Czechoslovak Mathematical Journal

Amirhesam Zaeim; Parvane Atashpeykar
Harmonic metrics on four dimensional non-reductive homogeneous manifolds

Czechoslovak Mathematical Journal, Vol. 68 (2018), No. 2, 475–490

Persistent URL: http://dml.cz/dmlcz/147230

HARMONIC METRICS ON FOUR DIMENSIONAL NON-REDUCTIVE HOMOGENEOUS MANIFOLDS

Amirhesam Zaeim, Tehran, Parvane Atashpeykar, Bonab

Received May 26, 2016. First published February 14, 2018.

Abstract. We study harmonic metrics with respect to the class of invariant metrics on non-reductive homogeneous four dimensional manifolds. In particular, we consider harmonic lifted metrics with respect to the Sasaki lifts, horizontal lifts and complete lifts of the metrics under study.

Keywords: harmonic metric; non-reductive homogeneous space; pseudo-Riemannian geometry

MSC 2010: 53C43, 53C55

1. Introduction

A (connected) pseudo-Riemannian manifold (M, g) is said to be homogeneous if it admits a group G of isometries, acting transitively on it. In this case, (M, g) can be identified with $(G/H, g)$, where H is the isotropy group at a fixed point of M and g is an invariant pseudo-Riemannian metric. A homogeneous pseudo-Riemannian manifold (M, g) is reductive if it can be realized as a coset space $M = G/H$, such that the Lie algebra $\mathfrak{g}$ can be decomposed into a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant subspace of $\mathfrak{g}$. It is well known that when H is connected, this condition is equivalent to the algebraic condition $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$. All homogeneous Riemannian manifolds are reductive, but there exist homogeneous pseudo-Riemannian manifolds which do not admit any reductive decomposition. Full classification of these spaces, up to isometry classes, have been done by Fels and Renner in [15]. They showed that any non-reductive homogeneous pseudo-Riemannian four manifold is isometric to one of 8 classes $A1, \dots, A5, B1, \dots B3$, which contain both Lorentzian and neutral signature examples. Non-reductive homogeneous four-dimensional manifolds were

DOI: 10.21136/CMJ.2018.0502-16

studied from several aspects. For example, Ricci solitons and geometry of these spaces were studied in [7] and [9], while Walker structures were considered in [10].

Harmonicity is a rich topic in differential geometry, analysis, theoretical physics etc. This topic was initiated by harmonic functions, i.e., real $\mathbb{C}^2$ functions belonging to the kernel of the Laplace operator, and then generalized to harmonic maps between (pseudo-)Riemannian manifolds, i.e., maps which satisfy the Euler-Lagrange systems, and harmonic exterior forms, i.e., forms that are simultaneously closed and co-closed. Today, harmonicity has been extended in several directions such as harmonic morphisms (see [1]), harmonic (pseudo-)Riemannian metrics (see [11]), harmonic sections (see [5], [12]), harmonic endomorphisms (see [2], [3]), harmonic connections (see [2]) etc.

In this paper, we study four-dimensional non-reductive homogeneous manifolds and completely classify invariant harmonic metrics in different classes of the spaces under consideration. We then study harmonic metrics with respect to the lifted metrics (i.e., Sasaki lifts, horizontal lifts and complete lifts) on the tangent bundle, proving that an invariant metric $\hat{g}$ being harmonic with respect to the invariant metric g is equivalent to its Sasaki lift metric $\hat{g}^S$ being harmonic with respect to g^S (respectively, horizontal lift $\hat{g}^H$ with respect to g^H and complete lift $\hat{g}^C$ with respect to g^C).

The paper is organized in the following way. In Section 2 we report some basic facts about harmonic functions and some related notation. We recall the classification of four-dimensional non-reductive homogeneous spaces in Section 3. Section 4 is devoted to the study of non-reductive homogeneous spaces which are lifted to the tangent bundle. Finally, invariant harmonic metrics with respect to the invariant non-reductive homogeneous metrics and harmonic lifted metrics (Sasaki lifts, horizontal lifts and complete lifts) with respect to the non-reductive homogeneous lifted metrics is considered in the last section.

2. Preliminaries

To study harmonic structures (especially harmonic metrics) over pseudo-Riemannian manifolds, we need to know some basic relevant facts. We recall some definitions in [11], [13].

Definition 2.1. For a C^2-map $(M,g) \to (N,h)$ between pseudo-Riemannian manifolds (M,g) and (N,h),

(1) the *Hilbert-Schmidt square norm* on $T^*M \otimes \varphi^{-1}TN$ induced by g and h is

$$|\,\mathrm{d}\varphi|^2 = \mathrm{tr}_g(\varphi^* h) = g^{ij} h(\,\mathrm{d}\varphi(X_i),\,\mathrm{d}\varphi(X_j)),$$

where X_i is an arbitrary local frame on M. In this case, the *energy density* of φ is defined by the identity

$$e_\varphi = \frac{1}{2}|\,\mathrm{d}\varphi|^2;$$

(2) if D is a compact domain of M, the *energy (integral)* of φ on D is the real number

$$E(\varphi; D) = \int_D e(\varphi) v_g,$$

where v_g is the volume measure of g;

(3) if φ is an extremal of the energy functionals $E(\cdot, D)$, for all compact domains D in M, φ is called a *harmonic map* between pseudo-Riemannian manifolds;

(4) if the *second fundamental form* $\nabla^{\varphi^{-1}TN}\,\mathrm{d}\varphi$ of φ is defined by

$$\begin{aligned}\nabla^{\varphi^{-1}TN}\,\mathrm{d}\varphi(X,Y) &= (\nabla_X^{\varphi^{-1}TN}\,\mathrm{d}\varphi)(Y)\\ &= \nabla^N_{\mathrm{d}\varphi(X)}\,\mathrm{d}\varphi(Y) - \mathrm{d}\varphi(\nabla^M_X Y), \quad \forall X, Y \in \Gamma(TM),\end{aligned}$$

where ∇^M and ∇^N denote the Levi-Civita connections on (M,g) and (N,h), the tension field $\tau(\varphi)$ is given by

$$\tau(\varphi) = \mathrm{div}(\,\mathrm{d}\varphi) = \mathrm{tr}_g(\nabla^{\varphi^{-1}TN}\,\mathrm{d}\varphi).$$

According to Eells and Sampson, the harmonicity condition can be described as follows:

Theorem 2.1. *Any smooth map $\varphi\colon (M,g) \to (N,h)$ between pseudo-Riemannian manifolds is harmonic if and only if*

$$\tau(\varphi) = 0, \tag{2.1}$$

that is the following Euler-Lagrange system is satisfied:

$$g^{ij}\Big(\frac{\partial^2\varphi^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\varphi^\alpha}{\partial x^k} + {}^N\Gamma^\alpha_{\alpha\beta}\frac{\partial\varphi^\beta}{\partial x^i}\frac{\partial\varphi^\sigma}{\partial x^j}\Big) = 0, \tag{2.2}$$

summing over $i, j, k = 1, \ldots, m$, $\beta, \sigma = 1, \ldots, n$, for every $\alpha = 1, \ldots, n$.

