Mohammad Reza Jabbarzadeh; Mehri Jafari Bakhshkandi
Centered weighted composition operators via measure theory

Mathematica Bohemica, Vol. 143 (2018), No. 2, 123–134

Persistent URL: `http://dml.cz/dmlcz/147240`

CENTERED WEIGHTED COMPOSITION OPERATORS VIA MEASURE THEORY

Mohammad Reza Jabbarzadeh, Mehri Jafari Bakhshkandi, Tabriz

Received September 9, 2016. First published May 31, 2017.
Communicated by Jiří Spurný

Abstract. We describe the centered weighted composition operators on $L^2(\Sigma)$ in terms of their defining symbols. Our characterizations extend Embry-Wardrop-Lambert's theorem on centered composition operators.

Keywords: Aluthge transform; Moore-Penrose inverse; weighted composition operator; conditional expectation; centered operator

MSC 2010: 47B20, 47B38

1. Introduction and preliminaries

Let (X,Σ,μ) be a σ-finite measure space and $\mathcal{A}$ be a sub-σ-algebra of Σ such that $(X,\mathcal{A},\varphi)$ is σ-finite. In measure space context, statements concerning function equality or inequality and set equality or inclusion are to be interpreted as holding up to sets of measure 0. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. For $f\in L^0(\Sigma)$ we let $\sigma(f)$ be the support of f so that $\sigma(f)=\{x\in X\colon f(x)\neq 0\}$. Let $\varphi\colon X\to X$ be a measurable transformation. Then $\mu\circ\varphi^{-1}(A):=\mu(\varphi^{-1}(A))$ is a measure on Σ. Let $\mu\circ\varphi^{-1}$ be absolutely continuous with respect to μ. Then for each $n\in\mathbb{N}$, $\mu\circ\varphi^{-n}$ is also absolutely continuous with respect to μ. Put $h_n:=\mathrm{d}(\mu\circ\varphi^{-n})/\mathrm{d}\mu$ the Radon-Nikodym derivative. Here we use the notation $L^p(\Sigma)$ for $L^p(X,\Sigma,\mu)$.

For a sub-σ-algebra $\mathcal{A}\subseteq\Sigma$, the conditional expectation mapping associated with $\mathcal{A}$ is a mapping $E^{\mathcal{A}}\colon f\mapsto E^{\mathcal{A}}f$ defined for all non-negative Σ-measurable functions f, where $E^{\mathcal{A}}f$ is the unique $\mathcal{A}$-measurable function satisfying

$$\int_A f\,\mathrm{d}\mu=\int_A E^{\mathcal{A}}f\,\mathrm{d}\mu\quad\forall\,A\in\mathcal{A}.$$

DOI: 10.21136/MB.2017.0080-16

Now, let f be a real valued Σ-measurable function on X. Then f is said to be conditionable if $\mu(\{x\colon E^{\mathcal{A}}(f^+(x)) = E^{\mathcal{A}}(f^-(x)) = \infty\}) = 0$. A complex valued function f is conditionable if both the real part and the imaginary part of f are conditionable and the respective expectations are not both infinite on the same set of positive measure. We show the set of all conditionable functions with respect to sub-σ-algebra $\mathcal{A}$ with $\mathcal{D}(E^{\mathcal{A}})$. As an operator on $L^2(\Sigma)$, $E^{\mathcal{A}}$ is an orthogonal projection and $E^{\mathcal{A}}(L^2(\Sigma)) = L^2(\mathcal{A})$. For each $n \in \mathbb{N}$ and $C \in \Sigma$ put $\mathcal{A}_C = \{A \cap C\colon A \in \mathcal{A}\}$, $\Sigma_n = \varphi^{-n}(\Sigma)$ and $E_n = E^{\Sigma_n}$. We use E instead of E_1. Let $f \in \mathcal{D}(E_n)$. Since $E_n(f)$ is a Σ_n-measurable function, there is a $g \in L^0(\Sigma)$ such that $E_n(f) = g \circ \varphi^n$. In general, g is not unique. This deficiency can be solved by assuming $\sigma(g) \subseteq \sigma(h_n)$ because for each $g_1, g_2 \in L^0(\Sigma)$, $g_1 \circ \varphi^n = g_2 \circ \varphi^n$ if and only if $g_1 = g_2$ on $\sigma(h_n)$. As a notation, we then write $g = E_n(f) \circ \varphi^{-n}$. With this setting, by the change of variables formula we obtain $\int_X f\,\mathrm{d}\mu = \int_X h_n E_n(f) \circ \varphi^{-n}\,\mathrm{d}\mu$, in the sense that if one of the integrals exists, then so does the other and they have the same value. Let $u \in \mathcal{D}(E^{\mathcal{A}})$. The weighted composition operator W on $L^2(\Sigma)$ induced by the pair (u, φ) is given by $W = M_u \circ C_\varphi$, where M_u is the multiplication operator and C_φ is the composition operator defined by $M_u f = uf$ and $C_\varphi f = f \circ \varphi$, respectively. It is a classical fact that W is a bounded linear operator on $L^2(\Sigma)$ if and only if $J := hE(|u|^2) \circ \varphi^{-1} \in L^\infty(\Sigma)$ (see [5]). It follows that $W^n = M_{u_n} \circ C_{\varphi^n}$ is a bounded operator on $L^2(\Sigma)$ precisely when $J_n := h_n E_n(|u_n|^2) \circ \varphi^{-n} \in L^\infty(\Sigma)$, where $n \geqslant 0$ and $u_n = u(u \circ \varphi)(u \circ \varphi^2) \ldots (u \circ \varphi^{n-1})$. Throughout this paper we assume that $W\colon L^2(\Sigma) \to L^2(\Sigma)$ is a weighted composition operator with non-negative weight function u. A good reference for information on composition operators on various function spaces is the monograph [9].

