} = 0$, $A_{24} = 0$. Using the matrix A_{13}, A_{24} and the linear of φ_0, we can get the coefficients of x_β, y_β, respectively. They are as follows:

$$\sum_{k=1}^{n-1} a_{\beta,k}t^k = \sum_{k=1}^{n-1} f_\beta \beta(h_k)t^k = f_\beta \sum_{k=1}^{n-1} \beta(h_k)t^k = 0, \quad \sum_{k=1}^{n-1} b_{\beta,k}t^k = f'_\beta \sum_{k=1}^{n-1} \beta(h_k)t^k = 0.$$

Since for $k = 1, \ldots, n-1$, $\beta(h_k)$ are integers, and t is an algebraic number of degree more than $n-1$, we have $\sum\limits_{k=1}^{n-1} \beta(h_k)t^k \neq 0$. Thus, $f_\beta = f'_\beta = 0$, i.e., $A_{13} = 0$, $A_{24} = 0$. Therefore, $\varphi_0(H) = 0$. □

Remark 4.3. For $\bar{h}$ above, there exists an element $w \in \mathfrak{q}(n)$ and $a \in \mathbb{C}$ such that $\varphi_0(\bar{h}) = [w, \bar{h}] + ad_1(\bar{h})$. Let $\varphi'_0 = \varphi_0 - \operatorname{ad} w - ad_1$, i.e., $\varphi'_0(\bar{h}) = 0$. Then by Lemma 4.2, we have $\varphi'_0(H) = 0$. In the following, we always suppose that $\varphi_0(H) = 0$ for every even local superderivation φ_0.

Lemma 4.4. *Let φ_0 be an even local superderivation on $\mathfrak{q}(n)$ such that $\varphi_0(H) = 0$, then $\varphi_0(x_{\pm\alpha}) = \pm c_\alpha x_{\pm\alpha}$, $\varphi_0(y_{\pm\alpha}) = \pm c'_\alpha y_{\pm\alpha}$ for all $\alpha \in \Delta^+$.*

P r o o f. For $\alpha = \varepsilon_i - \varepsilon_j \in \Delta_0$, there exists an element $u = h_u + \sum\limits_{\beta\in\Delta_0} u_\beta x_\beta$ such that

$$\varphi_0(x_\alpha) = [u, x_\alpha] = \alpha(h_u)x_\alpha - u_{-\alpha}h_\alpha + \sum_{\alpha+\beta\in\Delta} N_{\alpha,\beta}u_\beta x_{\alpha+\beta}.$$

Since the α-string through $\beta \neq -\alpha$ contains 2 roots, without loss of generality let us assume that $\beta + \alpha \in \Delta$, then $\beta - \alpha \notin \Delta$. And from the above equation, the coefficients of $x_\beta, x_{\alpha+\beta}$ are 0 and $N_{\alpha,\beta}u_\beta$, respectively. Take $h \in H_0$ such that

$$(\alpha+\beta)(h) = 0, \qquad \beta(h) = 1.$$

For the element $h+x_\alpha$, we have $\varphi_0(h+x_\alpha) = \Big[h_v + \sum_{\gamma\in\Delta_0} v_\gamma x_\gamma, h+x_\alpha\Big]$. The coefficients of x_β, $x_{\alpha+\beta}$ are $-\beta(h)v_\beta$ and $N_{\alpha,\beta}v_\beta - (\beta+\alpha)(h)$, respectively. Since $\varphi_0(x_\alpha) = \varphi_0(h+x_\alpha)$ and $N_{\alpha,\beta} \neq 0$, we obtain $u_\beta = v_\beta = 0$. Therefore,

$$\varphi_0(x_\alpha) = c_\alpha x_\alpha + e_\alpha h_\alpha.$$

Similarly, for y_α, it is easy to prove that

$$\varphi_0(y_\alpha) = c'_\alpha y_\alpha + e'_\alpha([x_{-\alpha}, y_\alpha]).$$