Theorem 2.2 ([11])**.** *Let (M,g) be a pseudo-Riemannian manifold. A pseudo-Riemannian structure (i.e., tensor field) h is called harmonic with respect to g, if and only if the identity map $I\colon (M,g) \to (M,h)$ is harmonic.*

According to the above definition, we can study a pseudo-Riemannian metric h which is harmonic with respect to the metric g on the pseudo-Riemannian manifold (M,g).

Theorem 2.3. *Let $(M, g_{a,b,c})$ be a four-dimensional Walker manifold of signature $(2,2)$. There exist adapted local coordinates (t, z, y, z) such that $g_{a,b,c}$ is described as*

$$g_{a,b,c} = 2(\mathrm{d}t\,\mathrm{d}y + \mathrm{d}x\,\mathrm{d}z) + a\,\mathrm{d}y^2 + b\,\mathrm{d}z^2 + 2c\,\mathrm{d}y\,\mathrm{d}z,$$

where a, b, c are smooth functions of the variables (t, x, y, z) (for more information see [6]*).*

Following the result of [4], a pseudo-Riemannian Walker metric $\hat{g}_{\hat{a},\hat{b},\hat{c}}$ is harmonic with respect to $g_{a,b,c}$ if and only if the smooth functions a, b, c and $\hat{a}$, $\hat{b}$, $\hat{c}$ satisfy the following relation:

$$\begin{cases} \hat{a} = a + \displaystyle\int \lambda_y\,\mathrm{d}x + \alpha(y,z,t), \\ \hat{b} = b + \displaystyle\int \lambda_x\,\mathrm{d}y + \beta(y,z,t), \\ \hat{c} = c - \lambda(y,z,t) \end{cases} \tag{2.3}$$

for some local smooth functions $\alpha(y,z,t)$, $\beta(x,z,t)$ and $\lambda(x,y,z,t)$.

The Euler-Lagrange system of the identity map $I\colon (M,g) \to (M,\hat{g})$ is expressed by

$$\operatorname{tr}(G^{-1}(\hat{\Gamma}^k - \Gamma^k)) = 0, \quad k = 1,\dots,4, \tag{2.4}$$

where G is the matrix of g, and Γ^k and $\hat{\Gamma}^k$ are the matrices of the Christoffel symbols of g and $\hat{g}$ respectively. The relation (2.4) is a key point for the study of harmonic metrics on various spaces. We study this problem on the non-reductive homogeneous space of dimension four through the upcoming sections.

3. Four-dimensional non-reductive homogeneous spaces

Let $M = G/H$ (with H connected) be a homogeneous space. We denote the Lie algebra of G by $\mathfrak{g}$ and the isotropy subalgebra by $\mathfrak{h}$, the factor space which identifies with a subspace of $\mathfrak{g}$ complementary to $\mathfrak{h}$ is denoted by $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$. The pair $(\mathfrak{g},\mathfrak{h})$ uniquely defines the isotropy representation

$$\psi\colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{m}), \quad \psi(x)(y) = [x,y]_{\mathfrak{m}} \quad \text{for all } x \in \mathfrak{g},\ y \in \mathfrak{m}.$$

Given a basis $\{h_1,\dots,h_r,u_1,\dots,u_n\}$ of $\mathfrak{g}$ where $\{h_j\}$ and $\{u_i\}$ are bases of $\mathfrak{h}$ and $\mathfrak{m}$, respectively, a bilinear form on $\mathfrak{m}$ is determined by the matrix g of its components with respect to the basis $\{u_i\}$ and is invariant if and only if ${}^t\psi(x)\circ g + g\circ\psi(x) = 0$ for all $x \in \mathfrak{g}$. Invariant pseudo-Riemannian metrics g on the homogeneous space $M = G/H$ are in a one-to-one correspondence with nondegenerate invariant symmetric bilinear forms g on $\mathfrak{m}$, see [17].

The invariant metric g then uniquely defines its invariant linear Levi-Civita connection, described in terms of the corresponding homomorphism of $\mathfrak{h}$-modules $\Lambda\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$, such that $\Lambda(x)(y_{\mathfrak{m}}) = [x, y]_{\mathfrak{m}}$ for all $x \in \mathfrak{h}$, $y \in \mathfrak{g}$. Explicitly, one has

$$\Lambda(x)(y_{\mathfrak{m}}) = \tfrac{1}{2}[x, y]_{\mathfrak{m}} + v(x, y) \quad \text{for all } x, y \in \mathfrak{g},$$

where $v\colon\ \mathfrak{g} \times \mathfrak{g} \to \mathfrak{m}$ is the $\mathfrak{h}$-invariant symmetric mapping uniquely determined by

$$2g(v(x, y), z_{\mathfrak{m}}) = g(x_{\mathfrak{m}}, [z, y]_{\mathfrak{m}}) + g(y_{\mathfrak{m}}, [z, x]_{\mathfrak{m}}) \quad \text{for all } x, y, z \in \mathfrak{g}. \tag{3.1}$$

Non-reductive homogeneous manifolds of dimension four were classified in [15], in terms of the corresponding non-reductive Lie algebra presentations. This classification contains 8 classes $A1, \dots, A5$, $B1, \dots, B3$. The classes $A1, \dots, A3$ contain both Lorentzian and neutral examples, classes $A4, A5$ are just Lorentzian and $B1, \dots, B3$ are always of neutral signature. Also the description in coordinates system for the invariant metrics on the spaces mentioned were obtained in [8].

We now recall these classifications here.

(A1) $\mathfrak{g} = \mathfrak{a}_1$ is a decomposable 5-dimensional Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{s}(2)$, where $\mathfrak{s}(2)$ is the 2-dimensional solvable algebra. There exists a basis $\{e_1, \dots, e_5\}$ of $\mathfrak{a}_1$ such that the nonzero products are

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_4, e_5] = e_4,$$

and the isotropy subalgebra is $\mathfrak{h} = \mathrm{Span}\{h_1 = e_3 + e_4\}$. So, we can take

$$\mathfrak{m} = \mathrm{Span}\{u_1 = e_1, u_2 = e_2, u_3 = e_5, u_4 = e_3 - e_4\}$$

and the invariant metrics in the local coordinates $(x_1, \dots, x_4)$ are

$$\begin{aligned} g = {} & (4bx_2^2 + a)\,\mathrm{d}x_1^2 + 4bx_2\,\mathrm{d}x_1\,\mathrm{d}x_2 - (4ax_2x_4 - 4cx_2 + a)\,\mathrm{d}x_1\,\mathrm{d}x_3 \\ & + 4ax_2\,\mathrm{d}x_1\,\mathrm{d}x_4 + b\,\mathrm{d}x_2^2 - 2(ax_4 - c)\,\mathrm{d}x_2\,\mathrm{d}x_3 + 2a\,\mathrm{d}x_2\,\mathrm{d}x_4 + d\,\mathrm{d}x_3^2 \end{aligned} \tag{3.2}$$

on the whole of $\mathbb{R}^4$, whenever $a(a - 4d) \neq 0$.