Let $\mathcal{H}$ be the infinite dimensional complex Hilbert spaces and let $B(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. We write $\mathcal{N}(T)$ and $\mathcal{R}(T)$ for the null-space and the range of an operator $T \in B(\mathcal{H})$, respectively. In [7], Morrel and Muhly introduced the concept of a centered operator. An operator T on a Hilbert space $\mathcal{H}$ is said to be centered if the doubly infinite sequence $\{T^nT^{*n}, T^{*m}T^m\colon n, m \geqslant 0\}$ consists of mutually commuting operators. For $T \in B(\mathcal{H})$ and $n \in \mathbb{N}$ let $V_n|T^n|$ be the polar decomposition of T^n. It is shown in [7], Theorem I, that T is centered if and only if $V_n = V_1^n$. The Aluthge transform of T is the operator $\widetilde{T}$ given by $\widetilde{T} := |T|^{1/2} V_1 |T|^{1/2}$. In [2], Embry-Wardrop and Lambert proved that the composition operator $C_\varphi \in B(L^2(\Sigma))$ is centered if and only if h is Σ_∞-measurable, where $\Sigma_\infty = \bigcap\limits_{n=1}^{\infty} \Sigma_n$. Recently, in [4] Giselsson introduced the concept of a half-centered operator. An operator $T \in B(\mathcal{H})$ is called half-centered if the sequence T^*T, $T^{*2}T^2$, $\ldots$ consists of mutually commuting operators. He proved that if M_u and C_φ are bounded on $L^2(\Sigma)$, then the operator W is always half-centered. Singh and Komal in [8] showed that the bounded composition operator C_φ on l^2, the Hilbert space of all square summable sequences, is centered if and only if h_k is constant on $\varphi^{-p}(\{n\})$ for every $k, p, n \in \mathbb{N}$.

In Section 2, we give some necessary and sufficient conditions for W acting on $L^2(\Sigma)$ or l^2 being centered. In Section 3, to avoid tedious calculations we consider only the composition case. We show that C_φ is centered if and only if $C_\varphi^\dagger$ is centered. In addition, we show that $\widetilde{C}_\varphi$ is centered whenever C_φ is centered.

2. Embry-Wardrop-Lambert's theorem on centered weighted composition operators

Definition 2.1. We say that the weight function u satisfies the support condition if $\sigma(J) \subseteq \sigma(u)$.

Note that if u satisfies the support condition, then $\sigma(u)$ is invariant under φ. Support condition for u provides an interesting situation for studying weighted composition operators with arbitrary weight function $u \in \mathcal{D}(E)$. From now on, we assume that u satisfies the support condition.

Put $X_\sigma = \sigma(u)$, $\Sigma_\sigma = \Sigma_{\sigma(u)}$, $\mu_\sigma = \mu|_{\Sigma_\sigma}$, $\varphi_\sigma = \varphi|_{\sigma(u)}$, $h_\sigma = \mathrm{d}(\mu_\sigma \circ \varphi_\sigma^{-1})/\mathrm{d}\mu_\sigma$ and $E_\sigma = E^{\varphi_\sigma^{-1}(\Sigma_\sigma)}$. It is easy to check that $(X_\sigma, \Sigma_\sigma, \mu_\sigma)$ is σ-finite, $\mu_\sigma \circ \varphi_\sigma^{-1} \ll \mu_\sigma$ and $L^2(\Sigma) = L^2(\Sigma_\sigma) \oplus L^2(\Sigma_{\sigma^c})$, where $\sigma^c = X \setminus \sigma(u)$.

Recall that for $T \in B(\mathcal{H})$ there is a unique factorization $T = U|T|$, where $\mathcal{N}(T) = \mathcal{N}(U) = \mathcal{N}(|T|)$, U is a partial isometry; i.e. $UU^*U = U$ and $|T| = (T^*T)^{1/2}$ is a positive operator. This factorization is called the polar decomposition of T. It is known that the parts U, $|W|$ of the polar decomposition for W are given by $U = M_{u/\sqrt{h \circ \varphi E(u^2)}} C_\varphi$ and $|W| = M_{\sqrt{J}}$. Note that $\sigma(h \circ \varphi^i) = X$, $\sigma(E(u^2)) = \sigma(E(u)) \supseteq \sigma(u)$ and $\sigma(J \circ \varphi^i) = \sigma(h \circ \varphi^i E(u^2) \circ \varphi^{i-1}) \supseteq \sigma(u \circ \varphi^{i-1})$. Also, the support condition $\sigma(J_n \circ \varphi^n) = \sigma(E_n(u_n^2)) \supseteq \sigma(u_n) = \sigma(u)$ holds for each $n \in \mathbb{N}$. The following lemma is checked by a direct calculation.