Since $\mathrm{span}_{\mathbb{C}}\{x_{\pm\alpha}, h_\alpha\} \cong \mathfrak{sl}_2(\mathbb{C})$ and every local derivation on $\mathfrak{sl}_2(\mathbb{C})$ is a derivation in [3], there exists a constant c_α such that $\varphi_0|_{\mathrm{span}_{\mathbb{C}}\{x_{\pm\alpha},h_\alpha\}} = \mathrm{ad}(c_\alpha h_\alpha)$. Thus,

$$\varphi_0(x_{\pm\alpha}) = \pm c_\alpha x_{\pm\alpha}.$$

For the element $x_\alpha + y_\alpha$, $\alpha = \varepsilon_i - \varepsilon_j \in \Delta$ we can find an element $u = h_u + h'_u + \sum_{\beta\in\Delta} (u_\beta x_\beta + v_\beta x_\beta) \in \mathfrak{q}(n)$ and $b \in \mathbb{C}$ such that

$$\begin{aligned}\varphi_0(x_\alpha + y_\alpha) = [u, x_\alpha + y_\alpha] + bd_1(x_\alpha + y_\alpha) &= u_{-\alpha}[x_{-\alpha}, y_\alpha] + v_{-\alpha}[y_{-\alpha}, x_\alpha] \\ &+ u_{-\alpha}(E_{ii} + E_{n+i,n+i} - E_{jj} - E_{n+j,n+j}) \\ &+ v_{-\alpha}(E_{ii} + E_{n+i,n+i} + E_{jj} + E_{n+j,n+j}) + S,\end{aligned}$$

where S contains no element of H. On the other hand,

$$\varphi_0(x_\alpha + y_\alpha) = c_\alpha x_\alpha + c'_\alpha y_\alpha + e'_\alpha([x_{-\alpha}, y_\alpha]).$$

The two equations combined give $u_{-\alpha} = v_{-\alpha} = 0$. Therefore, $e'_\alpha = 0$, i.e., $\varphi_0(y_\alpha) = c'_\alpha y_\alpha$.

Using the element $y_\alpha + y_{-\alpha}$, we have

$$\varphi_0(y_\alpha + y_{-\alpha}) = \Big[h + \sum_{\beta\in\Delta} u_\beta x_\beta, y_\alpha + y_{-\alpha}\Big] = \alpha(h)y_\alpha - \alpha(h)y_{-\alpha} = c'_\alpha y_\alpha + c'_{-\alpha}y_{-\alpha}.$$

Thus, $c'_{-\alpha} = c'_\alpha$, i.e., $\varphi_0(y_{\pm\alpha}) = \pm c'_\alpha y_{\pm\alpha}$. □

With these lemmas, we can determine the structure of the even local superderivations on $\mathfrak{q}(n)$.

Proposition 4.5. *Let φ_0 be an even local superderivation on $\mathfrak{q}(n)$ such that $\varphi_0(H) = 0$, then φ_0 is a superderivation.*

Proof. By Lemma 4.4, we have

$$\varphi_0(x_{\pm\alpha}) = \pm c_\alpha x_{\pm\alpha}, \quad \varphi_0(y_{\pm\alpha}) = \pm c'_\alpha y_{\pm\alpha}, \qquad \alpha \in \Delta^+.$$

Since $\dim(H_0) = n-1$ and for $1 \leqslant i, k \leqslant n-1$, $(\alpha_k(h_i))$ is an $(n-1)\times(n-1)$ invertible matrix, we can find an element $\hat{h} \in H_0$ such that $\varphi_0(x_{\alpha_i}) = [\hat{h}, x_{\alpha_i}]$ for all $\alpha_i \in \Pi$. Let $\varphi'_0 = \varphi_0 - \operatorname{ad}\hat{h}$, then $\varphi'_0(x_{\pm\alpha_i}) = \varphi'_0(H) = 0$.

Claim. If $\varphi'_0(x_\alpha) = \varphi'_0(x_\beta) = 0, \alpha, \beta \in \Delta^+$ and $\alpha+\beta \in \Delta$, then

$$\varphi'_0(x_{\alpha+\beta}) = \varphi'_0(y_{\alpha+\beta}) = 0.$$

A repeated application of this claim proves that $\varphi'_0 = 0$, i.e., $\varphi_0 = \operatorname{ad}\hat{h}$ is a superderivation.