The metric g has Lorentzian signature (either (3,1) or (1,3)) if and only if $a(a - 4d) < 0$, and neutral signature when $a(a - 4d) > 0$.

(A2) $\mathfrak{g} = \mathfrak{a}_2$ is the decomposable 5-dimensional Lie algebra $A_{5,30}$ of [18]. There exists a basis $\{e_1, \dots, e_5\}$ of $\mathfrak{a}_2$ such that the nonzero products are

$$\begin{array}{lll} [e_1, e_5] = (\alpha+1)e_2, & [e_2, e_4] = e_1, & [e_2, e_5] = \alpha e_2, \\ [e_3, e_4] = e_2, & [e_3, e_5] = (\alpha-1)e_3, & [e_4, e_5] = e_2 \end{array}$$

for any value of $\alpha \in \mathbb{R}$, and the isotropy is $\mathfrak{h} = \operatorname{Span}\{h_1 = e_4\}$. Hence, we take

$$\mathfrak{m} = \operatorname{Span}\{u_1 = e_1, u_2 = e_2, u_3 = e_3, u_4 = e_5\}$$

and the invariant metrics in the local coordinates $(x_1, \dots, x_4)$ are

$$g = -2ae^{2\alpha x_4}\,\mathrm{d}x_1\,\mathrm{d}x_3 + ae^{2\alpha x_4}\,\mathrm{d}x_2^2 + be^{2(\alpha-1)x_4}\,\mathrm{d}x_3^2 + 2ce^{(\alpha-1)x_4}\,\mathrm{d}x_3\,\mathrm{d}x_4 + d\,\mathrm{d}x_4^2 \tag{3.3}$$

on the whole of $\mathbb{R}^4$, whenever $ad \neq 0$.

The metric g has Lorentzian signature when $ad > 0$ and neutral signature if and only if $ad < 0$.

(A3) $\mathfrak{g} = \mathfrak{a}_3$ is the decomposable 5-dimensional Lie algebra $A_{5,37}$ or $A_{5,36}$ of [18]. There exists a basis $\{e_1, \dots, e_5\}$ of $\mathfrak{a}_3$ such that the nonzero products are

$$\begin{array}{lll} [e_1, e_3] = 2e_1, & [e_2, e_3] = e_1, & [e_2, e_4] = e_2, \\ [e_2, e_5] = -\varepsilon e_3, & [e_3, e_4] = e_3, & [e_3, e_5] = e_2, \end{array}$$

with $\varepsilon = 1$ for $A_{5,37}$ and $\varepsilon = -1$ for $A_{5,36}$, and the isotropy is $\mathfrak{h} = \operatorname{Span}\{h_1 = e_3\}$. Thus, we take

$$\mathfrak{m} = \operatorname{Span}\{u_1 = e_1, u_2 = e_2, u_3 = e_4, u_4 = e_5\}$$

and the invariant metrics in the local coordinates $(x_1, \dots, x_4)$ are

$$\text{if } \varepsilon = 1:\; g = 2ae^{2x_3}\,\mathrm{d}x_1\,\mathrm{d}x_4 + ae^{2x_3}\cos(x_4)^2\,\mathrm{d}x_2^2 + b\,\mathrm{d}x_3^2 + 2c\,\mathrm{d}x_3 dx_4 + d\,\mathrm{d}x_4^2, \tag{3.4}$$

on the open subset where $\cos(x_4) \neq 0$, whenever $ab \neq 0$;

$$\text{if } \varepsilon = -1:\; g = 2ae^{2x_3}\,\mathrm{d}x_1\,\mathrm{d}x_4 + ae^{2x_3}\cosh(x_4)2\,\mathrm{d}x_2^2 + b\,\mathrm{d}x_3^2 + 2c\,\mathrm{d}x_3\,\mathrm{d}x_4 + d\,\mathrm{d}x_4^2, \tag{3.5}$$

on the whole of $\mathbb{R}^4$, whenever $ab \neq 0$.

The metric g has Lorentzian signature if and only if $ab > 0$ and neutral signature when $ab < 0$.

(A4) $\mathfrak{g} = \mathfrak{a}_4$ is the decomposable 6-dimensional Schroedinger Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{n}(3)$, where $\mathfrak{n}(3)$ is the 3-dimensional Heisenberg algebra. There exists a basis $\{e_1,\dots,e_6\}$ of $\mathfrak{a}_4$, where the nonzero products are

$$\begin{array}{llll} [e_1,e_2]=2e_2, & [e_1,e_3]=-2e_3, & [e_2,e_3]=e_1, & [e_1,e_4]=e_4, \\ [e_1,e_5]=-e_5, & [e_2,e_5]=e_4, & [e_3,e_4]=e_5, & [e_4,e_5]=e_6, \end{array}$$

and the isotropy is $\mathfrak{h} = \mathrm{Span}\{h_1 = e_3\}$. Thus, we take

$$\mathfrak{m} = \mathrm{Span}\{u_1 = e_1, u_2 = e_2, u_3 = e_3 - e_6, u_4 = e_4\}$$

and the invariant metrics in the local coordinates $(x_1,\dots,x_4)$ are

$$\begin{aligned} g = {} & \Big(\frac{a}{2}x_4^2 + 4bx_2^2 + a\Big)\,\mathrm{d}x_1^2 + 4bx^2\,\mathrm{d}x_1\,\mathrm{d}x_2 + ax_2(4+x_4^2)\,\mathrm{d}x_1\,\mathrm{d}x_3 \\ & + a(1+2x_2x_3)x_4\,\mathrm{d}x_1\,\mathrm{d}x_4 + b\,\mathrm{d}x_2^2 + \frac{a}{2}(4+x_4^2)\,\mathrm{d}x_2\,\mathrm{d}x_3 \\ & + ax_3x_4\,\mathrm{d}x_2\,\mathrm{d}x_4 + \frac{a}{2}\,\mathrm{d}x_4^2 \end{aligned} \tag{3.6}$$

on the whole of $\mathbb{R}^4$, whenever $a \neq 0$.