Lemma 2.2. *Let $n \in \mathbb{N}$ and U_n, $|W^n|$ be the polar decomposition of W^n, the n-th iterate of W. Then*

$$U_n = M_{u_n/\sqrt{J_n \circ \varphi^n}} C_{\varphi^n}, \quad |W^n| = M_{\sqrt{J_n}}.$$

Moreover, for each $f \in L^2(\Sigma)$,

$$U^n f = \prod_{i=1}^{n} \Big(\frac{u \circ \varphi^{i-1}}{\sqrt{J \circ \varphi^i}} \Big) f \circ \varphi^n.$$

Lemma 2.3. *For each $n \in \mathbb{N}$ the following assertions hold.*

(a) *Let $g \in L^0(\Sigma)$ be a finite valued function such that $gf = 0$ for all $f \in L^2(\Sigma_n)$. Then $g = 0$.*

(b) $J_{n+1} = h_n E_n(Ju_n^2) \circ \varphi^{-n} = hE(J_n u^2) \circ \varphi^{-1}$.

Proof. (a) Let $\sigma(g) \supseteq B \in \Sigma$ with $0 < \mu(B) < \infty$. Put $B^{\#} = \sigma(E_n(\chi_B))$. Then $B \subseteq B^{\#} \in \Sigma_n$. If $\mu(X) < \infty$, then $\chi_{B^{\#}} \in L^2(\Sigma_n)$. Since $g\chi_{B^{\#}} = 0$, it follows that $\mu(B) \leqslant \mu(\sigma(g) \cap B^{\#}) = 0$. But this is a contradiction. Thus $\mu(\sigma(g)) = 0$. Now, let $X = \bigcup X_m$ with $X_m \in \Sigma_n$ and $\mu(X_m) < \infty$. Then $g = 0$ on each X_m and so $g = 0$ on X.

(b) It is enough to show that for every $A \in \Sigma$, $\int_A J_{n+1}\,\mathrm{d}\mu = \int_A h_n E_n(Ju_n^2)\circ\varphi^{-n}\,\mathrm{d}\mu$. For this let $A \in \Sigma$. Then

$$\begin{aligned}\int_A J_{n+1}\,\mathrm{d}\mu &= \int_{\varphi^{-(n+1)}(A)} u_{n+1}^2\,\mathrm{d}\mu = \int_{\varphi^{-(n+1)}(A)} E(u^2)u_n^2 \circ \varphi\,\mathrm{d}\mu \\ &= \int_{\varphi^{-n}(A)} hE(u^2)\circ\varphi^{-1}u_n^2\,\mathrm{d}\mu = \int_{\varphi^{-n}(A)} E_n(Ju_n^2)\,\mathrm{d}\mu \\ &= \int_A h_n E_n(Ju_n^2)\circ\varphi^{-n}\,\mathrm{d}\mu,\end{aligned}$$

and

$$\begin{aligned}\int_A J_{n+1}\,\mathrm{d}\mu &= \int_{\varphi^{-(n+1)}(A)} E_n(u_n^2)u^2 \circ \varphi^n\,\mathrm{d}\mu = \int_{\varphi^{-1}(A)} J_n u^2\,\mathrm{d}\mu \\ &= \int_A hE(J_n u^2)\circ\varphi^{-1}\,\mathrm{d}\mu.\end{aligned}$$

This completes the proof. □

As a generalization of Embry-Wardrop-Lambert's theorem [2], Theorem 5, we can now characterize this pair (u, φ), for which W is centered.

Theorem 2.4. *The weighted composition operator W is centered if and only if for each $n \in \mathbb{N}$, $J_n \circ \varphi^n = \prod\limits_{i=1}^{n} J \circ \varphi^i$ on $\sigma(u)$.*

Proof. Recall from [2] that W is centered if and only if for any positive integer n, $U_n = U^n$. By Lemma 2.2 this means that for every f in $L^2(\Sigma)$,

$$\left(\frac{u_n}{\sqrt{J_n \circ \varphi^n}} - \frac{u_n}{\prod_{i=1}^n \sqrt{J \circ \varphi^i}}\right) f \circ \varphi^n = 0.$$

Now, the desired conclusion follows from Lemma 2.3 (a). □

Corollary 2.5. *The composition operator C_φ is centered if and only if for each $n \in \mathbb{N}$, $h_n \circ \varphi^n = \prod_{i=1}^{n} h \circ \varphi^i$.*

Put

$$(h_\sigma)_n = \frac{\mathrm{d}(\mu_\sigma \circ \varphi_\sigma^{-n})}{\mathrm{d}\mu_\sigma}; \quad (\Sigma_\sigma)_n = \varphi_\sigma^{-n}(\Sigma_\sigma); \quad (\Sigma_\sigma)_\infty = \bigcap_{n=1}^{\infty} (\Sigma_\sigma)_n;$$
$$(E_\sigma)_n = E^{(\Sigma_\sigma)_n}; \quad (J_\sigma)_n = (h_\sigma)_n (E_\sigma)_n (u_n^2) \circ \varphi_\sigma^{-n}.$$

