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PROPER CONNECTION NUMBER OF BIPARTITE GRAPHS

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Abstract. An edge-colored graph G is proper connected if every pair of vertices is connected by a proper path. The proper connection number of a connected graph G, denoted by $\mathrm{pc}(G)$, is the smallest number of colors that are needed to color the edges of G in order to make it proper connected. In this paper, we obtain the sharp upper bound for $\mathrm{pc}(G)$ of a general bipartite graph G and a series of extremal graphs. Additionally, we give a proper 2-coloring for a connected bipartite graph G having $\delta(G) \geqslant 2$ and a dominating cycle or a dominating complete bipartite subgraph, which implies $\mathrm{pc}(G) = 2$. Furthermore, we get that the proper connection number of connected bipartite graphs with $\delta \geqslant 2$ and $\mathrm{diam}(G) \leqslant 4$ is two.

Keywords: proper coloring; proper connection number; bipartite graph; dominating set

MSC 2010: 05C15, 05C69, 05C75

1. Introduction

All graphs in this paper are finite, connected and simple. We follow the terminology and notation of Bondy and Murty [2]. An *k-edge-coloring* of a graph is an assignment of k colors to the edges of G. An edge coloring is *proper* if adjacent edges receive distinct colors. The minimum number of colors needed in a proper edge coloring of the graph G is referred to as *the edge chromatic number* of G and denoted by $\chi'(G)$. Except the classical vertex coloring and edge coloring, there are many kinds of colorings being studied, such as list coloring, star coloring and acyclic coloring. In addition, rainbow connection and rainbow vertex-connection

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are the new ones. For more details we refer to a survey paper [5] and a book [6].

Inspired by proper coloring and rainbow coloring in graphs, Borozan et al. [3] introduced the concept of proper coloring of a graph G. Let G be an edge-colored graph. A path P in G is called a *proper path* if no two adjacent edges of P are colored the same. An edge-coloring c is a *proper-path coloring* of a connected graph G if every pair of distinct vertices u, v of G are connected by a proper (u, v)-path in G. If k colors are used, then c is referred to as a *k-proper coloring*. The minimum number of colors needed to produce a proper coloring of G is called the *proper connection number* of G, denoted by $\mathrm{pc}(G)$. More generally, a graph G is said to be *k-proper connected* if any two vertices are connected by k internally pairwise vertex-disjoint proper paths. The *k-proper connection number* of a k-connected graph G, denoted by $\mathrm{pc}_k(G)$, is the smallest number of colors that are needed in order to make G k-proper connected. From the definition, it is easy to get that $1 \leqslant \mathrm{pc}(G) \leqslant \min\{\chi'(G), \mathrm{rc}(G)\} \leqslant m$, where m is the number of edges of G. Recently, a lot of results have been obtained with respect to several aspects, such as connectivity, minimum degree, complements, operations on graphs and so on. For details we refer to [1], [3] and a dynamic survey paper [4].

Borozan et al. in [3] showed that the proper connection number of a complete bipartite graph is two but the proper connection number of a general bipartite graph is still unknown. Also in [3], Borozan et al. proved that $\mathrm{pc}(G) = 2$ for a 2-connected bipartite graph G by induction. However, how to give a proper 2-coloring for a 2-connected bipartite graph? These two problems are interesting and meaningful. We focus on these two problems in this paper.

This paper is organized as follows. In Section 2, we introduce some basic definitions and useful lemmas on the proper connection number of a graph. In Section 3, the sharp upper bound of $\mathrm{pc}(G)$ for a general bipartite graph G together with a series of extremal graphs are given. Section 4 gives a proper 2-coloring for the connected bipartite graph G such that $\delta(G) \geqslant 2$ and G has a dominating cycle or a dominating complete bipartite subgraph. Furthermore, the proper connection number of a connected bipartite graph G with $\delta \geqslant 2$ and $\mathrm{diam}(G) \leqslant 4$ are obtained.

2. Preliminaries

In this section, we introduce some definitions and present several useful lemmas about the proper coloring of graphs. We begin with some basic conceptions.

Definition 2.1. Given a colored path $P = v_1v_2 \ldots v_{s-1}v_s$ between two vertices v_1 and v_s, we denote by $\mathrm{star}(P)$ the color of the first edge in the path, i.e. $c(v_1v_2)$, and by $\mathrm{end}(P)$ the last color, i.e. $c(v_{s-1}v_s)$.

Definition 2.2. Let $G = (V, E)$ be a connected graph. A (v, S)*-path*, where $v \in V$, $S \subseteq V$, is a path which starts at v and ends at one vertex in S. An (S_1, S_2)*-path*, where $S_1, S_2 \subseteq V$, is a path which starts at one vertex in S_1 and ends at one vertex in S_2.

Definition 2.3. Let $G = (V, E)$ be a connected graph. The *distance between two vertices* u *and* v *in* G, denoted by $d(u, v)$, is the length of a shortest path between them in G. The *diameter* of G is the maximum distance between two vertices of G. The *distance between a vertex* v *and a set* $S \subseteq V$ is $d(v, S) := \min_{x \in S} d(v, x)$. The k*-step neighborhood* of a set $S \subseteq V$ is $N^k(S) := \{x \in V \colon d(x, S) = k\}$, $k \in \{0, 1, 2, \ldots\}$. The *degree* of a vertex v is $\deg(v) := |N^1(v)|$. The *minimum degree* of G is $\delta(G) := \min_{x \in V} \deg(x)$. A vertex is called *pendant* if its degree is 1 and *isolated* if its degree is 0. We may use $N^k(v)$ in place of $N^k(\{v\})$.

Definition 2.4. Given a graph G, a set $D \subseteq V(G)$ is called a k*-step dominating set* of G if every vertex in G is at a distance at most k from D. Further, if D induces a connected subgraph of G, then it is called a *connected* k*-step dominating set* of G.

Definition 2.5. A two-step dominating set D of vertices in a graph G is called a *two-way two-step dominating set* if

(i) every pendant vertex of G is included in D,
(ii) every vertex in $N^2(D)$ has at least two neighbours in $N^1(D)$.

Further, if $G[D]$ is connected, D is called a *connected two-way two-step dominating set* of G.

For other notation and terminology, we refer to [2]. Next we state some known results on the proper coloring, which will be useful in the sequel.

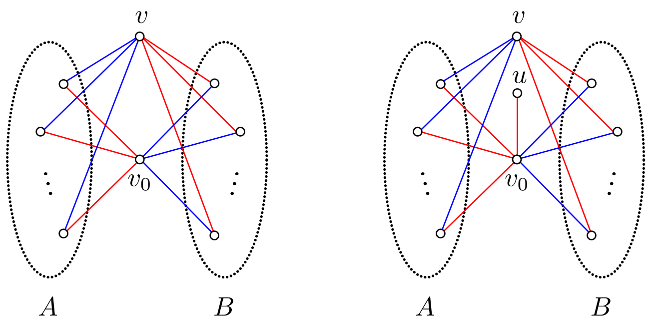
Lemma 2.1 ([3]). *If* $\mathrm{pc}(G) = 2$, *then* $\mathrm{pc}(G \cup v) = 2$ *as long as* $d(v) \geqslant 2$.

