ARCHIVUM MATHEMATICUM (BRNO)
Tomus 54 (2018), 65–89

ON A RESULT OF ZHANG AND XU CONCERNING THEIR OPEN PROBLEM

Sujoy Majumder and Rajib Mandal

Abstract. The motivation of this paper is to study the uniqueness of meromorphic functions sharing a nonzero polynomial with the help of the idea of normal family. The result of the paper improves and generalizes the recent result due to Zhang and Xu [24]. Our another remarkable aim is to solve an open problem as posed in the last section of [24].

1. Introduction, definitions and results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Suppose f and g are two non-constant meromorphic functions and $a \in \mathbb{C}$. We say that f and g share the value a with counting multiplicities (CM), provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share the value a with ignoring multiplicities (IM), provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. Moreover we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

In this paper we take up the standard notations and definitions of the value distribution theory (see [7]). For a non-constant meromorphic function f we denote by $S(r,f)$ any quantity satisfying the relation $S(r,f)=o(T(r,f))$ as $r\to\infty$ except possibly a set of finite linear measure.

We define $T(r)=\max\{T(r,f),T(r,g)\}$ and we use the notation $S(r)$ to denote any quantity satisfying the relation $S(r)=o(T(r))$ as $r\longrightarrow\infty$, outside of a possible exceptional set of finite linear measure.

A meromorphic function a is said to be a small function of f if $T(r,a)=S(r,f)$, i.e., if $T(r,a)=o(T(r,f))$ as $r\to\infty$ except possibly a set of finite linear measure.

If $f(z_0)=z_0$, where $z_0\in\mathbb{C}$, then z_0 is called a fixed point of $f(z)$. We use the following definition throughout this paper

$$\Theta(a;f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,a;f)}{T(r,f)}\,,$$

2010 *Mathematics Subject Classification*: primary 30D35; secondary 30D30.
Key words and phrases: normal families, uniqueness, meromorphic function, small functions.
Received May 2, 2017, revised November 2017. Editor M. Kolář.
DOI: 10.5817/AM2018-2-65

where $a \in \mathbb{C} \cup \{\infty\}$.

First we recall the following result due to W.K. Hayman.

Theorem A ([6])**.** *Let f be a transcendental meromorphic function and $n \in \mathbb{N} \setminus \{1, 2\}$. Then $f^n f' = 1$ has infinitely many solutions.*

Corresponding to Theorem A, C.C. Yang and X.H. Hua obtained the following result.

Theorem B ([19])**.** *Let f and g be two non-constant meromorphic functions, $n \in \mathbb{N}$ with $n \geq 11$. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

In 2002, using the idea of sharing fixed points, M.L. Fang and H.L. Qiu further generalized and improved Theorem B in the following manner.

Theorem C ([4])**.** *Let f and g be two non-constant meromorphic functions, and let $n \in \mathbb{N}$ with $n \geq 11$. If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three non-zero complex numbers satisfying $4(c_1 c_2)^{n+1} c^2 = -1$ or $f = tg$ for a complex number t such that $t^{n+1} = 1$.*

For the last couple of years a handful numbers of astonishing results have been obtained regarding the value sharing of non-linear differential polynomials which are mainly the k-th derivative of some linear expression of f and g.

In 2010, J.F. Xu, F. Lü and H.X. Yi studied the analogous problem corresponding to Theorem C where in addition to the fixed point sharing problem, sharing of poles are also taken under supposition. Thus the research has somehow been shifted to wards the following direction.

Theorem D ([16])**.** *Let f and g be two non-constant meromorphic functions, and let n, $k \in \mathbb{N}$ with $n > 3k + 10$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, f and g share ∞ IM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2 (c_1 c_2)^n c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Theorem E ([16])**.** *Let f and g be two non-constant meromorphic functions satisfying $\Theta(\infty, f) > \frac{2}{n}$, and let n, $k \in \mathbb{N}$ with $n \geq 3k + 12$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share z CM, f and g share ∞ IM, then $f \equiv g$.*

Recently, X.B. Zhang and J.F. Xu further generalized and improved the results obtained in [16] in the following manner.

Theorem F ([24])**.** *Let f and g be two transcendental meromorphic functions, let $p(z)$ be a non-zero polynomial with $\deg(p) = l \leq 5$, n, k, $m \in \mathbb{N}$ with $n > 3k+m+7$. Let $P^*(w) = a_m w^m + a_{m-1} w^{m-1} + \cdots + a_1 w + a_0$ be a non-zero polynomial. If $[f^n P^*(f)]^{(k)}$ and $[g^n P^*(g)]^{(k)}$ share p CM, f and g share ∞ IM then one of the following three cases hold:*

(1) *$f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD(n + m, \ldots, n + m - i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 1, 2, \ldots, m$,*

(2) *f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1,\omega_2) = \omega_1^n(a_m\omega_1^m + a_{m-1}\omega_1^{m-1} + \cdots + a_0) - \omega_2^n(a_m\omega_2^m + a_{m-1}\omega_2^{m-1} + \cdots + a_0)$;*

(3) *$P^*(z)$ reduces to a non-zero monomial, namely $P^*(z) = a_iz^i \not\equiv 0$ for some $i \in \{0,1,\ldots,m\}$; if $p(z)$ is not a constant, then $f(z) = c_1e^{cQ(z)}$, $g(z) = c_2e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)dt$, c_1, c_2 and c are constants such that $a_i^2(c_1c_2)^{n+i}[(n+i)c]^2 = -1$, if $p(z)$ is a non-zero constant b, then $f(z) = c_3e^{cz}$, $g(z) = c_4e^{-cz}$, where c_3, c_4 and c are constants such that $(-1)^ka_i^2(c_3c_4)^{n+i}[(n+i)c]^{2k} = b^2$.*

Zhang and Xu made the following commend in Remark 1.2 [24]:
"From the proof of Theorem 1.3, when $\deg(p)$ becomes large we can see that the computation will be very complicated and so we are not sure whether Theorem 1.3 holds for the general polynomial $p(z)$."

In addition they [24] posed the following open problem at the end of their paper.
Open problem. What happens to Theorem 1.3 [24] if the condition "$l \leq 5$" is removed?

Regarding the above result, the first author [13] asked the following question in 2016.
Question 1. Can the lower bound of n be further reduced in Theorem F?

Keeping in mind the above question, the first author obtained the following result.

Theorem G ([13]). *Let f and g be two transcendental meromorphic functions, let $p(z)$ be a nonzero polynomial with $\deg(p) \leq n-1$, $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 0)$ be three integers such that $n > 3k+m+6$ and $P^*(z)$ be defined as in Theorem F. If $[f^nP^*(f)]^{(k)}$, $[g^nP^*(g)]^{(k)}$ share p CM and f, g share ∞ IM then the conclusion of Theorem F holds.*

This paper is motivated by the following questions
Question 2. Can one remove the conditions "$l \leq 5$" and "$\deg(p) \leq n-1$" respectively in Theorems F and G?
Question 3. Can one deduce a generalized result in which Theorems F and G will be included?
Question 4. Can the lower bound of n be further reduced in Theorem G?
Our main objective to write this paper is to solve the above questions.