Theorem 2.6. *Let J_σ be $(\Sigma_\sigma)_\infty$-measurable. Then W is centered.*

Proof. Let $n \in \mathbb{N}$. By Theorem 2.4 it is enough to show that

$$(J_\sigma)_n \circ \varphi_\sigma^n = \prod_{i=1}^{n} J_\sigma \circ \varphi_\sigma^i. \tag{2.1}$$

Use induction on n and suppose (2.1) holds for some n. Since for each $n \in \mathbb{N}$ J_σ is $(\Sigma_\sigma)_n$-measurable, $(E_\sigma)_n(J_\sigma u_n^2) = J_\sigma (E_\sigma)_n (u_n^2)$. Now, by Lemma 2.3 (b) we obtain

$$\begin{aligned}
(J_\sigma)_{n+1} \circ \varphi_\sigma^{n+1} &= ((h_\sigma)_n (E_\sigma)_n (J_\sigma u_n^2) \circ \varphi_\sigma^{-n}) \circ \varphi_\sigma^{n+1} \\
&= (h_\sigma)_n \circ \varphi_\sigma^{n+1} (E_\sigma)_n (J_\sigma u_n^2) \circ \varphi_\sigma \\
&= ((h_\sigma)_n (E_\sigma)_n (u_n^2) \circ \varphi_\sigma^{-n}) \circ \varphi_\sigma^{n+1} J_\sigma \circ \varphi_\sigma \\
&= (J_{\sigma n} \circ \varphi_\sigma^{n+1}) J_\sigma \circ \varphi_\sigma = \prod_{i=1}^{n+1} J_\sigma \circ \varphi_\sigma^i.
\end{aligned}$$

Consequently, W is centered. □

Theorem 2.7. *If W is centered then, J_σ is $((\Sigma_\sigma)_\infty)_{\sigma(h_\sigma)}$-measurable.*

Proof. By hypothesis, (2.1) holds for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
(J_\sigma)_{n+1} \circ \varphi_\sigma^{n+1} &= \prod_{i=1}^{n+1} J_\sigma \circ \varphi_\sigma^i = J_\sigma \circ \varphi_\sigma \left(\prod_{i=1}^{n} J_\sigma \circ \varphi_\sigma^i \right) \circ \varphi_\sigma \\
&= J_\sigma \circ \varphi_\sigma (J_{\sigma n} \circ \varphi_\sigma^n) \circ \varphi_\sigma \\
&= (J_\sigma \circ \varphi_\sigma)((h_\sigma)_n \circ \varphi_\sigma^{n+1})((E_\sigma)_n (u_n^2)) \circ \varphi_\sigma.
\end{aligned}$$

On the other hand, from Lemma 2.3 (b),

$$(J_\sigma)_{n+1} \circ \varphi_\sigma^{n+1} = (h_\sigma)_n \circ \varphi_\sigma^{n+1} ((E_\sigma)_n (J u_n^2)) \circ \varphi_\sigma.$$

Since $\sigma((h_\sigma)_n \circ \varphi_\sigma^{n+1}) = \sigma(u)$, we have

$$(E_\sigma)_n(J_\sigma u_n^2) \circ \varphi_\sigma = J_\sigma \circ \varphi_\sigma (E_\sigma)_n(u_n^2) \circ \varphi_\sigma,$$

and so

$$(E_\sigma)_n(J_\sigma u_n^2) = J_\sigma (E_\sigma)_n(u_n^2)$$

on $\sigma(h_\sigma)$. Hence, for all $n \in \mathbb{N}$,

$$J_\sigma = \frac{(E_\sigma)_n(J_\sigma u_n^2)}{(E_\sigma)_n(u_n^2)}$$

on $\sigma(h_\sigma)$. Thus J_σ is $((\Sigma_\sigma)_\infty)_{\sigma(h_\sigma)}$-measurable. $\square$

An operator $T \in B(\mathcal{H})$ is said to be hyponormal if $T^*T - TT^*$ is positive. In [6], Lambert proved that $W_\sigma^* \in B(L^2(\Sigma_\sigma))$ is hyponormal if and only if $\Sigma_{\sigma(J)} \subseteq (\varphi^{-1}(\Sigma))_\sigma$ and $J \circ \varphi \geqslant J$. In the following theorem we give necessary conditions for cohyponormality of W_σ.

Theorem 2.8. *If W_σ^* is hyponormal, then*
(i) $\varphi_\sigma^{-1}(\Sigma_\sigma) = (\Sigma_\sigma)_\infty$,
(ii) *W_σ is centered.*

Proof. Let $A \in \Sigma_\sigma$. If $A \subseteq \sigma(u) \backslash \sigma(J_\sigma)$, then $\varphi_\sigma^{-1}(A) = \emptyset$ because $\varphi_\sigma^{-1}(\sigma(u)) \subseteq \sigma(u)$ and $\sigma(J_\sigma \circ \varphi_\sigma) = \sigma(u)$. Assume that $A \subseteq \sigma(J_\sigma)$. Since W_σ^* is hyponormal, $(\Sigma_\sigma)_{\sigma(J_\sigma)} \subseteq \varphi_\sigma^{-1}(\Sigma_\sigma)$ and so $A \in \varphi_\sigma^{-1}(\Sigma_\sigma)$. Thus, there is a set $B \in \Sigma_\sigma$ with $A = \varphi_\sigma^{-1}(B)$. Hence $\varphi_\sigma^{-1}(\Sigma_\sigma) = \varphi_\sigma^{-2}(\Sigma_\sigma)$. It follows that $\varphi_\sigma^{-1}(\Sigma_\sigma) = (\Sigma_\sigma)_\infty$.