We prove this claim. For the element $x_{-\alpha} + x_{-\beta} + x_{\alpha+\beta}$ we have

$$\begin{aligned}\varphi'_0(x_{-\alpha} + x_{-\beta} + x_{\alpha+\beta}) &= \Big[h_u + \sum_{\beta\in\Delta} u_\gamma x_\gamma, x_{-\alpha} + x_{-\beta} + x_{\alpha+\beta}\Big] \\ &= -\alpha(h_u)x_{-\alpha} - \beta(h_u)x_{-\beta} + (\alpha+\beta)(h_u)x_{\alpha+\beta} + S,\end{aligned}$$

where S does not contain $x_{-\alpha}, x_{-\beta}, x_{\alpha+\beta}$. Moreover,

$$\varphi'_0(x_{-\alpha} + x_{-\beta} + x_{\alpha+\beta}) = c_{\alpha+\beta}x_{\alpha+\beta}.$$

Therefore, these identities imply $c_{\alpha+\beta} = (\alpha+\beta)(h_u) = 0$, i.e., $\varphi'_0(x_{\alpha+\beta}) = 0$. Similarly, using the element $x_{-\alpha} + x_{-\beta} + y_{\alpha+\beta}$, we conclude that $\varphi'_0(y_{\alpha+\beta}) = 0$. □

Next, we will consider the odd local superderivations on $\mathfrak{q}(n)$. Suppose that φ_1 is an odd local superderivation on $\mathfrak{q}(n)$. For elements $h_i \in \mathfrak{q}(n)_0$, $h'_i \in \mathfrak{q}(n)_1$, $i = 1, \ldots, n-1$, there exist elements $u = \sum\limits_{k=1}^{n-1} u^i_k h'_k + \sum\limits_{\alpha\in\Delta_1} u^i_\alpha y_\alpha$, $u' = \sum\limits_{k=1}^{n-1} v^i_k h'_k + \sum\limits_{\alpha\in\Delta_1} v^i_\alpha y_\alpha$ and constants $a_i \in \mathbb{C}$, respectively, such that

$$\begin{aligned}(1)\qquad \varphi_1(h_i) &= [u, h_i] = -\sum_{\alpha\in\Delta_1} u^i_\alpha \alpha(h_i) y_\alpha, \\ \varphi_1(h'_i) &= [u', h'_i] + a_i d_1(h'_i) \\ &= \sum_{\alpha\in\Delta_0} v^i_\alpha N_{\alpha,i} x_\alpha + *(E_{ii} + E_{n+i,n+i}) + *(E_{i+1,i+1} + E_{n+i+1,n+i+1})\end{aligned}$$

where $*$ stands for the given constants. It is easy to see that $A_{11} = A_{12} = A_{22} = 0$, $A_{13} = A_{24} = 0$. Therefore, suppose that

$$\varphi_1(h_i) = \sum_{\alpha\in\Delta_1} a'_{\alpha,i} y_\alpha, \quad \varphi_1(h'_i) = \sum_{\alpha\in\Delta_0} b'_{\alpha,i} x_\alpha + h, \qquad h \in H_0,\ i = 1,\dots,n-1,$$

i.e., $A_{14} = (a'_{\alpha,i}), A_{23} = (b'_{\alpha,i})$.