2. Main result and definitions

Throughout this paper, we always use $P(z)$ to denote an arbitrary non-constant polynomial of degree n as follows

$$\begin{aligned} P(z) &= a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0 \\ &= a_n(z-e_1)^{d_1}(z-e_2)^{d_2}\ldots(z-e_s)^{d_s}, \end{aligned} \tag{2.1}$$

where $a_i \in \mathbb{C}$ $(i = 0, 1, \dots, n)$ with $a_n \neq 0$, $e_j (j = 1, 2, \dots, s)$ are distinct numbers in $\mathbb{C}$ and $d_1, d_2, \dots, d_s \in \mathbb{N} \cup \{0\}$, n, $s \in \mathbb{N}$ with

$$\sum_{i=1}^{s} d_i = n\,.$$

Let $d = \max\{d_1, d_2, \dots, d_s\}$ and e be the corresponding zero of $P(z)$ of multiplicity d. We set an arbitrary non-zero polynomial $P_1(z)$ by

$$P_1(z) = a_n \prod_{\substack{i=1 \\ d_i \neq d}}^{s} (z - e_i)^{d_i} = b_m z^m + b_{m-1} z^{m-1} + \dots + b_0\,, \tag{2.2}$$

where $a_n = b_m$ and $m = n - d$. Let $z_1 = z - e$. Then

$$P_1(z) = P_1(z_1 + e) = P_2(z_1) = c_m z_1^m + c_{m-1} z_1^{m-1} + \dots + c_1 z_1 + c_0\,,$$

where $c_m = b_m = a_n$. Obviously

$$P(z) = (z - e)^d P_1(z) = z_1^d P_2(z_1)\,. \tag{2.3}$$

Let

$$m_1 = \sum_{\substack{i=1 \\ d_i \neq d \\ d_i \leq k+1}}^{s} d_i\,,$$

where $k \in \mathbb{N}$. Suppose $\Gamma = m_1 + (k + 2)m_2$, where m_2 is the number of zeros of $P_1(z)$ with multiplicities $\geq k + 2$. Clearly $\Gamma \leq \deg(P_1) = m$.

Before going to our main result we now explain the following useful definition and notation.

Definition 1 ([10, 11])**.** Let $k \in \mathbb{N} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and k+1 times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively. If a is a small function, we define that f and g share (a, k) if $f - a$ and $g - a$ share $(0, k)$.

In this paper, taking the possible answers of the above questions into background we obtain the following result.

Theorem 1. *Let f and g be two transcendental meromorphic functions and let d, n, $k \in \mathbb{N}$ and m, $\Gamma \in \mathbb{N} \cup \{0\}$ such that $n > 2\Gamma + 3k + 6$ and $d > k$. Let $p(z)$ be a nonzero polynomial and $P(z)$ be defined as in* (2.1)*. If $[P(f)]^{(k)}$, $[P(g)]^{(k)}$ share (p, k_1) where $k_1 = \left[\frac{3+k}{n-k-1}\right] + 3$ and f, g share $(\infty, 0)$ then one of the following three cases holds*

(1) *$f(z) - e \equiv t(g(z) - e)$ for a constant t such that $t^{d_0} = 1$, where $d_0 = GCD(d + m, \ldots, d + m - i, \ldots, d)$, $c_{m-i} \neq 0$ for some $i = 1, 2, \ldots, m$,*

(2) *f_1 and g_1 satisfy the algebraic equation $R(f_1, g_1) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^d(c_m\omega_1^m + c_{m-1}\omega_1^{m-1} + \cdots + c_0) - \omega_2^d(c_m\omega_2^m + c_{m-1}\omega_2^{m-1} + \cdots + c_0)$, where $f_1 = f - e$ and $g_1 = g - e$;*

(3) *$P(z)$ takes the form $P(z) = c_i(z-e)^{d+i} \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$. Also if $p(z)$ is not a constant, then $f(z) = d_1e^{c^*Q(z)} + e$, $g(z) = d_2e^{-c^*Q(z)} + e$, where $Q(z) = \int_0^z p(t)dt$, d_1, d_2 and c^* are constants such that $c_i^2(d_1d_2)^{d+i}[(d + i)c^*]^2 = -1$, if $p(z)$ is a non-zero constant, say b, then $f(z) = d_3e^{c^*z} + e$, $g(z) = d_4e^{-c^*z} + e$, where d_3, d_4 and c^* are constants such that $(-1)^kc_i^2(d_3d_4)^{d+i}[(d + i)c^*]^{2k} = b^2$.*

Remark 1. In this paper we can able to remove the conditions "$l \leq 5$" and "$\deg(p) \leq n - 1$" respectively in Theorems F and G without imposing any other conditions and keeping all the conclusions intact. As a result both Theorems F and G hold for a general non-zero polynomial $p(z)$.

Remark 2. Let us take $d = n$, $e = 0$ and $P_1(z) = a_mz^m + a_{m-1}z^{m-1} + \cdots + a_1z + a_0$ in (2.3), where $a_0, a_1, \ldots, a_{m-1}$, a_m are complex constants. Then by replacing n by $d + m$ in Theorem 1, we can easily get a theorem which is the improvement of Theorems F and G.

We give the following definitions and notations which are used in the paper.

Definition 2 ([9])**.** Let $a \in \mathbb{C} \cup \{\infty\}$. For $p \in \mathbb{N}$ we denote by $N(r, a; f \mid\leq p)$ the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, a; f \mid\leq p)$ we denote the corresponding reduced counting function.

In an analogous manner we can define $N(r, a; f \mid\geq p)$ and $\overline{N}(r, a; f \mid\geq p)$.

Definition 3 ([11])**.** Let $k \in \mathbb{N} \cup \{\infty\}$. We denote by $N_k(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then $N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \cdots + \overline{N}(r, a; f \mid\geq k)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 4 ([2])**.** Let f and g be two non-constant meromorphic functions such that f and g share $(a, 0)$ for $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p and also an a-point of g with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ ($\overline{N}_L(r, a; g)$) the reduced counting function of those a-points of f and g, where $p > q \geq 1$ ($q > p \geq 1$). Also we denote by $\overline{N}_E^{(1}(r, a; f)$ the reduced counting function of those a-points of f and g, where $p = q \geq 1$.

Definition 5 ([10, 11])**.** Let f and g be two non-constant meromorphic functions such that f and g share $(a, 0)$. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

Definition 6 ([8])**.** Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g \neq b_1, b_2, \ldots, b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b_i-points of g for $i = 1, 2, \ldots, q$.

Definition 7. Let h be a meromorphic function in $\mathbb{C}$. Then h is called a normal function if there exists a positive real number M such that $h^{\#}(z) \leq M\ \forall\ z \in \mathbb{C}$, where

$$h^{\#}(z) = \frac{|h'(z)|}{1 + |h(z)|^2}$$

denotes the spherical derivative of h.

Definition 8. Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. We say that $\mathcal{F}$ is normal in D if every sequence $\{f_n\}_n \subseteq \mathcal{F}$ contains a subsequence which converges spherically and uniformly on the compact subsets of D (see [15]).

3. Lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We define the meromorphic functions H and V in the following manner

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big) \tag{3.1}$$

and

$$V = \Big(\frac{F'}{F-1} - \frac{F'}{F}\Big) - \Big(\frac{G'}{G-1} - \frac{G'}{G}\Big)\,. \tag{3.2}$$

Lemma 1 ([18])**.** *Let f be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0)$, $a_{n-1}(z), \ldots, a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, \ldots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0) = nT(r, f) + S(r, f)\,.$$

Lemma 2 ([23])**.** *Let f be a non-constant meromorphic function and p, $k \in \mathbb{N}$. Then*

$$N_p\big(r, 0; f^{(k)}\big) \leq T\big(r, f^{(k)}\big) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f)\,, \tag{3.3}$$

$$N_p\big(r, 0; f^{(k)}\big) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f)\,. \tag{3.4}$$

Lemma 3 ([12])**.** *If $N(r, 0; f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r, 0; f^{(k)} \mid f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; f \mid< k) + k\overline{N}(r, 0; f \mid\geq k) + S(r, f)\,.$$

Lemma 4 ([7, Theorem 3.10])**.** *Suppose that f is a non-constant meromorphic function, $k \in \mathbb{N} \setminus \{1\}$. If*

$$N(r, \infty, f) + N(r, 0; f) + N(r, 0; f^{(k)}) = S\Big(r, \frac{f'}{f}\Big)\,,$$

then $f(z) = e^{az+b}$, where $a \neq 0$, b are constants.