(A5) $\mathfrak{g} = \mathfrak{a}_5$ is the decomposable 7-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \ltimes A^1_{4,9}$, with $A^1_{4,9}$ as in [18]. It admits a basis $\{e_1,\dots,e_7\}$ such that the nonzero products are

$$\begin{array}{llll} [e_1,e_2]=2e_2, & [e_1,e_3]=-2e_3, & [e_1,e_5]=-e_5, & [e_1,e_6]=e_6, \\ [e_2,e_3]=e_1, & [e_2,e_5]=e_6, & [e_3,e_6]=e_5, & [e_4,e_7]=2e_4, \\ [e_5,e_6]=e_4, & [e_5,e_7]=e_5, & [e_6,e_7]=e_6. & \end{array}$$

The isotropy is $\mathfrak{h} = \mathrm{Span}\{h_1 = e_1 + e_7, h_2 = e_3 - e_4, h_3 = e_5\}$. So, we take

$$\mathfrak{m} = \mathrm{Span}\{u_1 = e_1 - e_7, u_2 = e_2, u_3 = e_3 + e_4, u_4 = e_6\},$$

and the invariant metrics in the local coordinates $(x_1,\dots,x_4)$ are

$$\begin{aligned} g = {} & -\frac{ax_4}{4x_2}\,\mathrm{d}x_1\,\mathrm{d}x_2 + \frac{a}{4}\,\mathrm{d}x_1\,\mathrm{d}x_4 + \frac{a(2+2x_4x_1+x_3^2)}{8x_2^2}\,\mathrm{d}x_2\,\mathrm{d}x_2 \\ & -\frac{ax_3}{4x_2}\,\mathrm{d}x_2\,\mathrm{d}x_3 - \frac{ax_1}{4x_2}\,\mathrm{d}x_2\,\mathrm{d}x_4 + \frac{a}{8}\,\mathrm{d}x_3^2, \end{aligned} \tag{3.7}$$

on the open subset where $x_2 \neq 0$.

(B1) $\mathfrak{g} = \mathfrak{b}_1$ is the decomposable 7-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ admitting a basis $\{e_1, \dots, e_5\}$, where the nonzero products are

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_4] = e_4,$$
$$[e_1, e_5] = -e_5, \quad [e_2, e_5] = e_4, \quad [e_3, e_4] = e_5,$$

and the isotropy is $\mathfrak{h} = \mathrm{Span}\{h_1 = e_3\}$. Thus, we take

$$\mathfrak{m} = \mathrm{Span}\{u_1 = e_1, u_2 = e_2, u_3 = e_4, u_4 = e_5\}$$

and the invariant metrics in the local coordinates $(x_1, \dots, x_4)$ are

$$\begin{aligned} g = {} & (d(x_3^2 + 4x_3x_2x_4 + 4x_2^2x_4^2) + 4cx_2x_3 + 8cx_2^2x_4 + 2ax_3 + 4bx_2^2)\,\mathrm{d}x_1^2 \\ & + 2(d(x_3x_4 + 2x_2x_4^2) + 4cx_2x_4 + cx_3 + 2bx_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ & + 2(d(x_3 + 2x_2x_4) + 2cx_2 + a)\,\mathrm{d}x_1\,\mathrm{d}x_3 + 4ax_2\,\mathrm{d}x_1\,\mathrm{d}x_4 \\ & + (dx_4^2 + 2cx_4 + b)\,\mathrm{d}x_2^2 + 2(dx_4 + c)\,\mathrm{d}x_2\,\mathrm{d}x_3 + 2a\,\mathrm{d}x_2\,\mathrm{d}x_4 + d\,\mathrm{d}x_3^2 \end{aligned} \tag{3.8}$$

on the whole of $\mathbb{R}^4$, whenever $\neq 0$.

(B2) $\mathfrak{g} = \mathfrak{b}_2$ is the 6-dimensional Schroedinger Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{n}(3)$, but with isotropy $\mathfrak{h} = \mathrm{Span}\{h_1 = e_3 - e_6, h_2 = e_5\}$. Then we take

$$\mathfrak{m} = \mathrm{Span}\{u_1 = e_1, u_2 = e_2, u_3 = e_3 + e_6, u_4 = e_4\},$$

and the invariant metrics in the local coordinates $(x_1, \dots, x_4)$ are

$$\begin{aligned} g = {} & \left(a - \frac{ax_2^4}{2} + 4bx_2^2\right)\mathrm{d}x_1^2 + 4bx_2\,\mathrm{d}x_1\,\mathrm{d}x_2 - ax_2(x_4^2 - 4)\,\mathrm{d}x_1\,\mathrm{d}x_3 \\ & - a(1 + 2x_2x_3)x_4\,\mathrm{d}x_1\,\mathrm{d}x_4 + b\,\mathrm{d}x_2^2 - \frac{1}{2}a(x_4^2 - 4)\,\mathrm{d}x_2\,\mathrm{d}x_3 \\ & - ax_3x_4\,\mathrm{d}x_2\,\mathrm{d}x_4 - \frac{1}{2}a\,\mathrm{d}x_4^2 \end{aligned} \tag{3.9}$$

on the open subset where $x_4 \neq 2$, whenever $a \neq 0$.

(B3) $\mathfrak{g} = \mathfrak{b}_3$ is the 6-dimensional Lie algebra $(\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2) \times \mathbb{R}$. It admits a basis $\{u_1, \dots, u_4, h_1 = u_5, h_2 = u_6\}$ such that $\mathfrak{h} = \mathrm{Span}\{h_1, h_2\}, \mathfrak{m} = \mathrm{Span}\{u_1, \dots, u_4\}$ and the nonzero products are

$$\begin{aligned} &[h_1, u_2] = u_1, && [h_1, u_3] = -u_4, && [h_2, u_2] = -2h_2, \\ &[h_2, u_3] = -u_2, && [h_2, u_4] = u_1, && [u_1, u_2] = -u_1, \\ &[u_1, u_3] = u_4, && [u_2, u_3] = -2u_3, && [u_2, u_4] = -u_4. \end{aligned}$$

and the invariant metrics in the local coordinates $(x_1, \ldots, x_4)$ are

$$
(3.10) \qquad \begin{aligned} g = &-2a\mathrm{e}^{-x_2} x_3 \,\mathrm{d}x_1 \,\mathrm{d}x_2 + 2a\mathrm{e}^{-x_2} \,\mathrm{d}x_1 \,\mathrm{d}x_3 + 2(2bx_3^2 - ax_4) \,\mathrm{d}x_2^2 \\ &- 4bx_3 \,\mathrm{d}x_2 \,\mathrm{d}x_3 + 2a \,\mathrm{d}x_2 \,\mathrm{d}x_4 + b \,\mathrm{d}x_3^2 \end{aligned}
$$

on the whole of $\mathbb{R}^4$, whenever $a \neq 0$.