Lemma 4.6. *Let φ_1 be an odd local superderivation on $\mathfrak{q}(n)$ such that $\varphi_1(\bar{h}) \in H_0$, where $\bar{h} = \sum\limits_{k=1}^{n-1} t^k h_k + \sum\limits_{k=1}^{n-1} t^k h'_k$ is defined in Lemma 4.2. Then $\varphi_1(H_0) = 0$ and*

$$\varphi_1(h'_i) = c_i(E_{ii} + E_{n+i,n+i}) - c_{i+1}(E_{i+1,i+1} + E_{n+i+1,n+i+1}), \quad i = 1,\dots,n-1.$$

P r o o f. For a fixed $\beta_0 \in \Delta$ there exists an integer l, $1 \leqslant l \leqslant n-1$, such that $\beta_0(h_l) \neq 0$, since $h_1,\dots,h_{n-1}$ is a basis of H_0. Let $r_{\beta_0} = a'_{\beta_0,l}/\beta_0(h_l)$. Similarly to the proof of Lemma 4.2, let $h' = \beta_0(h_l)h'_i - \beta_0(h_i)h'_l$ for $i \neq l$; we get

$$a'_{\beta_0,i} = r_{\beta_0}\beta_0(h_i), \quad i = 1,\dots,n-1.$$

For a fixed $\beta = \varepsilon_i - \varepsilon_j \in \Delta$ there exists $s = \min\{i,j\} - 1$ for $\min\{i,j\} > 1$ or $s = j$ for $i = 1$, $j < n$ or $s = 1$ for $i = 1$, $j = n$ such that $N_{\beta,s} \neq 0$. Let $r'_\beta = b'_{\beta,s}/N_{\beta,s}$. For $i \neq s$, take $h' = N_{\beta,s}h'_i - N_{\beta,i}h'_s$. Then the coefficient of x_β is $N_{\beta,s}b'_{\beta,i} - N_{\beta,i}b'_{\beta,s} = N_{\beta,s}(b'_{\beta,i} - r'_\beta N_{\beta,i})$. Moreover, we can find an element $u = \sum\limits_{k=1} u_k h'_k + \sum\limits_{\alpha\in\Delta_1} u_\alpha y_\alpha$ and a such that $\varphi_1(h') = [u, h'] + ad_1(h')$. Here the coefficient of x_β is $u_\beta(N_{\beta,s}N_{\beta,i} - N_{\beta,i}N_{\beta,s}) = 0$. Therefore,

$$b'_{\beta,i} = r'_\beta N_{\beta,i}, \qquad i = 1,\ \dots, n-1.$$

For the element $\varphi_1(\bar{h}) \in H_0$, the coefficients of y_β, x_β are

$$0 = \sum_{k=1}^{n-1} a'_{\beta,k}t^k = r_\beta \sum_{k=1}^{n-1} \beta(h_k)t^k, \qquad 0 = \sum_{k=1}^{n-1} b'_{\beta,k}t^k = r'_\beta \sum_{k=1}^{n-1} N_{\beta,k}t^k.$$

Since all $\beta(h_k)$, $N_{\beta,k}$ are integers and t is an algebraic number, we get $r_\beta = r'_\beta = 0$, i.e., $a'_{\alpha,i} = 0$, $b'_{\alpha,i} = 0$. Thus, $A_{14} = 0$, i.e., $\varphi_1(H_0) = 0$, and $A_{23} = 0$. Now we can assume that $\varphi_1(h'_i) = c^i_i(E_{ii} + E_{n+i,n+i}) - c^i_{i+1}(E_{i+1,i+1} + E_{n+i+1,n+i+1})$ by the equation (1).

For the element $\tilde{h} = \sum_{i=1}^{n-1} h'_i$ we have an element $u = \sum_{k=1}^{n-1} u_k h'_k + \sum_{\alpha \in \Delta_1} u_\alpha y_\alpha$ and $a \in \mathbb{C}$ such that

$$\varphi_1(\tilde{h}) = [u, \tilde{h}] + ad_1(\tilde{h}) = (2u_1 + a)(E_{11} + E_{n+1,n+1}) + (2u_{n-1} - a)(E_{nn} + E_{2n,2n}).$$

Further, $\varphi_1(\tilde{h}) = c_1^1(E_{11} + E_{n+1,n+1}) + \sum_{i=2}^{n-1} (c_i^i - c_i^{i-1})(E_{ii} + E_{n+i,n+i}) - c_n^{n-1}(E_{nn} + E_{2n,2n})$. Therefore, $c_i^i = -c_i^{i-1}$, $i = 2, \ldots, n-1$. Let $c_i^i = c_i$ for $i = 1, \ldots, n-1$, then $\varphi_1(h'_i) = c_i(E_{ii} + E_{n+i,n+i}) - c_{i+1}(E_{i+1,i+1} + E_{n+i+1,n+i+1})$. □