Lemma 5 ([5])**.** *Let $f(z)$ be a non-constant entire function and let $k \in \mathbb{N} \setminus \{1\}$. If $f(z)f^{(k)}(z) \neq 0$, then $f(z) = e^{az+b}$, where $a \neq 0$, b are constant.*

Lemma 6 ([20, Theorem 1.24])**.** *Let f be a non-constant meromorphic function and let $k \in \mathbb{N}$. Suppose that $f^{(k)} \not\equiv 0$, then*

$$N(r, 0; f^{(k)}) \leq N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f)\,.$$

Lemma 7. *Let f, g be non-constant meromorphic functions and let n, k, $\Gamma \in \mathbb{N}$ with $n > k + \Gamma + 2$. Let $P(z)$ be defined as in (2.1) and $a(z)(\not\equiv 0, \infty)$ be a small function of f. If $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share $(a, 0)$, then $T(r, f) = O(T(r, g))$, $T(r, g) = O(T(r, f))$.*

Proof. Let $f_1 = f - e$. Clearly $F = f_1^d P_1(f)$. By the second fundamental theorem for small functions (see [17]), we have

$$T(r, F^{(k)}) \leq \overline{N}(r, f) + \overline{N}(r, 0; F^{(k)}) + \overline{N}(r, a; F^{(k)}) + \big(\varepsilon + o(1)\big)T(r, f) \tag{3.5}$$

for all $\varepsilon > 0$. From (3.5) and Lemmas 1, 2 with $p = 1$ we have

$$\begin{aligned} n\, T(r, f) &\leq \overline{N}(r, f) + N_{k+1}(r, 0; F) + \overline{N}(r, a; F^{(k)}) + \big(\varepsilon + o(1)\big)T(r, f) \\ &\leq \overline{N}(r, f) + (k+1)\overline{N}(r, 0; f_1) + N_{k+1}\big(r, 0; P_1(f)\big) \\ &\quad + \overline{N}\big(r, a; [P(f)]^{(k)}\big) + \big(\varepsilon + o(1)\big)T(r, f) \\ &\leq \overline{N}(r, f) + (k+1)\overline{N}(r, 0; f_1) + N_{k+2}\big(r, 0; P_1(f)\big) \\ &\quad + \overline{N}\big(r, a; [P(f)]^{(k)}\big) + \big(\varepsilon + o(1)\big)T(r, f) \\ &\leq (k + \Gamma + 2)T(r, f) + \overline{N}\big(r, a; [P(g)]^{(k)}\big) + \big(\varepsilon + o(1)\big)T(r, f)\,, \end{aligned}$$

i.e.,

$$(n - k - \Gamma - 2)T(r, f) \leq \overline{N}\big(r, a; [P(g)]^{(k)}\big) + \big(\varepsilon + o(1)\big)T(r, f)\,.$$

Since $n > k + \Gamma + 2$, take $\varepsilon < 1$ and we have $T(r, f) = O(T(r, g))$. Similarly we have $T(r, g) = O(T(r, f))$. This completes the proof. □

Lemma 8. *Let f and g be two non-constant meromorphic functions. Let $P(z)$ be defined as in (2.1) and k, Γ, $n \in \mathbb{N}$ with $n > 3k + 2\,\Gamma$. If $[P(f)]^{(k)} \equiv [P(g)]^{(k)}$, then $P(f) \equiv P(g)$.*

Proof. We have $[P(f)]^{(k)} \equiv [P(g)]^{(k)}$. Integrating we get

$$[P(f)]^{(k-1)} \equiv [P(g)]^{(k-1)} + c_{k-1}\,.$$

If possible suppose $c_{k-1} \neq 0$. Now in view of Lemma 2 for $p = 1$ and using the second fundamental theorem we get

$$\begin{aligned} n\, T(r, f) &= T\big(r, P(f)\big) + O(1) \leq T\big(r, [P(f)]^{(k-1)}\big) - \overline{N}\big(r, 0; [P(f)]^{(k-1)}\big) \\ &\quad + N_k\big(r, 0; P(f)\big) + S(r, f) \\ &\leq \overline{N}(r, 0; [P(f)]^{(k-1)}) + \overline{N}(r, \infty; f) + \overline{N}\big(r, c_{k-1}; [P(f)]^{(k-1)}\big) \\ &\quad - \overline{N}(r, 0; [P(f)]^{(k-1)}) + N_k\big(r, 0; P(f)\big) + S(r, f) \end{aligned}$$

$$
\begin{aligned}
&\leq \overline{N}(r,\infty;f)+\overline{N}(r,0;[P(g)]^{(k-1)})+N_k(r,0;P(f))+S(r,f)\\
&\leq \overline{N}(r,\infty;f)+(k-1)\overline{N}(r,\infty;g)+N_k\big(r,0;P(g)\big)+N_k\big(r,0;P(f)\big)+S(r,f)\\
&\leq \overline{N}(r,\infty;f)+(k-1)\overline{N}(r,\infty;g)+k\,\overline{N}(r,0;g_1)+N_k\big(r,0;P_1(g)\big)\\
&\quad +k\,\overline{N}(r,0;f_1)+N_k(r,0;P_1(f))+S(r,f)\\
&\leq \overline{N}(r,\infty;f)+(k-1)\overline{N}(r,\infty;g)+k\,\overline{N}(r,0;g_1)+N_{k+2}\big(r,0;P_1(g)\big)\\
&\quad +k\,\overline{N}(r,0;f_1)+N_{k+2}\big(r,0;P_1(f)\big)+S(r,f)\\
&\leq (k+\Gamma+1)T(r,f)+(2k+\Gamma-1)T(r,g)+S(r,f)+S(r,g)\\
&\leq (3k+2\,\Gamma)T(r)+S(r)\,.
\end{aligned}
$$

Similarly we get

$$n\,T(r,g)\leq (3k+2\,\Gamma)T(r)+S(r)\,.$$

Combining we get

$$n\,T(r)\leq (3k+2\,\Gamma)T(r)+S(r)\,,$$

which is a contradiction since $n>3k+2\Gamma$. Therefore $c_{k-1}=0$ and so $[P(f)]^{(k-1)}\equiv [P(g)]^{(k-1)}$. Proceeding in this way after $(k-1)$-th step, we obtain $[P(f)]'\equiv[P(g)]'$. Integrating we get $P(f)\equiv P(g)+c_0$. If possible suppose $c_0\neq 0$. Now using the second fundamental theorem we get

$$
\begin{aligned}
n\,T(r,f)&=T\big(r,P(f)\big)+O(1)\\
&\leq \overline{N}\big(r,0;P(f)\big)+\overline{N}\big(r,\infty;P(f)\big)+\overline{N}\big(r,c_0;P(f)\big)+S(r,f)\\
&\leq \overline{N}\big(r,0;P(f)\big)+\overline{N}(r,\infty;f)+\overline{N}\big(r,0;P(g)\big)+S(r,f)\\
&\leq \overline{N}(r,0;f_1)+\overline{N}\big(r,0;P_1(f)\big)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g_1)\\
&\quad +\overline{N}\big(r,0;P_1(g)\big)+S(r,f)\\
&\leq \overline{N}(r,0;f_1)+N_{k+2}\big(r,0;P_1(f)\big)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g_1)\\
&\quad +N_{k+2}\big(r,0;P_1(g)\big)+S(r,f)\\
&\leq (\Gamma+2)T(r,f)+(\Gamma+1)T(r,g)+S(r,f)+S(r,g)\\
&\leq (2\Gamma+3)T(r)+S(r)\,.
\end{aligned}
$$

Similarly we get

$$n\,T(r,g)\leq (2\Gamma+3)T(r)+S(r)\,.$$

Combining these we get

$$(n-2\Gamma-3)T(r)\leq S(r)\,,$$

which is a contradiction since $n>2\Gamma+3$. Therefore $c_0=0$ and so $P(f)\equiv P(g)$. This proves the lemma. □

Lemma 9. *Let f, g be transcendental meromorphic functions and let $P(z)$ be defined as in (2.1). Let $d(\geq 1)$, $m(\geq 0)$ and $k(\geq 1)$ be three integers such that $d > k$. If $[P(f)]^{(k)}[P(g)]^{(k)} \equiv p^2$, where $p(z)$ is a non-zero polynomial and f, g share $(\infty, 0)$, then $P_2(z_1)$ is reduced to a non-zero monomial, namely $P_2(z_1) = c_i z_1^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$ and so $P(z)$ takes the form $P(z) = c_i(z-e)^{d+i} \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$.*

Proof. Suppose

$$[P(f)]^{(k)}[P(g)]^{(k)} \equiv p^2 \,,$$

i.e.,

$$\left[f_1^d P_2(f_1)\right]^{(k)}[g_1^d P_2(g_1)]^{(k)} \equiv p^2 \,, \tag{3.6}$$

where $f_1 = f - e$ and $g_1 = g - e$. Since f and g share $(\infty, 0)$, it follows that f and g are transcendental entire functions.