4. Non-reductive homogeneous spaces lifted to the tangent bundle

In this section we study the tangent bundle of a non-reductive homogeneous space. Let $(M = G/H, g)$ be a non-reductive homogeneous manifold of dimension four, we denote the Levi-Civita connection of the invariant metric g by ∇. The tangent bundle TM has the direct sum decomposition

$$
(4.1) \qquad TTM = VTM \oplus HTM
$$

into the vertical distribution $VTM = \ker d\pi$, where π is the natural projection of TM onto M, and HTM is the horizontal distribution defined by ∇. For more details concerning the geometry of the tangent bundle of a manifold see [16], [14], [19].

In the sequel, we get the local coordinates $p = (x^1, \ldots, x^4)$ on a non-reductive homogeneous manifolds of dimension four $(M = G/H, g)$ and the local coordinates on TM are denoted by $(p, u) = (x^1, \ldots, x^4, y^1, \ldots, y^4)$.

The adapted local frame field on TM is $\{\delta_{x^i}, \partial_{y^j}\}_{i,j=1,\ldots,4}$, where

$$
(4.2) \qquad \delta_{x^i} = \partial_{x^i} - \Gamma_{0i}^k \partial_{y^k} \quad \text{with } \Gamma_{0i}^k = y^h \Gamma_{hi}^k,
$$

and $\Gamma_{hi}^k(p)$ are the Christoffel symbols of the Levi-Civita connection ∇.

For any tensor field on the tangent bundle, we denote by i, j, k, h the indices corresponding to the horizontal components and by $\bar{\imath}$, $\bar{\jmath}$, $\bar{k}$, $\bar{h}$ those corresponding to the vertical ones. For an arbitrary vector field $X = X^i \partial_{x^i}$ on M, by X^V and X^H we denote the vertical and horizontal lifts of X respectively by

$$
(4.3) \qquad X^V = X^{\bar{\imath}} \partial_{y^{\bar{\imath}}}, \quad X^H = X^i \delta_{x^i},
$$

with respect to the adapted local frame $\{\delta_{x^i}, \partial_{y^{\bar{\jmath}}}, \}_{\bar{\imath},j=1,\ldots,4}$. Here we recall that the Sasaki metric g^S and the horizontal lift g^H on TM for all vector fields $X, Y \in \Gamma(TM)$ are defined by (we refer to [2] for more details)

$$
(4.4) \qquad \begin{cases} g^S_{(p,u)}(X^H, Y^H) = g_p(X, Y), \\ g^S_{(p,u)}(X^V, Y^H) = 0, \qquad \begin{cases} g^H_{(p,u)}(X^H, Y^H) = g^H_{(p,u)}(X^V, Y^V) = 0, \\ g^H_{(p,u)}(X^V, Y^H) = g_p(X, Y), \end{cases} \\ g^S_{(p,u)}(X^V, Y^V) = g_p(X, Y), \end{cases}
$$

respectively, and the matrices with respect to the adapted frames respectively are

$$ {}^{S}G = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix}, \quad {}^{H}G = \begin{pmatrix} 0 & G \\ G & 0 \end{pmatrix}. \tag{4.5} $$

If we denote the Levi-Civita connection of g^S by ${}^S\nabla$, then the following relations are valid (see [19] and references therein):

$$ \begin{cases} ({}^S\nabla_{X^H} Y^H)_{(p,u)} = (\nabla_X Y)^H_{(p,u)} - \frac{1}{2}(R_p(X,Y)u)^V, \\ ({}^S\nabla_{X^H} Y^V)_{(p,u)} = (\nabla_X Y)^V_{(p,u)} + \frac{1}{2}(R_p(u,Y)X)^H, \\ ({}^S\nabla_{X^V} Y^H)_{(p,u)} = \frac{1}{2}(R_p(u,X)Y)^H, \quad ({}^S\nabla_{X^V} Y^V)_{(p,u)} = 0, \end{cases} $$

for all vector fields $X, Y \in \Gamma(TM)$. Clearly the non-vanishing components of ${}^S\nabla$ are

$$ \begin{aligned} &{}^S\Gamma^k_{ij} =^S \Gamma^{\bar{k}}_{i\bar{j}} = \Gamma^k_{ij}, \quad {}^S\Gamma^k_{i\bar{j}} = \frac{1}{2} R^k_{hij} y^h, \\ &{}^S\Gamma^k_{\bar{i}j} = \frac{1}{2} R^k_{hij} y^h, \quad {}^S\Gamma^{\bar{k}}_{ij} = -\frac{1}{2} R^k_{ijh} y^h, \quad i,j,k = 1,\dots,4, \end{aligned} \tag{4.6} $$

where R is the curvature tensor of (M,g) and $R(\partial_{x^i}, \partial_{x^j})\partial_{x^h} = R^k_{ijh}\partial_{x^k}$. Similarly, if we denote the Levi-Civita connection of g^H by ${}^H\nabla$ then we have

$$ \begin{cases} ({}^H\nabla_{X^H} Y^H)_{(p,u)} = ({}^H\nabla_{X^H} Y^V)_{(p,u)} = (\nabla_X Y)^H_{(p,u)}, \\ ({}^H\nabla_{X^V} Y^H)_{(p,u)} = ({}^H\nabla_{X^V} Y^V)_{(p,u)} = 0, \end{cases} $$

for all vector filed $X, Y \in \Gamma(TM)$. By the above relations, the non-vanishing components of the Levi-Civita connection ${}^H\nabla$ are

$$ {}^H\Gamma^k_{ij} =^H \Gamma^k_{i\bar{j}} = \Gamma^k_{ij}, \quad i,j,k = 1,\dots,4. \tag{4.7} $$

The Euler-Lagrange system of the identity map for a Sasaki metric g^S and for the horizontal lift metric g^H is translated by replacing the corresponding components for g^S and g^H by their Levi-Civita connections in the equation (2.4).

The evaluation map is an important object on TM. For each one-form ω on M, we can introduce a function $\iota\omega\colon TM \to \mathbb{R}$ which is called the *evaluation map* and defined by $\iota\omega(p,u) = \omega_p(u)$. Specially, for each function $f \in \mathcal{F}(M)$, $f^C = \iota(df)$ is called the *complete lift* of the function f. The characterization of the vector fields on TM will be done by these complete lift functions. In fact, $\overline{X}(f^C) = \overline{Y}(f^C)$ for all $f \in \mathcal{F}(M)$ if and only if $\overline{X} = \overline{Y}$, where $\overline{X}$ and $\overline{Y}$ are two arbitrary vector fields on TM. For each vector field X on M, its complete lift X^C is the vector field on TM defined by $X^C(f^C) = (Xf)^C$. Remarkably, we can characterize tensor fields on TM by their action on complete lifts of vector fields on M. Hence, if (M, g) is a pseudo-Riemannian manifold, we can naturally equip the tangent bundle TM with the complete lift metric, which is defined by:

$$g^C(X^C, Y^C) = g(X, Y)^C. \tag{4.8}$$

A coordinate description of the complete lift metric is

$$^C G = \begin{pmatrix} y^k \partial g_{ij}/\partial x^k & G \\ G & 0 \end{pmatrix}, \tag{4.9}$$

The complete lift metric is always of neutral signature (m, m). We refer to [19] for more information on the geometry of the tangent bundle and the complete lift metric.