As the general case of Lemma 2.1, we give the following proposition.

Proposition 2.1. *If* $\mathrm{pc}(G) = k$, $k \geqslant 2$, *then* $\mathrm{pc}(G \cup v) \leqslant k$ *as long as* $d(v) \geqslant 2$.

Proof. Let u, w be two neighbours of v in G. Assume that c is a proper k-coloring of G, then there is a proper (u, w)-path P in G. Now color the edges uv and wv with colors in the color set of c such that $c(uv) \neq \mathrm{star}(P)$ and $c(wv) \neq \mathrm{end}(P)$. Now we will check that this is a proper k-coloring of the new graph $G \cup v$. If uw is an edge of G, then $c(uv) \neq c(uw)$ and $c(wv) \neq c(uw)$. Since every vertex has a proper path to u, every vertex has a proper path to v. And for the case that uw is not an edge of G, every vertex has a proper path to v through either u or w since every vertex has a proper path to any inner vertex of P. This completes the proof. $\square$

Remark 2.1. The value of $\mathrm{pc}(G) - \mathrm{pc}(G \cup v)$ can be arbitrarily large. We construct a new graph S_k' as follows. Take a star S_k, $k \geqslant 3$, as the base graph and the vertex v_0 as the center vertex of S_k. Add a new vertex v to S_k so that v is adjacent to k (if k is even) or $k-1$ (if k is odd) vertices in $V(S_k)-\{v_0\}$. Since $|N(v)|$ is even, we can divide $N(v)$ into two parts A and B so that $|A| = |B| = |N(v)|/2$ and $A \cap B = \emptyset$. For the case when k is odd we denote the only vertex in $V(S_k) - \{v_0\} - N(v)$ by u. Now we give a 2-proper coloring of the new graph S_k' as follows. Color all (v, A)-edges and (v_0, B)-edges by 1 and all (v, B)-edges and (v_0, A)-edges by 2. For the case when k is even, we have colored all edges in S_k' while for the case when k is odd, color the edge uv_0 with 1 or 2. One can check that this is a 2-proper coloring of S_k'. Then $\mathrm{pc}(S_k) - \mathrm{pc}(S_k') = k - 2$ and this difference can be arbitrarily large if k is large enough.



S_k' (k is even) S_k' (k is odd)

Figure 1. 2-proper colored graphs for Remark 2.1.

3. Proper connection number of bipartite graphs

In this section, we mainly consider the proper connection number of a general bipartite graph. First, we give some known results on the proper connection number of the complete bipartite graphs and the 2-connected (2-edge-connected) bipartite graphs.

Lemma 3.1 ([3])**.** *Let $G = K_{m,n}$, $m \geqslant n \geqslant 2k$ for $k \geqslant 1$. Then $\mathrm{pc}_k(G) = 2$.*

Lemma 3.2 ([3])**.** *Let G be a graph. If G is bipartite and 2-connected (2-edge-connected), then $\mathrm{pc}(G) = 2$ and there exists a 2-coloring of G that makes it properly connected with the following strong property. For any pair of vertices v, w there exist two paths P_1 and P_2 between them (not necessarily disjoint) such that $\mathrm{star}(P_1) \neq \mathrm{star}(P_2)$ and $\mathrm{end}(P_1) \neq \mathrm{end}(P_2)$.*

The above lemma implies that it suffices to consider a connected bipartite graph G containing bridges. The following theorem gives a sharp upper bound for $\mathrm{pc}(G)$.

Theorem 3.1. *Let G be a connected bipartite graph containing bridges. If b is the maximum number of bridges incident with a single vertex in G, then $\mathrm{pc}(G) \leqslant b+2$ and this upper bound is sharp.*

P r o o f. Let G be a connected bipartite graph containing bridges. The block decomposition of G includes its isolated vertices, bridges and maximal 2-connected subgraphs. In order to prove $\mathrm{pc}(G) \leqslant b+2$, we only need to give a $(b+2)$-proper coloring c of G. The coloring c is defined as follows. For the bridges which are incident to a single vertex v we color each bridge incident with v by a distinct fresh color in $[b] = \{1, 2, \ldots, b\}$. Since b is the maximum number of bridges incident with a single vertex, it is enough to color all bridges in G using b distinct colors. And for blocks which are maximal 2-connected subgraphs of G we give each of them a $\{b+1, b+2\}$-proper coloring having strong property as stated in Lemma 3.2. One can check that the above coloring c is a $(b+2)$-proper coloring of G.

Now we will show that this upper bound is sharp. For the case when $b = 1$, we find the graph G_1 (depicted in Figure 2) with $\mathrm{pc}(G_1) = 3$, which implies the sharpness of the upper bound. Since G_1 is not complete, then $\mathrm{pc}(G_1) \geqslant 2$. Suppose that 2 colors are enough to make G_1 proper connected. Note that there are only paths of length 2 or 4 between v_1 and v_2, v_1 and v_3, also v_2 and v_3. Hence, for any proper coloring c_1 of G_1, $c_1(u_1v_1)$, $c_1(u_2v_2)$ and $c_1(u_3v_3)$ are pairwisely different, a contradiction. So $\mathrm{pc}(G_1) = 3$.

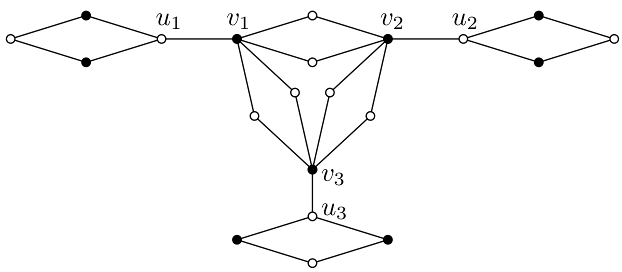


Figure 2. The graph G_1.

Next we will show that this upper bound is sharp for the case when $b \geqslant 2$. As depicted in Figure 3, the graph G_b has a cycle C of length 6. And each vertex of C is adjacent to b bridges, which lead to nontrivial blocks. Now we will show the sharpness of this upper bound by proving that $\mathrm{pc}(G_b) = b+2$.

Claim 1. The sharp upper bound of $\mathrm{pc}(G)$ is greater than b.