Lemma 4.7. *Let φ_1 be an odd local superderivation on $\mathfrak{q}(n)$ such that $\varphi_1(\bar{h}) \in H_0$. Then*

$$\varphi_1(x_{\pm\alpha}) = \pm a_\alpha y_{\pm\alpha}, \quad \varphi_1(y_{\pm\alpha}) = a'_\alpha x_{\pm\alpha}, \qquad \alpha \in \Delta^+.$$

P r o o f. Using the method of Lemma 4.4, we get

$$\varphi_1(x_\alpha) = a_\alpha y_\alpha + b_\alpha([y_{-\alpha}, x_\alpha]),\ \varphi_1(y_\alpha) = a'_\alpha x_\alpha + b'_\alpha([y_{-\alpha}, y_\alpha]).$$

Take the roots α, β such that $\alpha - \beta \in \Delta$, then $\alpha + \beta \notin \Delta$. For the element $x_\alpha + x_\beta$, we have $\varphi_1(x_\alpha + x_\beta) = a_\alpha y_\alpha + b_\alpha([y_{-\alpha}, x_\alpha]) + a_\beta y_\beta + b_\beta([y_{-\beta}, x_\beta])$. In addition, we can find an element $u = \sum_{k=1}^{n-1} u_k h'_k + \sum_{\gamma \in \Delta} u_\gamma y_\gamma \in \mathfrak{q}(n)$ such that

$$\begin{aligned} \varphi_1(x_\alpha + x_\beta) = [u, x_\alpha + x_\beta] &= *y_\alpha + *y_\beta - u_{-\alpha}(N_{\beta,-\alpha} y_{\beta-\alpha} + [y_{-\alpha}, x_\alpha]) \\ &\quad - u_{-\beta}(N_{-\alpha,\beta} y_{\alpha-\beta} + [y_{-\beta}, x_\beta]) + \ldots, \end{aligned}$$

where $*$ stands for certain coefficients. Therefore, $u_{-\alpha} = u_{-\beta} = 0$, which implies $b_\alpha = b_\beta = 0$. For the element $x_\alpha + x_{-\alpha}$, we have $a_{-\alpha} = -a_\alpha$. Thus, $\varphi_1(x_{\pm\alpha}) = \pm a_\alpha y_{\pm\alpha}$ for all $\alpha \in \Delta_0^+$.

For the element $x_\alpha + y_\alpha$ we have

$$\begin{aligned} \varphi_1(x_\alpha + y_\alpha) &= \left[h_u + h'_u + \sum_{\gamma \in \Delta} u_\gamma x_\gamma + \sum_{\gamma \in \Delta} v_\gamma y_\gamma, x_\alpha + y_\alpha\right] + bd_1(x_\alpha + y_\alpha) \\ &= *x_\alpha + *y_\alpha + u_{-\alpha}([x_{-\alpha}, x_\alpha + y_\alpha]) + v_{-\alpha}([y_{-\alpha}, x_\alpha + y_\alpha]) + \ldots, \end{aligned}$$

where $*$ stands for certain coefficients. On the other hand, we know that

$$\varphi_1(x_\alpha + y_\alpha) = a_\alpha y_\alpha + a'_\alpha x_\alpha + b'_\alpha([y_{-\alpha}, y_\alpha]).$$

Combining the equations, since $[x_{-\alpha}, x_\alpha]$ and $[y_{-\alpha}, y_\alpha]$ are linearly independent and $u_{-\alpha}[x_{-\alpha}, y_\alpha] + v_{-\alpha}[y_{-\alpha}, x_\alpha] = 0$, we obtain $u_{-\alpha} = v_{-\alpha} = 0$, which means $b'_\alpha = 0$. Thus, $\varphi_1(y_\alpha) = a'_\alpha x_\alpha$. Using the element $y_\alpha + y_{-\alpha}$, we conclude $a'_{-\alpha} = a'_\alpha$. Therefore, $\varphi_1(y_{\pm\alpha}) = a'_\alpha x_{\pm\alpha}$. □

With these results, we can determine the structure of the odd local superderivations on $\mathfrak{q}(n)$.