Remark 4.1. The complete lift g^C is a pseudo-Riemannian metric of neutral signature (m, m), $m = \dim M$, whose properties have a nice correspondence with those of (M, g). For instance, (M, g) is locally symmetric if and only if (TM, g^C) is so, (M, g) is a real space form if and only if (TM, g^C) is locally conformally flat, (M, g, J) is a complex space form if and only if (TM, g^C, J^C) is a Bochner-flat Kähler manifold.

Now we have all arguments for the tangent bundle of a non-reductive homogeneous four-manifold, endowed with the above lifted metrics.

5. Invariant harmonic metrics

Following the notation of Section 3, any invariant metric on a non-reductive homogeneous four manifold is described by some arbitrary coefficients $a, b, \ldots$. In this section, we take an arbitrary invariant metric $\hat{g}$, which is defined by arbitrary coefficients $\hat{a}, \hat{b}, \ldots$, and consider harmonicity of $\hat{g}$ with respect to g.

5.1. Harmonicity of $\hat{g}$ with respect to g.

Theorem 5.1. *Let $(M = G/H, g)$ be a non-reductive homogeneous manifold of dimension four, equipped with an invariant metric g. In this case, the invariant metric $\hat{g}$ is harmonic with respect to g if and only if one of the following cases occurs*

(1) *$(G/H, g)$ is of type A1 and $c\hat{a} = a\hat{c}$.*
(2) *$(G/H, g)$ is of type A2 and one of the following cases occurs*
   (i) $\alpha = 0$, $c\hat{a} = a\hat{c}$,
   (ii) $\alpha = 1/4$, $a\hat{d} = d\hat{a}$,
   (iii) $\alpha = \text{arbitary}$, $c\hat{a} = a\hat{c}$, $d\hat{a} = a\hat{d}$.

(3) $(G/H,g)$ *is of type* $A3$, $\varepsilon=\pm1$ *and* $b\hat{a}=a\hat{b}$, $b\hat{c}=c\hat{b}$.
(4) $(G/H,g)$ *is of type* $A4, A5, B2$ *or* $B3$.
(5) $(G/H,g)$ *is of type* $B1$ *and* $c\hat{a}=a\hat{c}$, $d\hat{a}=a\hat{d}$.

Proof. Let $(G/H,g)$ be a non-reductive homogeneous manifold of dimension four and $\hat{g}$ an arbitrary invariant metric which is harmonic with respect to the invariant metric g. We give details of computations for case (A1), the other cases could be treated in a similar way.

(A1) We refer to the invariant metric described in the equation (3.2), where a, b, c, d are arbitrary real constants and $a(a-4d)\neq0$. Putting $\Lambda[k]:=(\Gamma_{ij}^k)_{i,j=1,\dots,4}$ for all indices $k=1,\dots,4$, we find

$$\Lambda[1]=\begin{pmatrix} -\frac{32bdx_2^2}{a(a-4d)} & -\frac{16bdx_2}{a(a-4d)} & -\frac{2x_2(ax_4-c)}{a} & 2x_2\\ -\frac{16bdx_2}{a(a-4d)} & -\frac{8bd}{a(a-4d)} & -\frac{ax_4-c}{a} & 1\\ -\frac{2x_2(ax_4-c)}{a} & -\frac{ax_4-c}{a} & 0 & 0\\ 2x_2 & 1 & 0 & 0\end{pmatrix},$$

$$\Lambda[2]=\begin{pmatrix} \frac{64bdx_2^3}{a(a-4d)} & \frac{a^2-4ad+32bdx_2^2}{a(a-4d)} & \frac{(4ax_2x_4-4cx_2+a)x_2}{a} & -4x_2^2\\ \frac{a^2-4ad+32bdx_2^2}{a(a-4d)} & \frac{16bdx_2}{a(a-4d)}\,\frac{4ax_2x_4-4cx_2+a}{2a} & -2x_2 & \\ \frac{(4ax_2x_4-4cx_2+a)x_2}{a} & \frac{4ax_2x_4-4cx_2+a}{2a} & 0 & 0\\ -4x_2^2 & -2x_2 & 0 & 0\end{pmatrix},$$

$$\Lambda[3]=\begin{pmatrix} -\frac{16bx_2^2}{a-4d} & -\frac{8bx_2}{a-4d} & 0 & 0\\ -\frac{8bx_2}{a-4d} & -\frac{4b}{a-4d} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$\Lambda[4]=\begin{pmatrix} \frac{-4bx_2(a-4d+4ax_2x_4-4cx_2)}{a(a-4d)} & -\frac{b(a-4d+8ax_2x_4-8cx_2)}{a(a-4d)} & \frac{ax_4-c-bx_2}{a} & -1\\ -\frac{-4b(a-4d+8ax_2x_4-8cx_2)}{a(a-4d)} & -\frac{4b(ax_4-c)}{a(a-4d)} & -\frac{b}{2a} & 0\\ \frac{ax_4-c-bx_2}{a} & -\frac{b}{2a} & 0 & -\frac12\\ -1 & 0 & -\frac12 & 0\end{pmatrix}.$$

Now, the invariant metric $\hat{g}$ is

$$\begin{aligned}\hat{g}&=(4\hat{b}x_2^2+\hat{a})\,\mathrm{d}x_1^2+4\hat{b}x_2\,\mathrm{d}x_1\,\mathrm{d}x_2-(4\hat{a}x_2x_4-4\hat{c}x_2+\hat{a})\,\mathrm{d}x_1\,\mathrm{d}x_3\\&\quad+4\hat{a}x_2\,\mathrm{d}x_1\,\mathrm{d}x_4+\hat{b}\,\mathrm{d}x_2^2-2(\hat{a}x_4-\hat{c})\,\mathrm{d}x_2\,\mathrm{d}x_3+2\hat{a}\,\mathrm{d}x_2\,\mathrm{d}x_4+\hat{d}\,\mathrm{d}x_3^2,\end{aligned}$$

where $\hat{a},\dots,\hat{d}$ are arbitrary real constants and $\hat{a}(\hat{a}-4\hat{d})\neq0$. Clearly the Levi-

Civita components of $\hat{g}$ are $\hat{\Lambda}[1], \dots, \hat{\Lambda}[4]$, which are deduced by inserting respectively $\hat{a}, \dots, \hat{d}$ instead of $a, \dots, d$ in $\Lambda[1], \dots, \Lambda[4]$.