P r o o f of Claim 1. We will prove this claim by showing that b colors are not enough to make G_b (see Figure 3) proper connected. Assume that there is a $[b] = \{1, 2, \dots, b\}$-proper coloring c' for G_b. Obviously, the b bridges which are adjacent to u_i must be assigned distinct colors and without loss of generality, we set $c'(u_1u_{1i}) = c'(u_2u_{2i}) = i$, $1 \leqslant i \leqslant b$. In addition, $c'(u_1u_6) \neq c'(u_1u_2)$. Since $c'(u_1u_6)$, $c'(u_1u_2) \in [b]$ and if $c'(u_1u_6) = c'(u_1u_2)$, there is a vertex u_{1k} such that $c'(u_1u_{1k}) = c'(u_1u_6) = c'(u_1u_2)$, which contradicts c' being a proper coloring of G_b. So we can assume that $c'(u_1u_6) = 1$ and $c'(u_1u_2) = 2$. But there is no (u_{11}, u_{22})-proper path under the coloring c', a contradiction.

The above claim implies that the sharp upper bound of $\mathrm{pc}(G)$ is either $b+1$ or $b+2$. Suppose that G_b (see Figure 3) has a $[b+1]$-proper coloring c_b and assume that $c_b(u_1u_{1i}) = i$, $1 \leqslant i \leqslant b$.

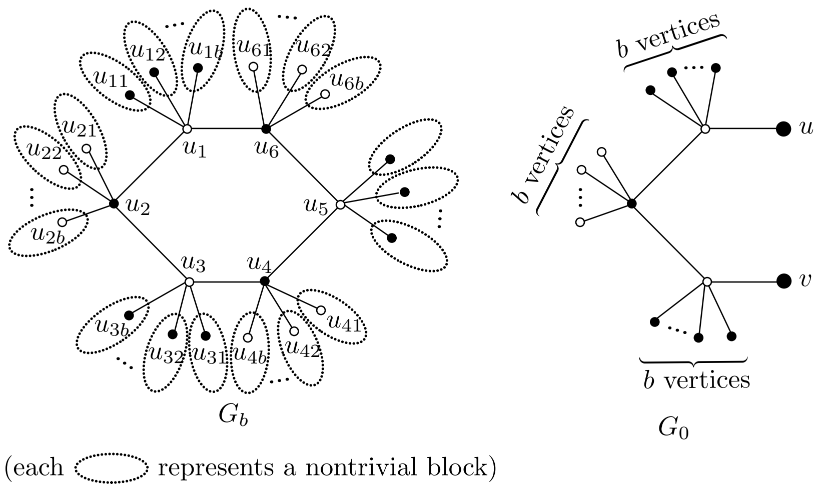


Figure 3. Graphs for the proof of Theorem 3.1.

Claim 2. Undering the proper coloring c_b, any two adjacent edges of $C := u_1u_2u_3u_4u_5u_6u_1$ are assigned different colors.

P r o o f of Claim 2. Suppose there are two adjacent edges on C with the same color. Without loss of generality, we can assume that $c_b(u_1u_2) = c_b(u_1u_6)$. Then $c_b(u_1u_2) = c_b(u_1u_6) = b+1$. If $b+1 \notin \{c_b(u_2u_{21}), c_b(u_2u_{22}), \dots, c_b(u_2u_{2b})\}$, then $\{c_b(u_2u_{21}), c_b(u_2u_{22}), \dots, c_b(u_2u_{2b})\} = [b]$. Since the path $u_2u_1u_6$ is not proper, $u_{2i}u_2u_3u_{3i}$, $1 \leqslant i \leqslant b$, must be proper and so $c_b(u_2u_3) = b+1$ and $\{c_b(u_3u_{31}), c_b(u_3u_{32}), \dots, c_b(u_3u_{3b})\} = [b]$. Additionally, $u_{3i}u_3u_4$, $1 \leqslant i \leqslant b$, must be proper since $c_b(u_2u_3) = b+1 = c_b(u_1u_2)$. So $c_b(u_3u_4) = b+1$ but both $u_{2i}u_2u_1u_6$ and $u_{2i}u_2u_3u_4$, $1 \leqslant i \leqslant b$, are not proper, which contradicts c_b being a proper coloring of G_b. Hence, $b+1 \in \{c_b(u_2u_{21}), c_b(u_2u_{22}), \dots, c_b(u_2u_{2b})\}$ and one of $[b]$

does not appear in $\{c_b(u_2u_{21}), c_b(u_2u_{22}), \ldots, c_b(u_2u_{2b})\}$. Without loss of generality, we assume $\{c_b(u_2u_{21}), c_b(u_2u_{22}), \ldots, c_b(u_2u_{2b})\} = [b+1] \setminus \{1\}$. Since $u_2u_1u_6$ is not proper, $u_{2i}u_2u_3u_{3i}$, $1 \leqslant i \leqslant b$, must be proper and so $c_b(u_2u_3) = 1$ and $\{c_b(u_3u_{31}), c_b(u_3u_{32}), \ldots, c_b(u_3u_{3b})\} = [b+1] \setminus \{1\}$. In addition, $u_{3i}u_3u_4$, $1 \leqslant i \leqslant b$, must be proper since $u_3u_2u_1u_6$ is not. So $c_b(u_3u_4) = 1$ but both $u_{2i}u_2u_1u_6$ and $u_{2i}u_2u_3u_4$, $1 \leqslant i \leqslant b$ are not proper, which contradicts c_b being a proper coloring of G_b. Thus, the result of Claim 2 is obtained.

From Claim 2 we know that for every vertex $u_j \in V(C)$ there is a bridge u_ju_{jk} such that either $c_b(u_ju_{jk}) = c_b(u_ju_{j+1}) \pmod 6$ or $c_b(u_ju_{jk}) = c_b(u_ju_{j-1}) \pmod 6$. Let us assume $c_b(u_1u_6) = c_b(u_1u_{11}) = 1$. It follows that the paths $u_{11}u_1u_2u_3u_{3i}$, $1 \leqslant i \leqslant b$, and $u_{11}u_1u_2u_3u_4$ must be proper and $c_b(u_1u_2) \in [b+1] \setminus \{1\}$.