Now, let $U \subseteq \mathbb{R}$ be an open set. If $0 \notin U$, then $J_\sigma^{-1}(U) \subseteq \sigma(J_\sigma)$. Because $J_\sigma^{-1}(U) \in \Sigma_\sigma$ and W_σ^* is hyponormal, $J_\sigma^{-1}(U) \in \varphi_\sigma^{-1}(\Sigma_\sigma)$. On the other hand, since $\sigma(J_\sigma) \in \varphi_\sigma^{-1}(\Sigma_\sigma)$, $J_\sigma^{-1}(U) = (J_\sigma^{-1}(U) \cap \sigma(J_\sigma)) \cup (\sigma(J_\sigma))^c \in \varphi_\sigma^{-1}(\Sigma_\sigma)$ whenever $0 \in U$. Hence from (i), J_σ is $(\Sigma_\sigma)_\infty$-measurable and thus by Theorem 2.6 W_σ is centered. $\square$

We now turn to the discrete versions of Embry-Wardrop-Lambert's theorem for W.

Lemma 2.9. *Let $m \in \mathbb{N}$. The bounded function $f\colon \mathbb{N} \to \mathbb{R}$ is Σ_m-measurable if and only if f is constant on $\varphi^{-m}(\{n\})$ for all $n \in \mathbb{N}$.*

Proof. Let f be a Σ_m-measurable function. Since for each $n \in \mathbb{N}$, $\varphi^{-m}(\{n\})$ is an atom in Σ_m, f is constant on $\varphi^{-m}(\{n\})$. Conversely, let $U \subseteq \mathbb{R}$ be an open set with $U \cap f(\mathbb{N}) = \{x_j\}_{j\in J}$. Then $f^{-1}(U) = \bigcup\limits_j A_j$, where $A_j = f^{-1}(\{x_j\})$ with $j \in J$. Since f is constant on $\varphi^{-m}(\{n\})$, $f^{-1}(\{x_j\}) = \bigcup\limits_{y\in A_j} \varphi^{-m}(\{\varphi^m(y)\})$, it follows that

$$f^{-1}(U) = \bigcup_{j\in J} \bigcup_{y\in A_j} \varphi^{-m}(\{\varphi^m(y)\}) \in \Sigma_m,$$

and the proof is complete. □

Theorem 2.10. *Let $W \in B(l^2)$. Then the following assertions hold.*

(i) *If for every $m, n \in \mathbb{N}$, J_σ is constant on $\varphi_\sigma^{-m}(\{n\})$, then W is centered.*

(ii) *If W is centered, then for every $n, m \in \mathbb{N}$, J_σ is constant on $\varphi_\sigma^{-m}(\{n\}) \cap \sigma(h_\sigma)$.*

Proof. (i) By Lemma 2.9, if for every $m, n \in \mathbb{N}$, J_σ on $\varphi_\sigma^{-m}(\{n\})$ is constant, then J_σ is $\Sigma_{\sigma m}$-measurable. It follows that J_σ is $(\Sigma_\sigma)_\infty$-measurable, and so by Theorem 2.6, W is centered.

(ii) By Theorem 2.7, for any $m \in \mathbb{N}$, J_σ is $(\Sigma_\sigma)_m$-measurable on $\sigma(h_\sigma)$. On the other hand, for any $n \in \mathbb{N}$, $\varphi_\sigma^{-m}(\{n\}) \cap \sigma(h_\sigma)$ is an atom in $((\Sigma_\sigma)_m)_{\sigma(h_\sigma)}$. Therefore J_σ is constant on $\varphi_\sigma^{-m}(\{n\}) \cap \varphi_\sigma(\sigma(u))$. □

3. Centered Moore-Penrose inverse and Aluthge transform of composition operators

Recall that $T \in B(\mathcal{H})$ has a generalized inverse if there exists an operator $S \in B(\mathcal{H})$ for which $TST = T$. It is well known that $T \in B(\mathcal{H})$ has a generalized inverse if and only if $\mathcal{R}(T)$ is closed (see [1]). In general, S is not unique. The generalized inverse S is called the Moore-Penrose inverse of T if $STS = S$ and the idempotents TS and ST are self-adjoint. In this case, S is unique and it is denoted by $T^\dagger$. Note that if $U|T|$ is the polar decomposition of T, then by definition, U^* is a generalized inverse of U and hence has closed range. Also, since $\mathcal{R}(U^*) = \mathcal{N}(U)^\perp$, U is isometry on $\mathcal{R}(U^*)$. It is easy to check that $U^*|T^*|^\dagger$ and $|T^\dagger|^{1/2}U^*|T^\dagger|^{1/2}$ are the polar decomposition and Aluthge transform of $T^\dagger$, respectively.