Proposition 4.8. *Let φ_1 be an odd local superderivation on $\mathfrak{q}(n)$ such that $\varphi_1(\bar{h}) \in H_0$. Then φ_1 is an odd superderivation.*

P r o o f. By Lemmas 4.6, 4.7 we have

$$\varphi_1(H_0) = 0, \quad \varphi_1(x_{\pm\alpha}) = \pm a_\alpha y_{\pm\alpha}, \quad \varphi_1(y_{\pm\alpha}) = a'_\alpha x_{\pm\alpha}, \qquad \alpha \in \Delta^+.$$

Since $\dim(H_1) = n-1$ and $(\alpha_k(h_i))$ is an $(n-1)\times(n-1)$ invertible matrix, we can find an element $\hat{h}'$ such that $\varphi_1(x_{\alpha_i}) = [\hat{h}', x_{\alpha_i}]$, $i = 1,\ldots,n-1$. Let $\varphi'_1 = \varphi_1 - \operatorname{ad}\hat{h}'$, then $\varphi'_1(x_{\pm\alpha_i}) = \varphi'_1(H_0) = 0$, $i = 1,\ldots,n-1$. It is easy to prove that if $\varphi'_1(x_\alpha) = \varphi'_1(x_\beta) = 0$, $\alpha,\beta \in \Delta^+$ and $\alpha+\beta \in \Delta$, then $\varphi'_1(x_{\alpha+\beta}) = 0$ by the proof of Proposition 4.5. Using this result, we obtain $\varphi'_1(x_\alpha) = 0$ for all $\alpha \in \Delta$.

Take $\alpha = \varepsilon_i - \varepsilon_j$, $\beta = \varepsilon_j - \varepsilon_k$, $\gamma = \varepsilon_k - \varepsilon_l$, $\delta = \varepsilon_l - \varepsilon_i$, where for $1 \leqslant i,j,k,l \leqslant n$, none is equal to another. For the element $X = x_\alpha + x_\beta + y_\gamma + y_\delta$, we can find the element $u = \sum\limits_{k=1} u_k h_k + \sum\limits_{\gamma\in\Delta} u_\gamma x_\gamma + \sum\limits_{k=1} v_k h'_k + \sum\limits_{\gamma\in\Delta} v_\gamma y_\gamma$ and $a \in \mathbb{C}$ such that

$$\begin{aligned}\varphi'_1(X) = [u,X] + ad_1(X) &= (v_i - v_{i-1} - v_j + v_{j-1})y_\alpha + (v_j - v_{j-1} - v_k + v_{k-1})y_\beta \\ &\quad + (v_k - v_{k-1} + v_l - v_{l-1} + a)x_\gamma + (v_l - v_{l-1} + v_i - v_{i-1} + a)x_\delta + \ldots.\end{aligned}$$

Meanwhile, $\varphi'_1(X) = a'_\gamma y_\gamma + a'_\delta y_\delta$. The two equations combined yield $a'_\gamma = a'_\delta$. Therefore,

$$\varphi'_1(y_\alpha) = a' x_\alpha \quad \forall\, \alpha \in \Delta.$$

By Lemma 4.6 we have

$$\varphi'_1(h'_i) = c_i(E_{ii} + E_{n+i,n+i}) - c_{i+1}(E_{i+1,i+1} + E_{n+i+1,n+i+1}), \quad i = 1,\ldots,n-1.$$