If $c_b(u_1u_2) = 2$ (the cases $c_b(u_1u_2) = 3, \ldots, b$ are all similar), then $\{c_b(u_2u_{21}), c_b(u_2u_{22}), \ldots, c_b(u_2u_{2b})\} = [b+1] \setminus \{2\}$ since $u_{11}u_1u_6$ is not a proper path and $u_{11}u_1u_2u_{2i}$, $1 \leqslant i \leqslant b$, must be proper. Additionally, since $u_2u_1u_{12}$ is not proper, then $u_{2i}u_2u_3$, $1 \leqslant i \leqslant b$, must be proper and so $c_b(u_2u_3) = 2$. However, this contradicts to the fact that the path $u_{11}u_1u_2u_3$ must be proper. If $c_b(u_1u_2) = b+1$, then $\{c_b(u_2u_{21}), c_b(u_2u_{22}), \ldots, c_b(u_2u_{2b})\} = [b]$ since $u_{11}u_1u_6$ is not a proper path and $u_{11}u_1u_2u_{2i}$, $1 \leqslant i \leqslant b$, must be proper. Take $c_b(u_2u_3) = 1$ into consideration and for $c_b(u_2u_3) = 2, \ldots, b$, the analyses are similar. Since $u_{11}u_1u_2u_3u_{3i}$, $1 \leqslant i \leqslant b$, are all proper paths, then $\{c_b(u_3u_{31}), c_b(u_3u_{32}), \ldots, c_b(u_3u_{3b})\} = [b+1] \setminus \{1\}$. Without loss of generality, we can assume $c_b(u_2u_{21}) = 1$ and then $u_{21}u_2u_3$ is not proper. And this implies that $u_{3i}u_3u_4$ is proper and $c_b(u_3u_4) = 1$. However, this contradicts $u_{11}u_1u_2u_3u_4$ being a proper path. As a result, the graph G_b has no proper $(b+1)$-coloring and $\mathrm{pc}(G_b) = b+2$. $\square$

From the proof of Theorem 3.1, we can find that the cycle C in G_b can be any even cycle. Furthermore, the structure of G_0 (see Figure 3) leads to the sharpness of the above upper bound. With similar analysis for Theorem 3.1, we can obtain the following result.

Theorem 3.2. *Let G be a connected bipartite graph containing bridges and $b \geqslant 2$ be an integer. If G contains G_0 as its induced subgraph and all paths in G joining any pair of vertices of $V(G_0)$ except $\{u, v\}$ appear in G_0, then* $\mathrm{pc}(G) = b + 2$.

Additionally, Andrews et al. in [1] gave a lower bound for any nontrivial connected graphs containing bridges, which is stated as follows.

Theorem 3.3 ([1])**.** *Let G be a nontrivial connected graph containing bridges. If b is the maximum number of bridges incident with a single vertex in G, then* $\mathrm{pc}(G) \geqslant b$.

Together with the results in Theorem 3.1 and Theorem 3.3, we can directly obtain the following corollary.

Corollary 3.1. *Let G be a connected bipartite graph containing bridges. If b is the maximum number of bridges incident with a single vertex in G, then* $\mathrm{pc}(G) \in \{b, b+1, b+2\}$.

4. Proper colorings for bipartite graphs

In [7], we found upper bounds of the proper connection number $\mathrm{pc}(G)$ with the help of two-way dominating sets or two-way two-step dominating sets of a graph G. This implies that the dominating set is a useful tool to help us to find a proper coloring and determine the proper connection number of a connected graph.

Theorem 4.1. *Let $G = (X, Y)$ be a connected bipartite graph such that $\delta(G) \geqslant 2$ and G has a dominating cycle or a dominating complete bipartite subgraph. Then* $\mathrm{pc}(G) = 2$.

Proof. In order to prove the theorem, we will distinguish two cases according to different dominating subgraphs of G.

Case 1. Suppose C_{2k}, $k \geqslant 2$, be a dominating cycle of G.

We claim that G is a 2-connected graph. If not, there exists a cut-vertex, say $x_0 \in X$. Then the graph $G - x_0$ can be composed by at least two bipartite subgraphs $G_1 = (X_1, Y_1)$, $G_2 = (X_2, Y_2)$, where $X_1, X_2 \in X$ and $Y_1, Y_2 \in Y$. Since G has a dominating cycle, $x_0 \in X \cap V(C_{2k})$ and one of X_1 and X_2 must be empty. Without loos of generality, we assume that $X_2 = \emptyset$. Then all the vertices in Y_2 are only adjacent x_0, which contradicts $\delta(G) \geqslant 2$. Then the graph G is a 2-connected bipartite graph, and $\mathrm{pc}(G) = 2$.

Now we give a proper 2-coloring for a spanning subgraph of G. Let C_{2a} be the dominating even cycle of G and color the edges of C_{2a} by $1, 2$ alternately, so that $121212\ldots$. For any vertex $v \in V(G) \setminus V(C_{2a})$ we have $|N(v) \cap V(C_{2a})| \geqslant 2$ or $|N(v) \cap V(C_{2a})| = 1$. If $|N(v) \cap V(C_{2a})| \geqslant 2$, then take two neighbours of v, say $u, w \in C_{2a}$, and color uv and wv so that $uvwC_{2a}u$ is a proper 2-coloring cycle, where the segment $wC_{2a}u$ has a greater length than the remaining segment of C_{2a}. If $|N(v) \cap V(C_{2a})| = 1$, v must on an open ear of length 3, denoted by $x_ivv'x_j$ ($x_i, x_j \in C_{2a}$ and $v, v' \notin C_{2a}$). Color this open ear by $1, 2$ alternately as $121212\ldots$, so that $x_ivv'x_jC_{2a}x_i$ is a proper 2-coloring cycle, where the segment $x_jC_{2a}x_i$ has a smaller length than the remaining segment of C_{2a}. One can check this is indeed a proper 2-coloring for a spanning subgraph of G.

Case 2. Let $D = (X, Y)$ be a dominating complete bipartite subgraph of G.

Subcase 2.1. $|X| = 1$ or $|Y| = 1$. Then D is a star. Without loss of generality, assume the vertex v_0 is the center vertex of D. Let $U = N_G^1(v_0)$ and $V = V(G) \setminus U$. Obviously, $v_0 \in V$ and $N_G^2(v_0) = V \setminus \{v_0\}$. Since G is bipartite and $\delta(G) \geqslant 2$, $\{v_0\}$ is a connected two-way two-step dominating set of G and for all $v \in N_G^1(v_0)$, $N(v) \cap N_G^2(v_0) \neq \emptyset$. The proof of Theorem 3.1 in [7] implies a proper 2-coloring of a spanning subgraph of G and so $\mathrm{pc}(G) = 2$.