To avoid tedious calculations we consider only the composition case. Suppose that $C_\varphi \in B(L^2(\Sigma))$ has closed range. Then h is bounded away from zero on $\sigma(h)$. Put $S = M_{\chi_{\sigma(h)}/h} C_\varphi^*$. Then $S \in B(L^2(\Sigma))$. Since $\sigma(h \circ \varphi) = X$, $C_\varphi^* C_\varphi = M_h$, and $C_\varphi C_\varphi^* = M_{h\circ\varphi} E$, we have $C_\varphi S C_\varphi = C_\varphi$ and $S C_\varphi S = S$. Also it is easy to check that $C_\varphi S = E = (C_\varphi S)^*$ and $S C_\varphi = M_{\chi_{\sigma(h)}} = (S C_\varphi)^*$. Hence, S is the Moore-Penrose inverse of C_φ. Also, it is easy to check that

$$(C_\varphi^\dagger)^* = C_\varphi M_{\chi_{\sigma(h)/h}} = M_{1/h\circ\varphi} C_\varphi,$$
$$(C_\varphi^\dagger)^* C_\varphi^\dagger = M_{1/h\circ\varphi} E = (M_{1/\sqrt{h\circ\varphi}} E)^2$$

and $VV^*V = V$, where $V(f) = \sqrt{h}E(f) \circ \varphi^{-1}$. These observations establish the following theorem.

Theorem 3.1. *Let $C_\varphi \in B(L^2(\Sigma))$ have closed range. Then $C_\varphi^\dagger = M_{\chi_{\sigma(h)}/h} C_\varphi^*$. Moreover, if $V|C_\varphi^\dagger|$ is the polar decomposition of $C_\varphi^\dagger$, then*

$$V(f) = \sqrt{h}E(f) \circ \varphi^{-1}, \quad |C_\varphi^\dagger|(f) = \frac{E(f)}{\sqrt{h \circ \varphi}}$$

for each f in $L^2(\Sigma)$.

It follows from Theorem 3.1 that if $V_n|(C_\varphi^\dagger)^n|$ is the polar decomposition of $(C_\varphi^\dagger)^n$, then

$$V_n(f) = \sqrt{h_n}E_n(f) \circ \varphi^{-n}, \quad |(C_\varphi^\dagger)^n|(f) = \frac{E_n(f)}{\sqrt{h_n \circ \varphi^n}},$$
$$V^n(f) = \sqrt{h}E(\sqrt{h}E(\ldots \sqrt{h}E(f) \circ \varphi^{-1} \ldots) \circ \varphi^{-1}) \circ \varphi^{-1}$$

for all $n \in \mathbb{N}$ and f in $L^2(\Sigma)$. Moreover, straightforward calculations show that

$$C_{\varphi^n}^\dagger(f) = E_n(f) \circ \varphi^{-n}$$

and

$$(C_\varphi^\dagger)^n(f) = E(E(\ldots E(f) \circ \varphi^{-1} \ldots) \circ \varphi^{-1}) \circ \varphi^{-1}.$$

But it is a classical fact that $(C_\varphi^\dagger)^n = C_{\varphi^n}^\dagger$. Thus

$$E(E(\ldots E(f) \circ \varphi^{-1} \ldots) \circ \varphi^{-1}) \circ \varphi^{-1} = E_n(f) \circ \varphi^{-n}. \tag{3.1}$$

Theorem 3.2. *Let $C_\varphi \in B(L^2(\Sigma))$ have closed range. Then C_φ is centered if and only if $C_\varphi^\dagger$ is centered.*

Proof. Recall that $C_\varphi^\dagger$ is centered if and only if $V^n = V_n$. Equivalently,

$$\sqrt{h}E(\sqrt{h}E(\ldots \sqrt{h}E(f) \circ \varphi^{-1} \ldots) \circ \varphi^{-1}) \circ \varphi^{-1} = \sqrt{h_n}E_n(f) \circ \varphi^{-n} \tag{3.2}$$

for all $n \in \mathbb{N}$ and $f \in L^2(\Sigma)$. Now suppose that C_φ is centered. Then by [2], Theorem 5, h is Σ_∞-measurable. So for $1 \leqslant i \leqslant n$, $h \circ \varphi^{i-n}$ is well-defined. It follows that the left hand side of equality (3.2) equals to

$$\sqrt{h \cdot h \circ \varphi^{-1} \ldots h \circ \varphi^{-(n-1)}}E(E(\ldots E(f) \circ \varphi^{-1} \ldots) \circ \varphi^{-1}) \circ \varphi^{-1}.$$

Since $h_n = h \cdot h \circ \varphi^{-1} \ldots h \circ \varphi^{-(n-1)}$ (see [2]), by (3.1), equality (3.2) holds.