Suppose on the contrary that, $P_2(z_1)$ does not reduce to a non-zero monomial. Then without loss of generality, we may assume that

$$P_2(z_1) = c_m z_1^m + c_{m-1} z_1^{m-1} + \cdots + c_1 z_1 + c_0 \,,$$

where $c_0 \neq 0, c_1, \ldots, c_{m-1}, c_m \neq 0$ are complex constants.

Since the number of zeros of $p(z)$ is finite, it follows that both f_1 and g_1 have finitely many zeros. Then $f_1(z)$ takes the form

$$f_1(z) = h(z)e^{\gamma(z)},$$

where h is a non-zero polynomial and γ is a non-constant entire function. Clearly

$$f_1^{d+i}(z) = h^{d+i}(z)e^{(d+i)\gamma(z)},$$

where $i = 0, 1, \ldots, m$. Then by induction we have

$$[c_i f_1^{d+i}(z)]^{(k)} = t_i\big(\gamma', \gamma'', \ldots, \gamma^{(k)}, h', h'', \ldots, h^{(k)}\big)e^{(d+i)\gamma(z)} \,, \tag{3.7}$$

where $t_i(\gamma', \gamma'', \ldots, \gamma^{(k)}, h', h'', \ldots, h^{(k)})(i = 0, 1, \ldots, m)$ are differential polynomials in

$\gamma', \gamma'', \ldots, \gamma^{(k)}, h', h'', \ldots, h^{(k)}$. Since $f_1(z)$ is a transcendental entire function, from (3.7) we see that

$$t_i\big(\gamma', \gamma'', \ldots, \gamma^{(k)}, h', h'', \ldots, h^{(k)}\big) \not\equiv 0 \,,$$

$i = 0, 1, \ldots, m$. Note that

$$\left[f_1^d P_2(f_1)\right]^{(k)} = \sum_{i=1}^{m} [c_i f_1^{d+i}]^{(k)} = \sum_{i=0}^{m} t_i e^{(d+i)\gamma} = e^{d\gamma} \sum_{i=0}^{m} t_i e^{i\gamma} \tag{3.8}$$

and so $[f_1^d P_2(f_1)]^{(k)} \not\equiv 0$. Note that $f_1 = he^\gamma$. So $f_1' = h'e^\gamma + \alpha' he^\gamma$. Therefore $\frac{f_1'}{f_1} = \frac{h'}{h} + \gamma'$. Since γ' is an entire function, we have

$$\begin{aligned} T(r, \gamma') = m(r, \gamma') = m\Big(r, \frac{f_1'}{f_1} - \frac{h'}{h}\Big) \leq m\Big(r, \frac{f_1'}{f_1}\Big) + m\Big(r, \frac{h'}{h}\Big) \\ = S(r, f) + O(\log r) = S(r, f) \,, \end{aligned}$$

i.e.,

$$T(r,\gamma') = S(r,f)\,.$$

Therefore

$$T(r,\gamma^{(i)}) = S(r,f)\,,$$

where $i = 1, 2, \ldots, k$. Since h are non-zero polynomial, it follows that $T(r,t_i) = S(r,f)$, where $i = 0, 1, \ldots, m$. Note that

$$\overline{N}(r,0;[f_1^d P_2(f_1)]^{(k)}) \leq N(r,0;p^2) \leq S(r,f)\,.$$

Now from (3.6) we have

$$\overline{N}(r,0;t_0 + t_1e^{\gamma} + \cdots + t_me^{m\gamma}) \leq S(r,f)\,. \tag{3.9}$$

Since $t_0 + t_1e^{\gamma} + \cdots + t_me^{m\gamma}$ is a transcendental entire function and $t_0(z)$ is a polynomial, it follows that t_0 is a small function of $t_0 + t_1e^{\gamma} + \cdots + t_me^{m\gamma}$. So from (3.9) and using the second fundamental theorem for small functions (see [17]), we obtain

$$\begin{aligned} m\,T(r,f_1) &= T(r,t_1e^{\gamma} + \cdots + t_me^{m\gamma}) + S(r,f_1) \\ &\leq \overline{N}(r,0;t_me^{m\gamma} + t_{m-1}e^{(m-1)\gamma} + \cdots + t_1e^{\gamma}) \\ &\quad + \overline{N}(r,0;t_me^{m\gamma} + t_{m-1}e^{(m-1)\gamma} + \cdots + t_1e^{\gamma} + t_0) + S(r,f_1) \\ &\leq \overline{N}(r,0;t_me^{(m-1)\gamma} + t_{m-1}e^{(m-2)\gamma} + \cdots + t_1) + S(r,f_1) \\ &\leq (m-1)T(r,f_1) + S(r,f_1)\,, \end{aligned}$$

which is a contradiction. Hence $P_2(z_1)$ is reduced to a non-zero monomial, namely $P_2(z_1) = c_iz_1^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$ and so $P(z)$ takes the form $P(z) = c_i(z-e)^{d+i} \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$. This proves the lemma. □

Lemma 10. *Let f, g be two transcendental meromorphic functions and let $P(z)$ be defined as in (2.1). Let $F = \frac{[P(f)]^{(k)}}{p}$, $G = \frac{[P(g)]^{(k)}}{p}$, where $p(z)$ is a non-zero polynomial and n, $k \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$ such that $n > 3k + 2\,\Gamma + 3$. If f, g share $(\infty, 0)$ and $H \equiv 0$, then either $[P(f)]^{(k)}[P(f)]^{(k)} \equiv p^2$, where $[P(f)]^{(k)}$ and $[P(f)]^{(k)}$ share p CM or $P(f) \equiv P(g)$.*

Proof. Since $H \equiv 0$, on integration we get

$$\frac{1}{F-1} = \frac{bG + a - b}{G-1}\,, \tag{3.10}$$

where a, b are constants and $a \neq 0$. From (3.10), we see that F and G share 1 CM. We now consider the following cases.

Case 1. Let $b \neq 0$ and $a \neq b$. If $b = -1$, then from (3.10) we have

$$F = \frac{-a}{G-a-1}\,.$$

Therefore

$$\overline{N}(r,a+1;G) = \overline{N}(r,\infty;F) = \overline{N}(r,\infty;f) + S(r,f)\,.$$

So in view of Lemmas 1 and 2 for $p = 1$ and using the second fundamental theorem we get

$$\begin{aligned} n\,T(r,g) &= T\big(r,P(g)\big) + S(r,g) \\ &\le T\big(r,[P(g)]^{(k)}\big) + N_{k+1}\big(r,0;P(g)\big) - \overline{N}\big(r,0;[P(g)]^{(k)}\big) + S(r,g) \\ &\le T(r,G) + N_{k+1}\big(r,0;P(g)\big) - \overline{N}(r,0;G) + S(r,g) \\ &\le \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,a+1;G) + N_{k+1}\big(r,0;P(g)\big) \\ &\quad - \overline{N}(r,0;G) + S(r,g) \\ &\le \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + N_{k+1}\big(r,0;P(g)\big) + S(r,f) + S(r,g) \\ &\le 2\,\overline{N}(r,\infty;g) + (k+1)\,\overline{N}(r,e;f) + \Gamma\,T(r,g) + S(r,f) + S(r,g) \\ &\le (k+\Gamma+3)\,T(r,g) + S(r,f) + S(r,g)\,, \end{aligned}$$

which is a contradiction since $n > k+\Gamma+3$. If $b \ne -1$, from (3.10) we obtain that

$$F - \Big(1+\frac{1}{b}\Big) = \frac{-a}{b^2[G+\frac{a-b}{b}]}\,.$$

So

$$\overline{N}\Big(r,\frac{b-a}{b};G\Big) = \overline{N}(r,\infty;F) = \overline{N}(r,\infty;f) + S(r,f)\,.$$

Using Lemmas 1, 2 and the same argument as used in the case when $b=-1$ we can get a contradiction.