Subcase 2.2. $|X| \geqslant 2$ or $|Y| \geqslant 2$. The dominating complete bipartite graph D must be an induced subgraph of G, since G is bipartite. Let $X = \{u_1, u_2, \ldots, u_{|X|}\}$, $Y = \{v_1, v_2, \ldots, v_{|Y|}\}$ be the bipartition of D. Set $D' = V(D) \cup \{w\colon |N(w) \cap V(D)| \geqslant 2$, $w \in V(G) \setminus V(D)\}$. Then we have $\mathrm{pc}(G[D']) = 2$ by Lemma 3.1 and Lemma 2.1. Now we give a proper $\{\text{dark}(1), \text{light}(2)\}$-coloring c for $G[D']$ (see Figure 4) as follows. Set $c(u_1v_1) = 1$, $c(u_iv_j) = 1$ and $c(u_1v_j) = c(v_1u_i) = 2$ for all $2 \leqslant i \leqslant |X|$, $2 \leqslant j \leqslant |Y|$. For any vertex $w \in D' \setminus V(D)$ it is obvious that $N(w) \cap V(D) \subseteq X$ or $N(w) \cap V(D) \subseteq Y$ because G is a bipartite graph. If $u_1w \in G$, then set $c(u_1w) = 1$. Take some $u_k \in N(w) \cap V(D)$, $2 \leqslant k \leqslant |X|$ and set $c(u_kw) = 2$. Similarly, if $v_1w \in G$, then set $c(v_1w) = 1$. Take some $v_l \in N(w) \cap V(D)$, $2 \leqslant l \leqslant |Y|$ and set $c(v_lw) = 2$. If $u_1w \notin G$ and $v_1w \notin G$, then there are $u_j, u_k \in N(w)$, $2 \leqslant j \neq k \leqslant |X|$ or v_l, $v_i \in N(w)$, $2 \leqslant l \neq i \leqslant |Y|$. In this case, if $u_j, u_k \in N(w)$, then color the edges u_jw and u_kw so that $\{c(u_jw), c(u_kw)\} = \{1, 2\}$; if v_l, $v_i \in N(w)$, then color the edges v_lw and v_iw so that $\{c(v_lw), c(v_iw)\} = \{1, 2\}$.

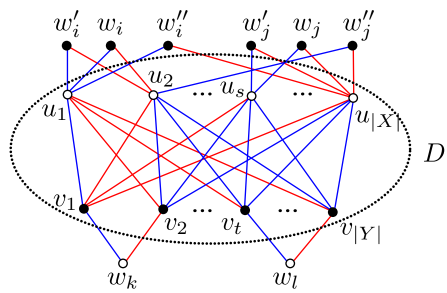


Figure 4. Proper 2-coloring of a subgraph of $G[D']$.

Let $S = V(G) \setminus D' = \{w\colon |N(w) \cap V(D)| = 1,\ w \in V(G) \setminus V(D)\}$. For any $w \in S$, if $w \in N(u_i)$ for some i, $1 \leqslant i \leqslant |X|$, then there is either $w' \in N(w)$ such that $w' \in S$ and $N(w') \cap V(D) \subseteq Y$ (i.e., there is a vertex v_j, $1 \leqslant j \leqslant |Y|$ such that $w'v_j \in E(G)$) or $w' \in D' \setminus V(D)$ and $N(w') \cap V(D) \subseteq Y$, since G is a bipartite graph and D is a dominating complete bipartite subgraph. Similarly, if $w \in N(v_k)$ for some k, $1 \leqslant k \leqslant |Y|$, then there is $w' \in N(w)$ such that $w' \in S$ and $N(w') \cap V(D) \subseteq X$ (i.e., there is either a vertex u_l, $1 \leqslant l \leqslant |X|$ such that

$w'u_l \in E(G)$ or $w' \in D' \setminus V(D)$) and $N(w') \cap V(D) \subseteq X$. For the vertex $w \in S$ we call the vertex w of type I in S if $N(w) \cap (D' \setminus V(D)) = \emptyset$ (i.e., $N(w) \setminus V(D) \subseteq S$), the rest elements of S are called type II ones.

Now we extend the coloring c of $G[D']$ to G. For any vertex $w \in S$ of type I, there is a 4-cycle $wu_iv_jw'w$, $1 \leqslant i \leqslant |X|$, $1 \leqslant j \leqslant |Y|$ in G. Set $c(ww') = c(u_iv_j)$ and $c(wu_i) = c(w'v_j) \in \{1,2\} \setminus c(u_iv_j)$. This coloring implies that each vertex $w \in S$ of type I is contained in a proper 4-cycle except $w_0 \in S$ in shapes (a), (b), (c), (d), which is depicted in Figure 5. We call all these vertices singular vertices.

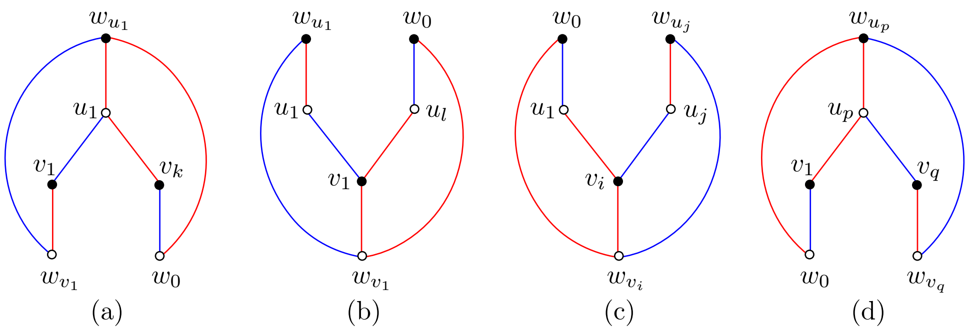


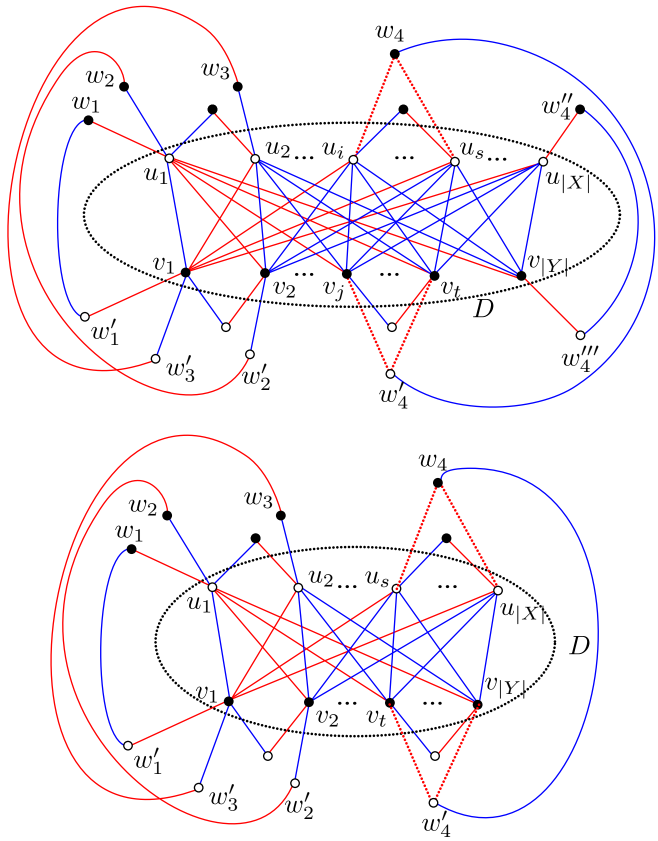
Figure 5. Proper 2-coloring for shapes (a), (b), (c), (d).