Conversely, suppose that $C_\varphi^\dagger$ is centered. It is easy to verify that

$$V^{n+1}(f) = \sqrt{h_n}E_n(\sqrt{h}E(f)\circ\varphi^{-1})\circ\varphi^{-n}$$

and

$$V_{n+1}(f) = \sqrt{h_{n+1}}E_{n+1}(f)\circ\varphi^{-(n+1)}.$$

It follows that

$$\sqrt{h_{n+1}}E_{n+1}(f)\circ\varphi^{-(n+1)} = \sqrt{h_n}E_n(\sqrt{h}E(f)\circ\varphi^{-1})\circ\varphi^{-n}.$$

Now let $A \in \Sigma$ with $\mu(A) < \infty$. Then $\mu(\varphi^{-(n+1)}(A)) < \infty$ because $h_{n+1} \in L^\infty(\Sigma)$ and so $f := \chi_{\varphi^{-(n+1)}(A)}$ is in $L^2(\Sigma)$. It follows that

$$\sqrt{h_{n+1}}\chi_A = \sqrt{h_n}E_n(\sqrt{h})\circ\varphi^{-n}\chi_A$$

and so $h_{n+1} = h_n(E_n(\sqrt{h}))^2\circ\varphi^{-n}$. But $h_{n+1} = h_nE_n(h)\circ\varphi^{-n}$. Hence

$$h_n\circ\varphi^nE_n(h) = h_n\circ\varphi^n(E_n(\sqrt{h}))^2.$$

Consequently, $E_n(\sqrt{h}^2) = (E_n(\sqrt{h}))^2$ because $\sigma(h_n\circ\varphi^n) = X$. This implies that h is Σ_n-measurable for all $n \in \mathbb{N}$. Thus, h is Σ_∞-measurable and so C_φ is centered. $\square$

At this stage, we consider the Aluthge transformation of C_φ. Recall that the Aluthge transformation of C_φ is defined by $\widetilde{C}_\varphi := |C_\varphi|^{1/2}U|C_\varphi|^{1/2}$. It is easy to check that

$$(\widetilde{C}_\varphi)^n(f) = \Big(\frac{h}{h\circ\varphi^n}\Big)^{1/4} f\circ\varphi^n, \quad n\in\mathbb{N},\ f\in L^2(\Sigma).$$

Performing some direct computations we get the following lemma.

Lemma 3.3. *Let $\widetilde{V}_n|(\widetilde{C}_\varphi)^n|$ be the polar decomposition of $(\widetilde{C}_\varphi)^n$ for $n \in \mathbb{N}$. Then for each $f \in L^2(\Sigma)$ we have*

$$\widetilde{V}_n(f) = \frac{h^{1/4}}{\sqrt{h_n\circ\varphi^nE_n(\sqrt{h})}}f\circ\varphi^n,$$

$$|(\widetilde{C}_\varphi)^n|(f) = \sqrt{h_nE_n\Big(\frac{h}{h\circ\varphi^n}\Big)^{1/2}\circ\varphi^{-n}}f$$

and

$$\widetilde{V}^n(f) = \Big(\frac{h^{1/2}}{h^{1/2}\circ\varphi\dots h^{1/2}\circ\varphi^{n-1}h^{1/2}\circ\varphi^nE(\sqrt{h})\dots E(\sqrt{h})\circ\varphi^{n-1}}\Big)^{1/2}f\circ\varphi^n.$$

Note that $\widetilde{C}_\varphi$ is centered if and only if $\widetilde{V}^n = \widetilde{V}_n$ for all $n \in \mathbb{N}$. Therefore using Lemma 3.3 we get the following corollary.

Corollary 3.4. *$\widetilde{C}_\varphi$ is centered if and only if for every $n \in \mathbb{N}$*

$$h_n \circ \varphi^n E_n(\sqrt{h}) = h^{1/2} \circ \varphi \ldots h^{1/2} \circ \varphi^{n-1} h^{1/2} \circ \varphi^n E(\sqrt{h}) \ldots E(\sqrt{h}) \circ \varphi^{n-1}$$

on $\sigma(h)$.

Theorem 3.5. *If $C_\varphi \in B(L^2(\Sigma))$ is a centered operator, then so is $\widetilde{C}_\varphi$.*

Proof. Since C_φ is centered, h is Σ_∞-measurable, and so for all $n \in \mathbb{N}$, $E_n(\sqrt{h}) = \sqrt{h}$. Then from Corollary 2.5 we have

$$\begin{aligned} h^{1/2} \circ \varphi \ldots h^{1/2} \circ \varphi^{n-1} h^{1/2} \circ \varphi^n E(\sqrt{h}) \ldots E(\sqrt{h}) \circ \varphi^{n-1} &= \sqrt{h} \prod_{i=1}^{n} h \circ \varphi^i \\ &= E_n(\sqrt{h}) h_n \circ \varphi^n. \end{aligned}$$

Now the desired conclusion follows from Corollary 3.4. □

The following example shows that the converse of Theorem 3.5 is in general not true.