Now, we apply the equation (2.4) to the identity map $I\colon (G/H, g) \to (G/H, \hat{g})$. Straightforward calculations yield that

$$\frac{4(a\hat{c} - c\hat{a})}{\hat{a}a(a - 4d)} = 0,$$

which immediately gives the first statement. Similar arguments used in the remaining cases will complete the proof. □

5.2. Harmonicity of $\hat{g}^S$ with respect to g^S and $\hat{g}^H$ with respect to g^H.

Theorem 5.2. *Let g be an invariant metric on a four-dimensional, non-reductive homogeneous pseudo-Riemannian manifold $M = G/H$. Then the Sasaki metric $\hat{g}^S$ is harmonic with respect to g^S (and the horizontal lift $\hat{g}^H$ is harmonic with respect to g^H) if and only if one of the following cases occurs*

(1) *$(G/H, g)$ is of type $A1$ and $c\hat{a} = a\hat{c}$.*

(2) *$(G/H, g)$ is of type $A2$ and one of the following cases occurs*

  (i) $\alpha = 0$, $c\hat{a} = a\hat{c}$,

  (ii) $\alpha = 1/4$, $a\hat{d} = d\hat{a}$,

  (iii) $\alpha = \text{arbitary}$, $c\hat{a} = a\hat{c}$, $d\hat{a} = a\hat{d}$.

(3) *$(G/H, g)$ is of type $A3$ for $\varepsilon = \pm 1$ and $b\hat{a} = a\hat{b}$, $b\hat{c} = c\hat{b}$.*

(4) *$(G/H, g)$ is of type $A4, A5, B2$ or $B3$.*

(5) *$(G/H, g)$ is of type $B1$ and $c\hat{a} = a\hat{c}$, $d\hat{a} = a\hat{d}$.*

Proof. Let $(G/H, g)$ be a non-reductive homogeneous manifold of dimension four and $\hat{g}$ an arbitrary invariant metric which is harmonic with respect to the invariant metric g.

(**A1**) Putting $\Lambda[i] := \Lambda(\delta_{x^i})$ and $\Lambda[\bar{j}] := \Lambda(\partial_{y^{\bar{j}}})$ for all indices $i, \bar{j} = 1, \dots, 4$, we find the Levi-Civita components of g^S and $\hat{g}^S$ (g^H and $\hat{g}^H$) using the relations (4.6) (respectively, (4.7)).

A straightforward calculation after substituting the equations (4.5) and (4.6) in the equation (2.4) shows that the Sasaki metric $\hat{g}^S$ is harmonic with respect to g^S (the horizontal lift metric $\hat{g}^H$ is harmonic with respect to g^H) if and only if the following equations are satisfied:

$$\begin{cases} \operatorname{tr}\big(g^{ij}(\hat{\Gamma}^k_{ij} - \Gamma^k_{ij})\big) = 0, \\ \operatorname{tr}\big(g^{ij}(\hat{R}^k_{ijh} - R^k_{ijh})y^h\big) = 0, \quad k = 1, \dots, 4, \end{cases}$$

(respectively, $\operatorname{tr}(g^{ij}(\hat{\Gamma}^k_{ij} - \Gamma^k_{ij})) = 0,\ k = 1, \ldots, 4)$. Now, we get the following equations for $\hat{g}^S$ (or $\hat{g}^H$) being harmonic with respect to g^S (respectively, g^H):

$$\begin{cases} x_2(a\hat{c} - \hat{a}c) = 0, \\ (\hat{a} - 4\hat{d} - 2x_2(\hat{c} + \hat{a}x_4))(a\hat{c} - \hat{a}c) = 0, \\ (a\hat{c} - \hat{a}c)\hat{d}x_2 = 0, \\ (a\hat{c} - \hat{a}c)\hat{d}x_2^2 = 0, \end{cases}$$

so $c\hat{a} = a\hat{c}$ which shows the validity of the first statement. By the same method used for each of the classes $A2, \ldots, A5$, $B1, \ldots, B3$, the remaining assertions are deduced and this ends the proof. $\square$

5.3. Harmonicity of $\hat{g}^C$ with respect to g^C.

Theorem 5.3. *Let g be an invariant metric on a four-dimensional, non-reductive homogeneous pseudo-Riemannian manifold $M = G/H$. Then the complete metric $\hat{g}^C$ is harmonic with respect to g^C if and only if one of the following cases occurs*

(1) *$(G/H, g)$ is of type $A1$ and $c\hat{a} = a\hat{c}$.*
(2) *$(G/H, g)$ is of type $A2$ and one of the following cases occurs*
  (i) $\alpha = 0,\ c\hat{a} = a\hat{c}$,
  (ii) $\alpha = 1/4,\ a\hat{d} = d\hat{a}$,
  (iii) $\alpha = \text{arbitary},\ c\hat{a} = a\hat{c},\ d\hat{a} = a\hat{d}$.
(3) *$(G/H, g)$ is of type $A3$ for $\varepsilon = \pm 1$ and $b\hat{a} = a\hat{b},\ b\hat{c} = c\hat{b}$.*
(4) *$(G/H, g)$ is of type $A4, A5, B2$ or $B3$.*
(5) *$(G/H, g)$ is of type $B1$ and $c\hat{a} = a\hat{c},\ d\hat{a} = a\hat{d}$.*

P r o o f. Let $(G/H, g)$ be a non-reductive homogeneous manifold of dimension four and g^C the complete lift of the invariant metric g. We will show the details of the case **(A1)**.

(**A1**) Using the equations (3.2) and (4.9), straightforward computations yield:

$${}^C G = \left(\begin{array}{cccc|c} 8bx_2y_2 & 2by_2 & -2ax_4y_2 + 2cy_2 - 2ax_2y_4 & 2ay_2 & \\ 2by_2 & 0 & -ay_4 & 0 & G \\ -2ax_4y_2 + 2cy_2 - 2ax_2y_4 & -ay_4 & 0 & 0 & \\ 2ay_2 & 0 & 0 & 0 & \\ & & G & & 0 \end{array} \right),$$

in the local coordinates $(x^1, \ldots, x^4, y^1, \ldots, y^4)$ on the tangent bundle.

Putting $\Lambda[i] := \Lambda(\delta_{x^i})$ and $\Lambda[\bar{j}] := \Lambda(\partial_{y^{\bar{j}}})$ for all indices $i, \bar{j} = 1, \ldots, 4$, we obtain the Levi-Civita componenets of g^C and $\hat{g}^C$.