For each vertex $w \in S$ of type II there exists at least one vertex $w_i^0 \in D' \setminus D$ such that $ww_i^0 \in G$. If $wu_1 \in G$, then set $c(wu_1) = 1$ and $c(ww_i^0) = 2$. If $wu_{j_1} \in G$, $2 \leqslant j_1 \leqslant |X|$, then set $c(wu_{j_1}) = 2$ and $c(ww_i^0) = 1$. If $wv_1 \in G$, then set $c(wv_1) = 1$ and $c(ww_i^0) = 2$. If $wv_{j_2} \in G$ ($2 \leqslant j_2 \leqslant |Y|$), then set $c(wv_{j_2}) = 2$ and $c(ww_i^0) = 1$. Thus, we give a 2-coloring c of a spanning subgraph of G and the above analysis implies that this spanning subgraph is 2-connected. Therefore, the graph G has a 2-connected bipartite spanning subgraph and Lemma 3.2 implies that $\mathrm{pc}(G) = 2$. Now we will prove that the 2-coloring c defined as above is indeed a proper coloring of G.

Based on the proper 2-coloring c of $G[D']$ as above, vertices of type I in S can be divided into five classes, as depicted in Figure 6, $\{w_1, w_1'\}$, $\{w_2, w_3'\}$, $\{w_3, w_2'\}$, $\{w_4, w_4'\}$ and $\{w_4'', w_4'''\}$. We only take the first four items into consideration (Figure 6 (0)), since the case of $\{w_4'', w_4'''\}$ is the same as that of $\{w_4, w_4'\}$.

Case 3. For any pair of vertices w, w' of type I in S there is a proper 2-coloring (w, w')-path which excludes any $(D' \setminus V(D), V(D))$-edges.

For any pair of vertices w, w' of type I in S, if neither w nor w' is w_0 in shape (a), (b), (c) or (d), then $\{w, w'\} \subseteq \{w_1, w_2, w_3, w_4, w_1', w_2', w_3', w_4'\}$. So it suffices to show that there is a proper path between any pair of vertices in $\{w_1, w_2, w_3, w_4, w_1', w_2', w_3', w_4'\}$. One can check that $w_1u_1w_2$, $w_1u_1v_1u_2w_3$,



(0)

Figure 6. Proper 2-coloring for edges between vertices of type I in S and D'.

$w_1w_1'v_1u_1v_2u_sw_4$ (or $w_1w_1'v_1u_1v_2u_{|X|}w_4$), w_1w_1', $w_1w_1'v_1u_1v_2w_2'$, $w_1w_1'v_1w_3'$ and $w_1u_1v_1u_2v_tw_4'$ (or $w_1u_1v_1u_2v_{|Y|}w_4'$) are proper (w_1, w_2), (w_1, w_3), (w_1, w_4), (w_1, w_1'), (w_1, w_2'), (w_1, w_3') and (w_1, w_4')-paths, respectively. One can similarly find proper paths between any other pair of vertices in $\{w_1, w_2, w_3, w_4, w_1', w_2', w_3', w_4'\}$.

Now we assume that both w and w' are singular vertices in shape (a), (b), (c) or (d). We only need to show that every pair of singular vertices can reach each other through proper paths. If w, w' both are singular vertices of shape (a), then $wv_ku_1v_1w_{v_1}w_{u_1}w'$ is a proper (w, w')-path. If w is a singular vertex of shape (a) and w' is a singular vertex of shape (b), then $wv_ku_1v_1u_lw'$ is a proper (w, w')-path. If w is a singular vertex of shape (a) and w' is a singular vertex of shape (c), then wv_ku_1w' is a proper (w, w')-path. If w is a singular vertex of shape (a) and w' is a singular vertex of shape (d), then $ww_{u_1}w_{v_1}v_1w'$ is a proper (w, w')-path. The cases of other pairs of singular vertices are similar and we omit the details.

Finally, we consider the case that one of w, w' is a singular vertex while the other is not. Without loss of generality, we assume that w is a singular vertex in shape (a), (b), (c) or (d). Thus, it suffices to show that there is a proper path between w and every vertex in $\{w_1, w_2, w_3, w_4, w_1', w_2', w_3', w_4'\}$. If w is a singular vertex of shape (a), then $wv_ku_1v_1w_1'w_1$, $wv_ku_1w_2$, $wv_ku_1v_1u_2w_3$, $ww_{u_1}w_{v_1}v_1u_1v_2u_sw_4$ (or $ww_{u_1}w_{v_1}v_1u_1v_2u_{|X|}w_4$), $wv_ku_1v_1w_1'$, $ww_{u_1}w_{v_1}v_1u_1v_2w_2'$, $ww_{u_1}w_{v_1}v_1w_3'$ and $ww_{u_1}w_{v_1}v_1u_1v_tw_4'$ (or $ww_{u_1}w_{v_1}v_1u_1v_{|Y|}w_4'$) are proper (w, w_1), (w, w_2), (w, w_3), (w, w_4), (w, w_1'), (w, w_2'), (w, w_3') and (w, w_4')-paths, respectively. If w is a singular vertex of shape (b), then $wu_lv_1u_1w_1$, $ww_{v_1}w_{u_1}u_1w_2$, $ww_{v_1}w_{u_1}u_1v_1u_2w_3$, wu_lw_4 (if $u_l = u_s$ or $u_l = u_{|X|}$) or $wu_lv_1u_1v_2u_s$ (or $u_{|X|}$)w_4, $wu_lv_1u_1w_1w_1'$, $wu_lv_1u_1v_2w_2'$, $wu_lv_1w_3'$ and $ww_{v_1}w_{u_1}u_1v_1u_2v_t$ (or $v_{|Y|}$)w_4' are proper (w, w_1), (w, w_2), (w, w_3), (w, w_4), (w, w_1'), (w, w_2'), (w, w_3') and (w, w_4')-paths, respectively. For the cases when w is a singular vertex of shape (c) or (d), one can check it in the similar way.

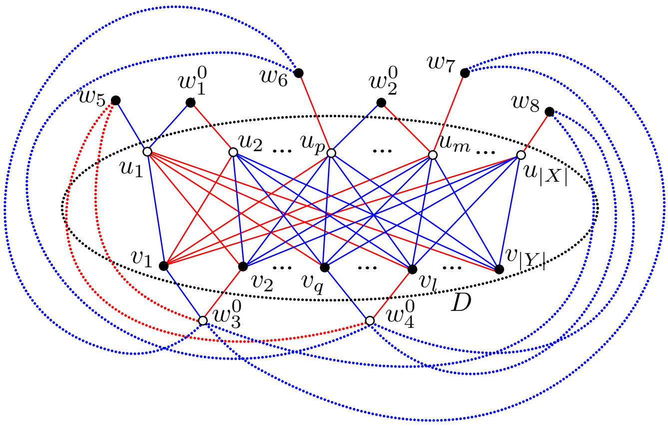
By checking the proper paths above between any pair of vertices of type I in S, we obtain a direct observation that there is a proper path under the coloring c between any pair of vertices of type I in S such that these proper paths do not contain any $(D' \setminus V(D), V(D))$-edges. And then, we complete the proof of Case 3.