Case 2. Let $b \ne 0$ and $a = b$. If $b = -1$, then from (3.10) we have $FG \equiv 1$, i.e.,

$$[P(f)]^{(k)}[P(g)]^{(k)} \equiv p^2\,,$$

where $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$ share p CM. If $b \ne -1$, from (3.10) we have

$$\frac{1}{F} = \frac{bG}{(1+b)G-1}\,.$$

Therefore

$$\overline{N}\Big(r,\frac{1}{1+b};G\Big) = \overline{N}(r,0;F)\,.$$

So in view of Lemmas 1 and 2 for $p = 1$ and using the second fundamental theorem we get

$$\begin{aligned} n\,T(r,g) &= T\big(r,P(g)\big) + S(r,g) \\ &\le T\big(r,[P(g)]^{(k)}\big) + N_{k+1}\big(r,0;P(g)\big) - \overline{N}\big(r,0;[P(g)]^{(k)}\big) + S(r,g) \\ &\le T(r,G) + N_{k+1}\big(r,0;P(g)\big) - \overline{N}(r,0;G) + S(r,g) \\ &\le \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}\Big(r,\frac{1}{1+b};G\Big) + N_{k+1}\big(r,0;P(g)\big) \\ &\quad - \overline{N}(r,0;G) + S(r,g) \end{aligned}$$

$$\begin{aligned}
&\leq \overline{N}(r,\infty;g) + N_{k+1}\big(r,0;P(g)\big) + \overline{N}(r,0;F) + S(r,g)\\
&\leq \overline{N}(r,\infty;g) + N_{k+1}\big(r,0;P(g)\big) + N_{k+1}\big(r,0;P(f)\big)\\
&\quad + k\,\overline{N}(r,\infty;f) + S(r,f) + S(r,g)\\
&\leq (k+\Gamma+2)\,T(r,g) + (2k+\Gamma+1)\,T(r,f) + S(r,f) + S(r,g)\,.
\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leq T(r,g)$ for $r \in I$. So for $r \in I$ we have

$$(n-3k-2\,\Gamma-3)\,T(r,g) \leq S(r,g),$$

which is a contradiction since $n > 3k + 2\,\Gamma + 3$.

Case 3. Let $b = 0$. From (3.10) we obtain

$$F = \frac{G+a-1}{a}\,. \tag{3.11}$$

If $a \neq 1$ then from (3.11) we obtain $\overline{N}(r,1-a;G) = \overline{N}(r,0;F)$. We can similarly deduce a contradiction as in Case 2. Therefore $a = 1$ and from (3.11) we obtain $F \equiv G$, i.e., $[P(f)]^{(k)} \equiv [P(g)]^{(k)}$. Then by Lemma 8 we have $P(f) \equiv P(g)$. This completes the proof. □

Lemma 11 ([7, Lemma 3.5])**.** *Suppose that F is meromorphic in a domain D and set $f = \frac{F'}{F}$. Then for $n \geq 1$,*

$$\frac{F^{(n)}}{F} = f^n + \frac{n(n-1)}{2} f^{n-2} f' + a_n f^{n-3} f'' + b_n f^{n-4} (f')^2 + P_{n-3}(f)\,,$$

where $a_n = \frac{1}{6}n(n-1)(n-2)$, $b_n = \frac{1}{8}n(n-1)(n-2)(n-3)$ and $P_{n-3}(f)$ is a differential polynomial with constant coefficients, which vanishes identically for $n \leq 3$ and has degree $n-3$ when $n > 3$.

Lemma 12 ([3])**.** *Let f be a meromorphic function on $\mathbb{C}$ with finitely many poles. If f has bounded spherical derivative on $\mathbb{C}$, then f is of order at most 1.*

Lemma 13 ([20, Theorem 2.11])**.** *Let f be a transcendental meromorphic function in the complex plane such that $\rho(f) > 0$. If f has two distinct Borel exceptional values in the extended complex plane, then $\mu(f) = \rho(f)$ and $\rho(f)$ is a positive integer or ∞.*

Lemma 14 ([22])**.** *Let F be a family of meromorphic functions in the unit disc Δ such that all zeros of functions in F have multiplicity greater than or equal to l and all poles of functions in F have multiplicity greater than or equal to j and α be a real number satisfying $-l < \alpha < j$. Then F is not normal in any neighborhood of $z_0 \in \Delta$, if and only if there exist*

(i) *points $z_n \in \Delta$, $z_n \to z_0$,*

(ii) *positive numbers ρ_n, $\rho_n \to 0^+$ and*

(iii) *functions $f_n \in F$,*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) \to g(\zeta)$ spherically locally uniformly in $\mathbb{C}$, where g is a non-constant meromorphic function. The function g may be taken to satisfy the normalisation $g^{\#}(\zeta) \leq g^{\#}(0) = 1 (\zeta \in \mathbb{C})$.

Remark 3. Suppose in Lemma 14 that F is a family of holomorphic functions in the domain D and there exists a number $A \geq 1$ such that $|f^{(k)}(z)| \leq A$, whenever $f = 0$. Then the real number α in Lemma 14 can be such that $0 \leq \alpha \leq k$. In that case also $f_n(z_n + \rho_n\zeta) \rightarrow g(\zeta)$ spherically locally uniformly in $\mathbb{C}$, where g is a non-constant holomorphic function. The function g may be taken to satisfy the normalisation $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1 (\zeta \in \mathbb{C})$.

Lemma 15 ([20]). *Let f_j $(j = 1, 2, 3)$ be meromorphic functions, where f_1 be non-constant. Suppose that*

$$\sum_{j=1}^{3} f_j \equiv 1$$

and

$$\sum_{j=1}^{3} N(r, 0; f_j) + 2\sum_{j=1}^{3} \overline{N}(r, \infty; f_j) < (\lambda + o(1))T(r),$$

as $r \longrightarrow +\infty$, $r \in I$, $\lambda < 1$ and $T(r) = \max_{1\leq j\leq 3} T(r, f_j)$. Then $f_2 \equiv 1$ or $f_3 \equiv 1$.

Lemma 16. *Let f, g be two transcendental entire functions such that f and g have no zeros and p be a non-constant polynomial. Suppose $(f^n)'(g^n)' \equiv p^2$, where $n \in \mathbb{N}$. Now*
(i) if $p(z)$ is not a constant, then $f(z) = d_1 e^{cQ(z)}$, $g(z) = d_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)dt$, d_1, d_2 and c are constants such that $(nc)^2(d_1d_2)^n = -1$,
(ii) if $p(z)$ is a non-zero constant, say b, then $f(z) = d_3 e^{cz}$, $g(z) = d_4 e^{-cz}$, where d_3, d_4 and c are constants such that $(-1)^k(d_3d_4)^n(nc)^{2k} = b^2$.

Proof. Proof of lemma follows from proof of Theorem 1.3 [24]. □

Lemma 17. *Let f, g be two transcendental meromorphic functions such that f, g share $(\infty, 0)$ and p be a non-zero polynomial. Let n, $k \in \mathbb{N}$ such that $n > k$. Suppose $(f^n)^{(k)}(g^n)^{(k)} \equiv p^2$, where $(f^n)^{(k)} - p(z)$ and $(g^n)^{(k)} - p(z)$ share 0 CM. Now*
(i) *if $p(z)$ is not a constant, then $f(z) = d_1 e^{cQ(z)}$, $g(z) = d_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)dt$, d_1, d_2 and c are constants such that $(nc)^2(d_1d_2)^n = -1$,*
(ii) *if $p(z)$ is a non-zero constant, say b, then $f(z) = d_3 e^{cz}$, $g(z) = d_4 e^{-cz}$, where d_3, d_4 and c are constants such that $(-1)^k(d_3d_4)^n(nc)^{2k} = b^2$.*

Proof. Suppose

$$(f^n)^{(k)}(g^n)^{(k)} \equiv p^2\,. \tag{3.12}$$

Since f and g share $(\infty, 0)$, from (3.12) one can easily say that f and g are transcendental entire functions. Let

$$F_1 = \frac{(f^n)^{(k)}}{p} \quad \text{and} \quad G_1 = \frac{(g^n)^{(k)}}{p}\,. \tag{3.13}$$