Example 3.6. Let $X = \{a_i \colon i \in \mathbb{N}\} \cup \{b_i \colon i \in \mathbb{N}\} \cup \{c_i \colon i \in \mathbb{N}\}$, $\Sigma = 2^X$, $m(\{a_i\}) = m_i$, $m(\{b_i\}) = n_i$, $m(\{c_i\}) = k_i$ and let φ be a transformation on X such that

$$\varphi(a_{i+1}) = a_i, \quad \varphi(b_{i+1}) = b_i, \quad \varphi(a_1) = \varphi(b_1) = c_1, \quad \varphi(c_i) = c_{i+1}.$$

Let u be a nonzero real-valued function on X that satisfies the support condition $\sigma(u) \subseteq \varphi^{-1}(\sigma(u))$. Then $\sigma(u) \cap \{c_i \colon i \in \mathbb{N}\} \neq \emptyset$. Direct computation shows that

$$J(a_i) = \frac{u^2(a_{i+1}) m_{i+1}}{m_i}, \quad J(b_i) = \frac{u^2(b_{i+1}) n_{i+1}}{n_i},$$

$$J(c_1) = \frac{u^2(a_1) m_1 + u^2(b_1) n_1}{k_1}, \quad J(c_i) = \frac{u^2(c_{i-1}) k_{i-1}}{k_i}, \quad i = 2, 3, \ldots$$

Also, it is easy to check that $\varphi_\sigma^{-n}(\{a_i\}) = \{a_{i+n}\}$, $\varphi_\sigma^{-n}(\{b_i\}) = \{b_{i+n}\}$ and

$$\varphi_\sigma^{-n}(\{c_i\}) = \begin{cases} \{c_{i-n}\}, & i - n \geqslant 1, \\ \{a_{n+1-i}, b_{n+1-i}\}, & i - n < 1. \end{cases}$$

By Theorem 2.10, W_σ is centered whenever for every $i \in \mathbb{N}$, J_σ is constant on $\{a_i, b_i\} \cap \sigma(h)$. But

$$J_\sigma(a_i) = \frac{u^2(a_{i+1})m_{i+1}}{m_i} = \frac{u^2(b_{i+1})n_{i+1}}{n_i} = J_\sigma(b_i)$$

for each $n \in \mathbb{N}$. Hence W_σ is centered. Note that if $\sigma(u) = \sigma(h) = X$, then W is centered if and only if $J(a_i) = J(b_i)$ for all $i \in \mathbb{N}$.

Example 3.7. Let X and Σ be as in the above example. Put $u = \sqrt[4]{h/(h \circ \varphi)}$. Then $W = \widetilde{C}_\varphi$. We set $n_i = 1$, $k_i = 1$ for $i \neq 4$, $k_4 = 2$, $m_1 = m_3 = 1$, $m_2 = 2$, and for $i \geqslant 2$, $m_{i+2} = m_i$. Note that $\sigma(u) = \sigma(h) = X$. Since

$$h(a_1) = \frac{m_2}{m_1} = 2 \neq 1 = \frac{n_2}{n_1} = h(b_1),$$

C_φ is not centered. However, since for every $i \in \{4, 5, 6, \ldots\}$,

$$J(a_i) = \sqrt{\frac{m_{i+2}}{m_i}} = J(b_i)$$

and for $i = 1, 2, 3$, $J(a_i) = J(b_i) = 1$, by Theorem 2.10, $\widetilde{C}_\varphi$ is centered. Moreover, since

$$(J \circ \varphi)(c_3) = \frac{\sqrt[4]{m(\{c_4\})}}{m(\{c_4\})} = \frac{\sqrt[4]{2}}{2} < 1 = J(c_3),$$

$\widetilde{C}_\varphi^*$ is not hyponormal. So the converse of [3], Lemma 2 does not hold in general.

Acknowledgment. The authors would like to thank the referee for very helpful comments and valuable suggestions.

References

[1] *S. R. Caradus*: Generalized Inverses and Operator Theory. Queen's Pap. Pure Appl. Math. 50. Queen's University, Kingston, Ontario, 1978. zbl MR

[2] *M. Embry-Wardrop, A. Lambert*: Measurable transformations and centred composition operators. Proc. R. Ir. Acad., Sect. A *90* (1990), 165–172. zbl MR

[3] *M. Embry-Wardrop, A. Lambert*: Subnormality for the adjoint of a composition operator on L^2. J. Oper. Theory *25* (1991), 309–318. zbl MR

[4] *O. Giselsson*: Half-centered operators. Online `https://arxiv.org/pdf/1602.05081v1.pdf`, 44 pages.

[5] *T. Hoover, A. Lambert, J. Quinn*: The Markov process determined by a weighted composition operator. Stud. Math. *72* (1982), 225–235. zbl MR

[6] *A. Lambert*: Hyponormal composition operators. Bull. Lond. Math. Soc. *18* (1986), 395–400. zbl MR doi

[7] *B. B. Morrel, P. S. Muhly*: Centered operators. Studia Math. *51* (1974), 251–263. zbl MR

[8] *R. K. Singh, B. S. Komal*: Composition operators. Bull. Aust. Math. Soc. *18* (1978), 439–446. zbl MR doi
[9] *R. K. Singh, J. S. Manhas*: Composition Operators on Function Spaces. North-Holland Mathematics Studies 179. North-Holland Publishing, Amsterdam, 1993. zbl MR doi

Authors' address: *Mohammad Reza Jabbarzadeh*, *Mehri Jafari Bakhshkandi*, Faculty of Mathematical Sciences, University of Tabriz, P. O. Box: 5166615648, Tabriz, Iran, e-mail: `mjabbar@tabrizu.ac.ir`, `m_jafari@tabrizu.ac.ir`.