Considering the proper 2-coloring c of $G[D']$, we divide vertices of type II in S into four classes, as depicted in Figure 7, $\{w_5, w_5'\}$, $\{w_6, w_6'\}$, $\{w_7, w_7'\}$ and $\{w_8, w_8'\}$. For $i \in \{5, 6, 7, 8\}$ (see Figure 7) it is possible that only one of $\{w_iw_3^0, w_iw_4^0\}$ appears in G, so we use dotted lines to label the edges $w_iw_3^0$ and $w_iw_4^0$. And we use dotted edges for $\{w_i'w_5^0, w_i'w_6^0\}$ for the same reason in Figure 7.

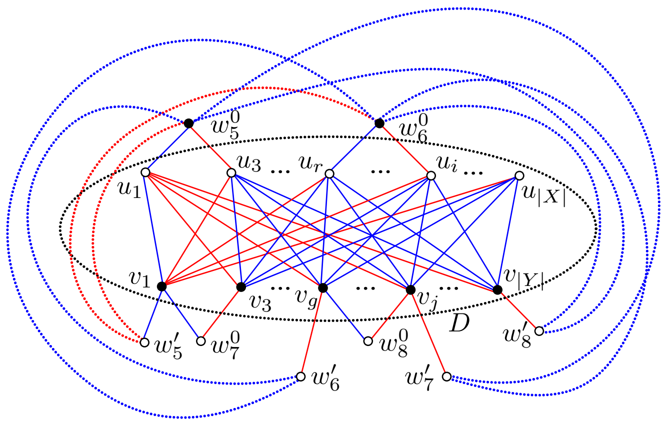
Case 4. For any pair of vertices w, w' of type II in S there is a proper 2-coloring (w, w')-path such that all edges except the first and the last ones of this path are contained in $G[D']$.

At first, we illustrate that there is a proper path between each pair of vertices in $\{w_5, w_6, w_7, w_8, w_5', w_6', w_7', w_8'\}$. If $w_5w_3^0 \in G$, as depicted in Figure 7 (1), $w_5u_1v_2u_pw_6$, $w_5u_1v_2u_mw_7$, $w_5u_1v_2u_{|X|}w_8$, $w_5u_1v_3u_3v_1w_5'$, $w_5w_3^0v_1u_3v_gw_6'$, $w_5w_3^0v_1 u_3v_jw_7'$ and $w_5w_3^0v_1u_3v_{|Y|}w_8'$ are, respectively, proper (w_5, w_6), (w_5, w_7), (w_5, w_8), (w_5, w_5'), (w_5, w_6'), (w_5, w_7') and (w_5, w_8')-paths. Otherwise, there is a vertex like w_4^0 (as depicted in Figure 7 (1)) such that $w_5w_4^0 \in G$. Thus, $w_5u_1v_2u_pw_6$, $w_5u_1v_2u_mw_7$, $w_5u_1v_2u_{|X|}w_8$, $w_5u_1v_3u_3v_1w_5'$, $w_5w_4^0v_q(=v_g)w_6'$ or $w_5w_4^0v_qu_1v_1u_2v_gw_6'$, $w_5w_4^0v_q(=v_j)w_7'$ or $w_5w_4^0v_qu_1v_1u_2v_jw_7'$, and $w_5w_4^0v_q(=v_{|Y|})w_8'$ or $w_5w_4^0v_qu_1v_1u_2 v_{|Y|}w_8'$ are proper (w_5, w_6), (w_5, w_7), (w_5, w_8), (w_5, w_5'), (w_5, w_6'), (w_5, w_7') and (w_5, w_8')-paths, respectively. For any other pair of vertices in $\{w_5, w_6, w_7, w_8, w_5', w_6', w_7', w_8'\}$, one can find proper paths in the same way.

Additionally, we need to show that for two vertices w, w' which both are like w_i or w_i', $5 \leqslant i \leqslant 8$, there is a proper (w, w')-path. Take the case when w, w' are both like w_5 as an example. Suppose $ww^0, ww'^0 \in G$ and the edge coloring is defined



(1)



(2)

Figure 7. Proper 2-coloring for edges between vertices of type II in S and D'.

as c, where $w^0, w'^0 \in D' \setminus V(D)$. Then $ww^0v_1u_2v_2u_1w'$ or $ww^0v_2u_1w'$ is a proper (w, w')-path in G.

By verifying the proper paths above between any pair of vertices of type II in S, we obtain a direct observation that there is a proper path under the coloring c between any pair of vertices of type II in S such that all edges except the first ones and the last ones in these proper paths are contained in $G[D']$. And this implies the result in Case 4.

Case 5. For any pair of vertices w, w' of type I and II in S, respectively, there is a proper 2-coloring (w, w')-path which does not contain any $(D' \setminus V(D), V(D))$-edge.

As depicted in Figure 8, $\{w_1, w_2, w_3, w_4, w_1', w_2', w_3', w_4', w_1^1, w_2^1, w_3^1, w_4^1, w_5^1, w_6^1\}$ are all distinct classes vertices of type I in S, where $\{w_1^1\}, \{w_2^1, w_3^1\}, \{w_4^1\}$ and $\{w_5^1, w_6^1\}$

are, respectively, singular vertices in shapes (c), (b), (d) and (a). In addition, $\{w_5, w_6, w_7, w_8\}$ are all vertices of distinct classes with type II in S. Here, we need to check that there is a proper (w, w')-path for any $w \in \{w_1, w_2, w_3, w_4, w'_1, w'_2, w'_3, w'_4, w^1_1, w^1_2, w^1_3, w^1_4, w^1_5, w^1_6\}$ and $w' \in \{w_5, w_6, w_7, w_8\}$. If $w = w_1$, then $w_1u_1w_5$, $w_1w'_1v_1u_1v_2u_rw_6$, $w_1w'_1v_1u_1v_2u_sw_7$ and $w_1w'_1v_1u_1v_2u_{|X|}w_8$ are proper (w, w_5), (w, w_6), (w, w_7) and (w, w_8)-paths, respectively. If $w = w_4$, then $w_4u_rv_2u_1w_5$ (if $w_4u_r \in G$) or $w_4u_sv_2u_1w_5$ (if $w_4u_s \in G$), $w_4w'_4v_g$(or v_t)u_rw_6, $w_4w'_4v_g$(or v_t)u_sw_7 and $w_4w'_4v_g$(or v_t)$u_{|X|}w_8$ are proper (w, w_5), (w, w_6), (w, w_7) and (w, w_8)-paths, respectively. If $w = w^1_1$, then $w^1_1w_{v_i}w_{u_j}u_jv_iu_1w_5$, $w^1_1u_1v_2u_rw_6$, $w^1_1u_1v_2u_sw_7$, $w^1_1u_1v_2u_sw_7$ and $w^1_1u_1v_2u_{|X|}w_8$ are proper (w, w_5), (w, w_6), (w, w_7) and (w, w_8)-paths, respectively. With similar analysis, we can find proper paths between other pairs of vertices and as a direct observation, these paths do not contain any $(D' \setminus V(D), V(D))$-edge.