Straightforward calculations according to equation (2.4) (by replacing the necessary components of g^C) we get the following equations for $\hat{g}^C$ being harmonic with respect to g^C:

$$\frac{a\hat{c} - c\hat{a}}{a(\hat{a} - 4\hat{d})(a - 4d)} = 0,$$

so $c\hat{a} = a\hat{c}$ which shows the validity of the first statement. By straightforward computations we use the same method for each of the classes $A2, \ldots, A5$, $B1, \ldots, B3$. □

Summarizing the arguments above, we give the proof of the main Theorem 5.4 as following.

Theorem 5.4. *Let (M, g) be a non-reductive homogeneous four-manifold. The following statements are equivalent:*

(i) *A pseudo-Riemannian metric $\hat{g}$ on M is harmonic with respect to the pseudo-Riemannian metric g.*

(ii) *The Sasaki metric $\hat{g}^S$ of a pseudo-Riemannian metric $\hat{g}$ is harmonic with respect to the Sasaki metric g^S of the pseudo-Riemannian metric g.*

(iii) *The horizontal lift $\hat{g}^H$ of a pseudo-Riemannian metric $\hat{g}$ is harmonic with respect to the horizontal lift g^H of the pseudo-Riemannian metric g.*

(iv) *The complete lift $\hat{g}^C$ of a pseudo-Riemannian metric $\hat{g}$ is harmonic with respect to the complete lift g^C of the pseudo-Riemannian metric g.*

Proof. In order to show the equivalence between statements (i), (ii), (iii) and (iv), we must completely consider all cases which are presented in the classification of non-reductive homogeneous four manifolds. Following Theorems 5.1, 5.2 and 5.3, we immediately observe that harmonicity conditions of a pseudo-Riemannian metric $\hat{g}$ with respect to the pseudo-Riemannian metric g, the Sasaki metric $\hat{g}^S$ with respect to the Sasaki metric g^S, the horizontal lift $\hat{g}^H$ with respect to the horizontal lift g^H and the complete lift $\hat{g}^C$ with respect to the complete lift g^C coincide. Thus, the statements (i), ..., (iv) are equivalent. □

References

[1] *P. Baird, J. C. Wood*: Harmonic Morphisms Between Riemannian Manifolds. London Mathematical Society Monographs New Series 29, Oxford University Press, Oxford, 2003. zbl MR doi

[2] *C.-L. Bejan, S.-L. Druţă-Romaniuc*: Connections which are harmonic with respect to general natural metrics. Differ. Geom. Appl. *30* (2012), 306–317. zbl MR doi

[3] *C.-L. Bejan, S.-L. Druţă-Romaniuc*: Harmonic almost complex structures with respect to general natural metrics. Mediterr. J. Math. *11* (2014), 123–136. zbl MR doi

[4] *C.-L. Bejan, S.-L. Druţă-Romaniuc*: Structures which are harmonic with respect to Walker metrics. Mediterr. J. Math. *12* (2015), 481–496. zbl MR doi
[5] *M. Benyounes, E. Loubeau, C. M. Wood*: Harmonic sections of Riemannian vector bundles, and metrics of Cheeger-Gromoll type. Differ. Geom. Appl. *25* (2007), 322–334. zbl MR doi
[6] *M. Brozos-Vázquez, E. García-Río, P. Gilkey, S. Nikčević, R. Vázquez-Lorenzo*: The Geometry of Walker Manifolds. Synthesis Lectures on Mathematics and Statistics 5, Morgan & Claypool Publishers, San Rafael, 2009. zbl MR doi
[7] *G. Calvaruso, A. Fino*: Ricci solitons and geometry of four-dimensional non-reductive homogeneous spaces. Can. J. Math. *64* (2012), 778–804. zbl MR doi
[8] *G. Calvaruso, A. Fino, A. Zaeim*: Homogeneous geodesics of non-reductive homogeneous pseudo-Riemannian 4-manifolds. Bull. Braz. Math. Soc. (N.S.) *46* (2015), 23–64. zbl MR doi
[9] *G. Calvaruso, A. Zaeim*: A complete classification of Ricci and Yamabe solitons of non-reductive homogeneous 4-spaces. J. Geom. Phys. *80* (2014), 15–25. zbl MR doi
[10] *G. Calvaruso, A. Zaeim*: Geometric structures over non-reductive homogeneous 4-spaces. Adv. Geom. *14* (2014), 191–214. zbl MR doi
[11] *B.-Y. Chen, T. Nagano*: Harmonic metrics, harmonic tensors, and Gauss maps. J. Math. Soc. Japan *36* (1984), 295–313. zbl MR doi
[12] *C. T. J. Dodson, M. Trinidad Pérez, M. E. Vázquez-Abal*: Harmonic-Killing vector fields. Bull. Belg. Math. Soc. Simon Stevin *9* (2002), 481–490. zbl MR
[13] *J. Eells*, Jr., *J. H. Sampson*: Harmonic mappings of Riemannian manifolds. Am. J. Math. *86* (1964), 109–160. zbl MR doi
[14] *M. Falcitelli, A. M. Pastore*: On the tangent bundle and on the bundle of the linear frames over an affinely connected manifold. Math. Balk., New Ser. *2* (1988), 336–356. zbl MR
[15] *M. E. Fels, A. G. Renner*: Non-reductive homogeneous pseudo-Riemannian manifolds of dimension four. Can. J. Math. *58* (2006), 282–311. zbl MR doi
[16] *O. Kowalski, M. Sekizawa*: Almost Osserman structures on natural Riemann extensions. Differ. Geom. Appl. *31* (2013), 140–149. zbl MR doi
[17] *B. O'Neill*: Semi-Riemannian Geometry. With Applications to Relativity. Pure and Applied Mathematics 103, Academic Press, New York, 1983. zbl MR
[18] *J. Patera, R. T. Sharp, P. Winternitz, H. Zassenhaus*: Invariants of real low dimension Lie algebras. J. Math. Phys. *17* (1976), 986–994. zbl MR doi
[19] *K. Yano, S. Ishihara*: Tangent and Cotangent Bundles. Differential Geometry. Pure and Applied Mathematics 16, Marcel Dekker, New York, 1973. zbl MR

Authors' addresses: Amirhesam Zaeim (corresponding author), Department of Mathematics, Payame Noor University, Nakhl St., P. O. Box 19395-3697, Tehran, Iran, e-mail: zaeim@pnu.ac.ir; Parvane Atashpeykar, Department of Mathematics, Basic Sciences Faculty, University of Bonab, Velayat Highway, P. O. Box 5551761167, Bonab 55517, Iran, e-mail: parvanehatashpeykar@gmail.com.