From (3.12) we get

$$F_1G_1 \equiv 1\,. \tag{3.14}$$

If $F_1 \equiv c_1^* G_1$, where $c_1^* \in \mathbb{C} \setminus \{0\}$, then from (3.14) we have F_1 is a constant and so f is a polynomial, which contradicts our assumption. Hence $F_1 \not\equiv c_1^* G_1$.
Let

$$\Phi = \frac{(f^n)^{(k)} - p}{(g^n)^{(k)} - p} \,. \tag{3.15}$$

We deduce from (3.15) that

$$\Phi \equiv e^{\gamma_1} \,, \tag{3.16}$$

where γ_1 is an entire function. Let $f_1 = F_1$, $f_2 = -e^{\gamma_1} G_1$ and $f_3 = e^{\gamma_1}$. Here f_1 is transcendental. Now from (3.15) and (3.16), we have

$$f_1 + f_2 + f_3 \equiv 1 \,.$$

Hence by Lemma 6 we get

$$\begin{aligned}\sum_{j=1}^{3} N(r, 0; f_j) + 2\sum_{j=1}^{3} \overline{N}(r, \infty; f_j) &\le N(r, 0; F_1) + N(r, 0; e^{\gamma_1} G_1) + O(\log r)\\ &\le \big(\lambda + o(1)\big) T(r) \,,\end{aligned}$$

as $r \longrightarrow +\infty$, $r \in I$, $\lambda < 1$ and $T(r) = \max_{1 \le j \le 3} T(r, f_j)$.
So by Lemma 15, we get either $e^{\gamma_1} G_1 \equiv -1$ or $e^{\gamma_1} \equiv 1$. But here the only possibility is that $e^{\gamma_1} G_1 \equiv -1$, i.e., $(g^n)^{(k)} \equiv -e^{-\gamma_1} p(z)$ and so from (3.12) we get

$$(f^n)^{(k)} \equiv c_2^* e^{\gamma_1} p \,, \qquad (g^n)^{(k)} \equiv c_2^* e^{-\gamma_1} p \,, \tag{3.17}$$

where $c_2^* = \pm 1$. This shows that $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 0 CM. Let z_p be a zero of $f(z)$ of multiplicity p and z_q be a zero of $g(z)$ of multiplicity q. Since $n > k$, it follows that z_p will be a zero of $(f^n(z))^{(k)}$ of multiplicity $np - k$ and z_q will be a zero of $(g^n(z))^{(k)}$ of multiplicity $nq - k$. Since $(f^n(z))^{(k)}$ and $(g^n(z))^{(k)}$ share 0 CM, it follows that $z_p = z_q$ and $p = q$. Consequently $f(z)$ and $g(z)$ share 0 CM. Since $N(r, 0; f) = O(\log r)$ and $N(r, 0; g) = O(\log r)$, so we can take

$$f(z) = h_1(z) e^{\alpha(z)} \,, \qquad g(z) = h_1(z) e^{\beta(z)} \,, \tag{3.18}$$

where h_1 is a non-zero polynomial and α, β are two non-constant entire functions. We consider the following cases.
Case 1. Suppose 0 is a Picard exceptional value of both f and g.
We now consider the following sub-cases.
Sub-case 1.1. Let $\deg(p) = l \in \mathbb{N}$.
Since $N(r, 0; f) = 0$ and $N(r, 0; g) = 0$, so we can take

$$f(z) = e^{\alpha(z)} \,, \qquad g(z) = e^{\beta(z)} \,, \tag{3.19}$$

where α and β are two non-constant entire functions.

We deduce from (3.12) and (3.19) that either both α and β are transcendental entire functions or both are polynomials. We consider the following sub-cases.

Sub-case 1.1.1. Let $k \in \mathbb{N} \setminus \{1\}$.
First we suppose both α and β are transcendental entire functions. Note that

$$S(r, n\alpha') = S\Big(r, \frac{(f^n)'}{f^n}\Big), \quad S(r, n\beta') = S\Big(r, \frac{(g^n)'}{g^n}\Big).$$

Moreover we see that

$$N(r, 0; (f^n)^{(k)}) \le N(r, 0; p^2) = O(\log r).$$

$$N(r, 0; (g^n)^{(k)}) \le N(r, 0; p^2) = O(\log r).$$

From these and using (3.19) we have

$$N(r, \infty; f^n) + N(r, 0; f^n) + N\big(r, 0; (f^n)^{(k)}\big) = S(r, n\alpha') = S\Big(r, \frac{(f^n)'}{f^n}\Big) \tag{3.20}$$

and

$$N(r, \infty; g^n) + N(r, 0; g^n) + N\big(r, 0; (g^n)^{(k)}\big) = S(r, n\beta') = S\Big(r, \frac{(g^n)'}{g^n}\Big). \tag{3.21}$$

Then from (3.20), (3.21) and Lemma 4 we must have

$$f(z) = e^{a_3^* z + b_3^*}, \qquad g(z) = e^{c_3^* z + d_3^*}, \tag{3.22}$$

where $a_3^* \neq 0$, b_3^*, $c_3^* \neq 0$ and d_3^* are constants. But these types of f and g do not agree with the relation (3.12).

Next we suppose α and β are both non-constant polynomials.
Also from (3.12) we get $\alpha + \beta \equiv C_1$, i.e., $\alpha' \equiv -\beta'$. Therefore $\deg(\alpha) = \deg(\beta)$. If $\deg(\alpha) = \deg(\beta) = 1$, then we again get a contradiction from (3.12). Next we suppose $\deg(\alpha) = \deg(\beta) \ge 2$. Now from (3.19) and Lemma 11 we see that

$$(f^n)^{(k)} = \Big[n^k(\alpha')^k + \frac{k(k-1)}{2} n^{k-1}(\alpha')^{k-2}\alpha'' + P_{k-1}(\alpha')\Big] e^{n\alpha}.$$

Similarly we have

$$\begin{aligned}(g^n)^{(k)} &= \Big[n^k(\beta')^k + \frac{k(k-1)}{2} n^{k-1}(\beta')^{k-2}\beta'' + P_{k-1}(\beta')\Big] e^{n\beta} \\ &= \Big[(-1)^k n^k(\alpha')^k - \frac{k(k-1)}{2} n^{k-1}(-1)^{k-2}(\alpha')^{k-2}\alpha'' + P_{k-1}(-\alpha')\Big] e^{n\alpha}.\end{aligned}$$

Since $\deg(\alpha) \ge 2$, we observe that $\deg((\alpha')^k) \ge k\ \deg(\alpha')$ and so $(\alpha')^{k-2}\alpha''$ is either a non-zero constant or $\deg((\alpha')^{k-2}\alpha'') \ge (k-1)\ \deg(\alpha') - 1$. Also we see that

$$\deg\big((\alpha')^k\big) > \deg\big((\alpha')^{k-2}\alpha''\big) > \deg\big(P_{k-2}(\alpha')\big) \quad (\text{or}\ \ \deg\big(P_{k-2}(-\alpha')\big)).$$

Let

$$[\alpha(z)]' = e_{1t}z^t + e_{1t-1}z^{t-1} + \dots + e_{10},$$

where $e_{1t} \in \mathbb{C} \setminus \{0\}$. Then we have

$$\big([\alpha(z)]'\big)^i = e_{1t}^i z^{it} + i e_{1t}^{i-1} e_{1t-1} z^{it-1} + \dots,$$

where $i \in \mathbb{N}$. Therefore we have

$$(f^n)^{(k)} = \big[n^k e_{1t}^k z^{kt} + kn^k e_{1t}^{k-1} e_{1t-1} z^{kt-1} + \dots + (D_1 + D_2) z^{kt-t-1} + \dots\big] e^{n\alpha}$$

and

$$\begin{aligned}(g^n)^{(k)} = \big[&(-1)^k n^k e_{1t}^k z^{kt} + (-1)^k kn^k e_{1t}^{k-1} e_{1t-1} z^{kt-1} + \dots \\ &+ \{(-1)^k D_1 + (-1)^{k-1} D_2\} z^{kt-t-1} + \dots\big] e^{n\beta},\end{aligned}$$

where $D_1,\ D_2 \in \mathbb{C}$ such that $D_2 = \frac{k(k-1)}{2} tn^{k-1} e_{1t}^{k-1}$. Since $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 0 CM, we have

$$\begin{aligned}n^k e_{1t}^k z^{kt} &+ kn^k e_{1t}^{k-1} e_{1t-1} z^{kt-1} + \dots + (D_1 + D_2) z^{kt-t-1} + \dots \\ &= d_1^* \big\{(-1)^k n^k e_{1t}^k z^{kt} + (-1)^k kn^k e_{1t}^{k-1} e_{1t-1} z^{kt-1} + \cdots \\ &\quad + \{(-1)^k D_1 + (-1)^{k-1} D_2\} z^{kt-t-1} + \dots\big\}\end{aligned} \tag{3.23}$$

where $d_1^* \in \mathbb{C} \setminus \{0\}$. From (3.23) we get $D_2 = 0$, i.e.,

$$\frac{k(k-1)}{2} tn^{k-1} e_{1t}^{k-1} = 0,$$

which is impossible for $k \geq 2$.