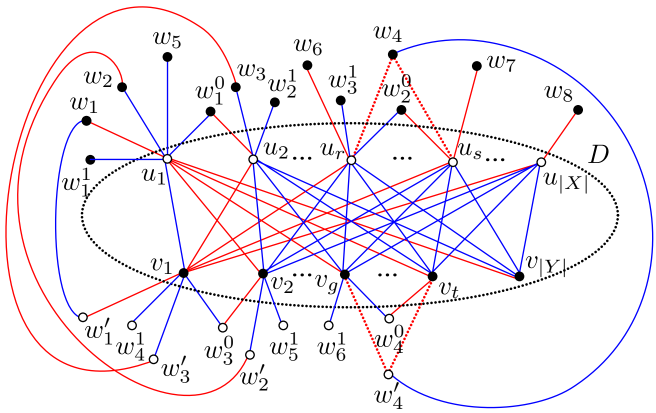


Figure 8. Proper 2-coloring for edges between vertices of types I and II in S.

Case 6. For any pair of vertices $w \in S$ and $w' \in D'$, there is a proper 2-coloring (w, w')-path such that all edges except the first edge of this proper path are contained in $G[D']$.

Now we give proper (w, w')-paths for any $w \in S$ of type I and $w' \in D'$ first. As depicted in Figure 6 (0), w_1u_1, $w_1u_1v_1u_i$, $2 \leqslant i \leqslant |X|$, $i \neq s$ or $i \neq |X|$, $w_1u_1v_1$, $w_1u_1v_1u_2v_j$, $2 \leqslant j \leqslant |Y|$, $w_1u_1w^0_1$, $w_1u_1v_1u_sw^0_2$, $w_1u_1v_1u_2v_2w^0_3$ and $w_1u_1v_1u_2v_{|Y|}w^0_4$ are proper (w_1, X), (w_1, Y) and $(w_1, D' \setminus V(D))$-paths, respectively. If $w = w_4$, then w_4u_s(or $u_{|X|}$)v_2u_1, w_4u_s (or $u_{|X|}$), w_4u_s(or $u_{|X|}$)$v_2u_1v_1u_i$, $2 \leqslant i \leqslant |X|$, w_4u_s(or $u_{|X|}$)$v_2u_1v_1$, w_4u_s(or $u_{|X|}$)v_j, $2 \leqslant j \leqslant |Y|$, w_4u_s(or $u_{|X|}$)$v_2u_1w^0_1$, w_4u_s(or $u_{|X|}v_2u_1v_1u_s$)w^0_2, w_4u_s(or $u_{|X|}$)$v_2w^0_3$ and w_4u_s(or $u_{|X|}$)$v_{|Y|}W^0_4$ are, respectively, proper (w_4, X), (w_4, Y) and $(w_4, D' \setminus V(D))$-paths. Through the same way we can find proper (w_2, D'), (w_3, D'), (w'_1, D'), (w'_2, D'), (w'_3, D') and (w'_4, D')-paths and all edges except the first ones of these proper paths are contained in $G[D']$. This implies that we can also find proper (w_0, D')-paths, when w_0 is a singular vertex of shape (a) (or (b), (c), (d)) in similar way as that for w'_2 (or w_3, w_2, w'_3).

Next we will give proper (w, w')-paths for any $w \in S$ of type II and $w' \in D'$. As depicted in Figure 7 (1), w_5u_1, $w_5u_1v_2u_i$, $2 \leqslant i \leqslant |X|$, $w_5u_1v_2u_2v_1$, $w_5u_1v_j$, $2 \leqslant j \leqslant |Y|$, $w_5u_1v_2u_2w_1^0$, $w_5u_1v_2u_mw_2^0$, $w_5u_1v_2u_2v_1w_3^0$ and $w_5u_1v_qw_4^0$ are proper (w_5, X), (w_5, Y) and $(w_5, D'\setminus V(D))$-paths, respectively. If $w = w_8$, then $w_8u_{|X|}v_2u_1$, $w_8u_{|X|}$, $w_8u_{|X|}v_2u_1v_1u_i$, $2 \leqslant i < |X|$, $w_8u_{|X|}v_2u_1v_1$, $w_8u_{|X|}v_j$, $2 \leqslant j \leqslant |Y|$, $w_8u_{|X|}v_2u_1w_1^0$, $w_8u_{|X|}v_2u_1v_1u_pw_2^0$, $w_8u_{|X|}v_2w_3^0$ and $w_8u_{|X|}v_lw_4^0$ are proper (w_8, X), (w_8, Y) and $(w_8, D' \setminus V(D))$-paths, respectively. Similarly, we can find proper (w_6, D'), (w_7, D'), (w'_5, D'), (w'_6, D'), (w'_7, D') and (w'_8, D')-paths and all edges except the first ones of these proper paths are contained in $G[D']$, which implies the result in Case 6.

Therefore, c is indeed a proper 2-coloring of the graph G. $\square$

Corollary 4.1. *Let $G = (X, Y)$ be a connected noncomplete bipartite graph with minimum degree $\delta(G) \geqslant 2$ and $\operatorname{diam}(G) \leqslant 4$. Then $\operatorname{pc}(G) = 2$.*

Proof. If G is 2-connected, then Lemma 3.2 implies the result. Otherwise, we can assume that $v_0 \in X$ be a cut vertex of G and $G_1, G_2, \ldots, G_t$, $t \geqslant 2$ be the connected components of $G \setminus \{v_0\}$, where $G_i = (X_i, Y_i)$, $X_i \subseteq X$, $Y_i \subseteq Y$. It follows that $Y_i \neq \emptyset$, where $1 \leqslant i \leqslant t$. Also, $X_i \neq \emptyset$, $1 \leqslant i \leqslant t$ since $\delta(G) \geqslant 2$. In addition, $\operatorname{diam}(G) \leqslant 4$ implies that v_0 is adjacent to all vertices of $\bigcup_{i=1}^{t} Y_i$. For each vertex $v \in X_i$, $N_{G\setminus\{v_0\}}(v) \subseteq Y_i$ and $|N_{G\setminus\{v_0\}}(v)| = |N_G(v)| \geqslant 2$. Hence, the star $G\Big[\{v_0\} \bigcup_{i=1}^{t} Y_i\Big]$ is a dominating set of the bipartite graph G and Theorem 4.1 suggests that $\operatorname{pc}(G) = 2$. $\square$

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