Let $\alpha = \varepsilon_s - \varepsilon_{s+2}$, $\beta = \varepsilon_{s+2} - \varepsilon_{s+1}$, $1 \leqslant s \leqslant n-2$. For the element $Y = h'_s + y_\alpha + y_\beta$, we can find an element $u = \sum\limits_{k=1} u_k h'_k + \sum\limits_{\gamma\in\Delta} u_\gamma y_\gamma \in \mathfrak{q}(n)_1$ and $a \in \mathbb{C}$ such that $\varphi'_1(Y) = [u,Y] + ad_1(Y) = (u_s - u_{s-1} + u_{s+2} - u_{s+1} + u_\alpha + a)x_\alpha + (u_{s+2} - u_s - u_\beta + a)x_\beta + (u_\alpha + u_\beta)x_{\alpha+\beta} + (u_{-\alpha} + u_{-\beta})(E_{s+2,s+2} + E_{n+s+2,n+s+2}) + (2u_s - 2u_{s-1} + u_{-\alpha} + a)(E_{ss} + E_{n+s,n+s}) - (2u_{s+1} - 2u_s - u_{-\beta} + a)(E_{s+1,s+1} + E_{n+s+1,n+s+1})$. Moreover,

$$\varphi'_1(Y) = a'(x_\alpha + x_\beta) + c_s(E_{ss} + E_{n+s,n+s}) - c_{s+1}(E_{s+1,s+1} + E_{n+s+1,n+s+1}).$$

These equations lead to

$$u_\alpha + u_\beta = u_{-\alpha} + u_{-\beta} = 0, \qquad u_s - u_{s-1} = u_{s+1} - u_s.$$

They imply $c_s = c_{s+1}$, $s = 1,\dots,n-2$. Similarly, using the element $h'_{n-1} + B_{n-1,n-3} + B_{n-3,n}$, we obtain $c_{n-1} = c_n$. Therefore, assume that

$$\varphi'_1(h'_i) = bh_i, \quad i = 1,\dots,n-1.$$

For the element $h'_1 + y_{\alpha_1}$ we can find an element $u = \sum\limits_{k=1} u_k h'_k + \sum\limits_{\gamma\in\Delta} u_\gamma y_\gamma \in \mathfrak{q}(n)_1$ and $\lambda \in \mathbb{C}$ such that $\varphi'_1(h'_1 + y_{\alpha_1}) = [u, h'_1 + y_{\alpha_1}] + \lambda d_1(h'_1 + y_{\alpha_1})$, i.e.,

$$\begin{aligned} bh_1 + a' x_{\alpha_1} &= b(E_{11} + E_{n+1,n+1}) - b(E_{22} + E_{n+2,n+2}) + a' x_{\alpha_1} \\ &= (2u_1 + u_{-\alpha_1} + \lambda)(E_{11} + E_{n+1,n+1}) \\ &\quad - (2u_2 - 2u_1 - u_{-\alpha_1} + \lambda)(E_{22} + E_{n+2,n+2}) + (u_2 + \lambda) x_{\alpha_1}. \end{aligned}$$

This equation implies $b = a'$. Therefore, we obtain

$$\varphi'_1(y_\alpha) = a' x_\alpha, \varphi'_1(h'_k) = a' h_k, \quad k = 1,\dots,n-1,\ \alpha \in \Delta.$$

Now, we know that $\varphi'_1 = a' d_1$, i.e., $\varphi_1 = \operatorname{ad} \hat{h}' + a' d_1$ is an odd superderivation. □

Now we will prove Theorem 4.1.

P r o o f of Theorem 4.1. Since every local superderivation can be written as the sum of an even and an odd local superderivation, we only need to prove that every even or odd local superderivation is a superderivation. By Proposition 4.5 and Remark 4.3, we prove that every even local superderivation on $\mathfrak{q}(n)$ is a superderivation. For an odd local superderivation φ_1, there exists an element u and a constant $a \in \mathbb{C}$ such that $\varphi_1(\bar{h}) = [u, \bar{h}] + a d_1(\bar{h})$. Let $\varphi'_1 = \varphi_1 - \operatorname{ad} u$, then $\varphi'_1(\bar{h}) = a d_1(\bar{h}) \in H_0$. By Lemma 4.8, we conclude that φ'_1 is a superderivation. Thus, φ_1 is also a superderivation. □

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