Sub-case 1.1.2. Let $k = 1$. Remaining part follows from Lemma 16.

Sub-case 1.2. Let $p(z) = b \in \mathbb{C} \setminus \{0\}$. Since $n > 2k$, we have $f \neq 0$ and $g \neq 0$. Now using Sub-case 1.1 we can prove that $f = e^\alpha$ and $g = e^\beta$, where α and β are non-constant entire functions. We now consider the following two sub-cases.

Sub-case 1.2.1. Let $k \geq 2$. We see that $N(r, 0; (f^n)^{(k)}) = 0$. Clearly

$$f^n(z)\big(f^n(z)\big)^{(k)} \neq 0, \quad g^n(z)\big(g^n(z)\big)^{(k)} \neq 0. \tag{3.24}$$

Then from (3.24) and Lemma 5 we must have $f(z) = e^{az+b}$, $g(z) = e^{cz+d}$, where $a \neq 0$, b, $c \neq 0$ and d are constants. From (3.12) it is clear that $a + c = 0$. Therefore f and g take the forms $f(z) = d_3 e^{cz}$, $g(z) = d_4 e^{-cz}$, where d_3, d_4, $c \in \mathbb{C}$ such that $(-1)^k (d_3 d_4)^n (nc)^{2k} = b^2$.

Sub-case 1.2.2. Let $k = 1$. Remaining part follows from Lemma 16.

Case 2. Suppose 0 is not a Picard exceptional value of f and g.

Let $H = f^n$, $\hat{H} = g^n$, $F = \frac{H}{p}$ and $G = \frac{\hat{H}}{p}$. Let $\mathcal{F} = \{F_\omega\}$ and $\mathcal{G} = \{G_\omega\}$, where $F_\omega(z) = F(z+\omega) = \frac{H(z+\omega)}{p(z+\omega)}$ and $G_\omega(z) = G(z+\omega) = \frac{\hat{H}(z+\omega)}{p(z+\omega)}$, $z \in \mathbb{C}$. Clearly $\mathcal{F}$ and $\mathcal{G}$ are two families of meromorphic functions defined on $\mathbb{C}$. We now consider following two sub-cases.

Sub-case 2.1. Suppose that one of the families $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{F}$, is normal on $\mathbb{C}$. Then by Marty's theorem $F^{\#}(\omega) = F^{\#}_\omega(0) \leq M$ for some $M > 0$ and for all $\omega \in \mathbb{C}$. Hence by Lemma 12 we have F is of order at most 1. Now from (3.12) we have

$$\begin{aligned}\rho(f) = \rho\Big(\frac{f^n}{p}\Big) &= \rho(f^n) = \rho\big((f^n)^{(k)}\big) = \rho\big((g^n)^{(k)}\big) \\ &= \rho(g^n) = \rho\Big(\frac{g^n}{p}\Big) = \rho(g) \leq 1.\end{aligned} \tag{3.25}$$

Noting that f and g are transcendental entire functions, we observe from (3.25) and Lemma 13 that $\mu(f)=\rho(f)=1$. Now from (3.18) we have

$$f=h_1e^{\alpha}\,,\quad g=h_1e^{\beta}\,, \tag{3.26}$$

where α and β are non-constant polynomials with degree 1. From (3.12) we see that $\alpha+\beta\equiv C_1$ where C_1 is a constant and so $\alpha'+\beta'\equiv 0$. Again from (3.26) we have

$$\big(f^n(z)\big)^{(k)}=e^{n\alpha}\sum_{i=0}^{k}{}^kC_i(n\alpha')^{k-i}\big(h_1^n(z)\big)^{(i)},$$

where we define $(h_1^n(z))^{(0)}=h_1^n(z)$. Similarly we have

$$\big(g^n(z)\big)^{(k)}=e^{n\beta}\sum_{i=0}^{k}{}^kC_i(-1)^{k-i}(n\alpha')^{k-i}\big(h_1^n(z)\big)^{(i)}.$$

Since $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 0 CM, it follows that

$$\sum_{i=0}^{k}{}^kC_i(n\alpha')^{k-i}\big(h_1^n(z)\big)^{(i)}\equiv d_2^*\sum_{i=0}^{k}{}^kC_i(-1)^{k-i}(n\alpha')^{k-i}\big(h_1^n(z)\big)^{(i)}\,, \tag{3.27}$$

where $d_2^*\in\mathbb{C}\setminus\{0\}$. But from (3.27) we arrive at a contradiction.

Sub-case 2.2. Suppose that one of the families $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{F}$ is not normal on $\mathbb{C}$. Then there exists at least one $z_0\in\Delta$ such that $\mathcal{F}$ is not normal z_0, we assume that $z_0=0$. Now by Marty's theorem there exists a sequence of meromorphic functions $\{F(z+\omega_j)\}\subset\mathcal{F}$, where $z\in\{z:|z|<1\}$ and $\{\omega_j\}\subset\mathbb{C}$ is some sequence of complex numbers such that

$$F^{\#}(\omega_j)\to\infty\,,$$

as $|\omega_j|\to\infty$. Note that p has only finitely many zeros. So there exists a $r>0$ such that $p(z)\neq 0$ in $D=\{z:|z|\geq r\}$. Since $p(z)$ is a polynomial, for all $z\in\mathbb{C}$ satisfying $|z|\geq r$, we have

$$0\leftarrow\Big|\frac{p'(z)}{p(z)}\Big|\leq\frac{M_1}{|z|}<1\,,\quad p(z)\neq 0\,. \tag{3.28}$$

Also since $w_j\to\infty$ as $j\to\infty$, without loss of generality we may assume that $|w_j|\geq r+1$ for all j. Let $D_1=\{z:|z|<1\}$ and

$$F(w_j+z)=\frac{H(w_j+z)}{p(w_j+z)}\,.$$

Since $|w_j+z|\geq|w_j|-|z|$, it follows that $w_j+z\in D$ for all $z\in D_1$. Also since $p(z)\neq 0$ in D, it follows that $p(\omega_j+z)\neq 0$ in D_1 for all j. Observing that $F(z)$ is analytic in D, so $F(\omega_j+z)$ is analytic in D_1. Therefore all $F(\omega_j+z)$ are analytic in D_1. Also from (3.17) we see that every zeros of $h_1(z)$ must be the zeros of $p(z)$. Thus we have structured a family $\{F(\omega_j+z)\}$ of holomorphic functions such that $F(\omega_j+z)\neq 0$ in D_1 for all j.

Then by Lemma 14 there exist

(i) points z_j, $|z_j|<1$,

(ii) positive numbers ρ_j, $\rho_j \to 0^+$,

(iii) a subsequence $\{F(\omega_j + z_j + \rho_j\zeta)\}$ of $\{F(\omega_j + z)\}$

such that

$$h_j(\zeta) = \rho_j^{-k} F(\omega_j + z_j + \rho_j\zeta) \to h(\zeta)\,,$$

i.e.,

$$h_j(\zeta) = \rho_j^{-k} \frac{H(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h(\zeta) \tag{3.29}$$

spherically locally uniformly in $\mathbb{C}$, where $h(\zeta)$ is some non-constant holomorphic function such that $h^{\#}(\zeta) \leq h^{\#}(0) = 1$. Now from Lemma 12 we see that $\rho(h) \leq 1$. By Hurwitz's theorem we can see that $h(\zeta) \neq 0$. In the proof of Zalcman's lemma (see [14, 21]) we see that

$$\rho_j = \frac{1}{F^{\#}(b_j)} \tag{3.30}$$

and

$$F^{\#}(b_j) \geq F^{\#}(\omega_j)\,, \tag{3.31}$$

where $b_j = \omega_j + z_j$. Note that

$$\frac{p'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to 0\,, \tag{3.32}$$

as $j \to \infty$. We now prove that

$$\big(h_j(\zeta)\big)^{(k)} = \frac{H^{(k)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h^{(k)}(\zeta)\,. \tag{3.33}$$

Note that from (3.29)

$$\begin{aligned}\rho_j^{-k+1} \frac{H'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} &= h_j'(\zeta) + \rho_j^{-k+1} \frac{p'(\omega_j + z_j + \rho_j\zeta)}{p^2(\omega_j + z_j + \rho_j\zeta)} H(\omega_j + z_j + \rho_j\zeta)\\ &= h_j'(\zeta) + \rho_j \frac{p'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} h_j(\zeta)\,. \qquad (3.34)\end{aligned}$$

Now from (3.29), (3.32) and (3.34) we observe that

$$\rho_j^{-k+1} \frac{H'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h'(\zeta)\,.$$

Suppose

$$\rho_j^{-k+l} \frac{H^{(l)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h^{(l)}(\zeta)\,.$$

Let

$$G_j(\zeta) = \rho_j^{-k+l} \frac{H^{(l)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)}\,.$$

Then

$$G_j(\zeta) \to h^{(l)}(\zeta)\,.$$

Note that

$$\rho_j^{-k+l+1}\frac{H^{(l+1)}(\omega_j+z_j+\rho_j\zeta)}{p(\omega_j+z_j+\rho_j\zeta)} = G_j'(\zeta)+\rho_j^{-k+l+1}\frac{p'(\omega_j+z_j+\rho_j\zeta)}{p^2(\omega_j+z_j+\rho_j\zeta)}H^{(l)}(\omega_j+z_j+\rho_j\zeta)$$

$$= G_j'(\zeta)+\rho_j\frac{p'(\omega_j+z_j+\rho_j\zeta)}{p(\omega_j+z_j+\rho_j\zeta)}G_j(\zeta)\,. \tag{3.35}$$

So from (3.32) and (3.35) we see that

$$\rho_j^{-k+l+1}\frac{H^{(l+1)}(\omega_j+z_j+\rho_j\zeta)}{p(\omega_j+z_j+\rho_j\zeta)} \to G_j'(\zeta)\,,$$

i.e.,

$$\rho_j^{-k+l+1}\frac{H^{(l+1)}(\omega_j+z_n+\rho_j\zeta)}{p(\omega_j+z_j+\rho_j\zeta)} \to h_j^{(l+1)}(\zeta)\,.$$

Then by mathematical induction we get the desired result (3.33). Let

$$\big(\hat{h}_j(\zeta)\big)^{(k)} = \frac{\hat{H}^{(k)}(\omega_j+z_j+\rho_j\zeta)}{p(\omega_j+z_j+\rho_j\zeta)}\,. \tag{3.36}$$

From (3.12) we have

$$\frac{H^{(k)}(\omega_j+z_j+\rho_j\zeta)}{p(\omega_j+z_j+\rho_j\zeta)}\frac{\hat{H}^{(k)}(\omega_j+z_j+\rho_j\zeta)}{p(\omega_j+z_j+\rho_j\zeta)} = 1$$

and so from (3.33) and (3.36) we get

$$\big(h_j(\zeta)\big)^{(k)}\big(\hat{h}_j(\zeta)\big)^{(k)} = 1\,. \tag{3.37}$$

Now from (3.33), (3.37) and the formula of higher derivatives we can deduce that

$$\hat{h}_j(\zeta) \to \hat{h}(\zeta)$$

i.e.,

$$\frac{\hat{H}(\omega_j+z_j+\rho_j\zeta)}{p(\omega_j+z_j+\rho_j\zeta)} \to \hat{h}(\zeta)\,, \tag{3.38}$$

spherically locally uniformly in $\mathbb{C}$, where $\hat{h}(\zeta)$ is some non-constant holomorphic function in the complex plane. By Hurwitz's theorem we can see that $\hat{h}(\zeta) \neq 0$. Therefore (3.38) can be rewritten as

$$\big(\hat{h}_j(\zeta)\big)^{(k)} \to \big(\hat{h}(\zeta)\big)^{(k)} \tag{3.39}$$

spherically locally uniformly in $\mathbb{C}$. From (3.33), (3.37) and (3.39) we get

$$\big(h(\zeta)\big)^{(k)}\big(\hat{h}(\zeta)\big)^{(k)} \equiv 1\,. \tag{3.40}$$

Now from (3.40) and $\rho(h) \le 1$ we see that

$$\rho(h) = \rho(h^{(k)}) = \rho(\hat{h}^{(k)}) = \rho(\hat{h}) \le 1\,. \tag{3.41}$$

Noting that $\bar{h}$ and $\hat{h}$ are transcendental entire functions, we observe from (3.41) and Lemma 13 that $\mu(h) = \rho(\bar{h}) = 1$. Therefore we have

$$h(z) = c_1 e^{cz}\,, \quad \hat{h}(z) = \hat{c}_2 e^{-cz}\,, \tag{3.42}$$

where c_1, $\hat{c}_2$ and c are non-zero constants satisfying $(-1)^k(c_1\hat{c}_2)(c)^{2k} = 1$. Also from (3.42) we have

$$\frac{h_j'(\zeta)}{h_j(\zeta)} = \rho_j \frac{F'(w_j + z_j + \rho_j\zeta)}{F(w_j + z_j + \rho_j\zeta)} \to \frac{h'(\zeta)}{h(\zeta)} = c\,, \tag{3.43}$$

spherically locally uniformly in $\mathbb{C}$. From (3.30) and (3.43) we get

$$\begin{aligned}\rho_j \Big|\frac{F'(\omega_j + z_j)}{F(\omega_j + z_j)}\Big| &= \frac{1 + |F(\omega_j + z_j)|^2}{|F'(\omega_j + z_j)|}\frac{|F'(\omega_j + z_j)|}{|F(\omega_j + z_j)|}\\ &= \frac{1 + |F(\omega_j + z_j)|^2}{|F(\omega_j + z_j)|} \to \Big|\frac{h'(0)}{h(0)}\Big| = |c|\,,\end{aligned}$$

which implies that

$$\lim_{j\to\infty} F(\omega_j + z_j) \neq 0\,,\ \infty\,. \tag{3.44}$$

From (3.29) and (3.44) we see that

$$h_j(0) = \rho_j^{-k} F(\omega_j + z_j) \to \infty\,. \tag{3.45}$$

Again from (3.29) and (3.42) we have

$$h_j(0) \to h(0) = c_1\,. \tag{3.46}$$

Now from (3.45) and (3.46) we arrive at a contradiction. This completes the lemma. □

Lemma 18. *Let f and g be two transcendental meromorphic functions and let $d(\geq 1), m(\geq 0), k(\geq 1)$ be three integers such that $d > k$. Let $P(z)$ be defined as in (2.1) and $p(z)$ be a non-zero polynomial. Suppose $[P(f)]^{(k)}[P(g)]^{(k)} \equiv p^2$, where $[P(f)]^{(k)}$, $[P(g)]^{(k)}$ share p CM and f, g share $(\infty, 0)$, then $P_2(z_1)$ is reduced to a non-zero monomial, namely $P_2(z_1) = c_i z_1^i \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$ and so $P(z)$ takes the form $P(z) = c_i(z - e)^{d+i} \not\equiv 0$ for some $i \in \{0, 1, \ldots, m\}$; if $p(z)$ is not a constant, then $f(z) - e = d_1 e^{c^*Q(z)}$, $g(z) - e = d_2 e^{-c^*Q(z)}$, where $Q(z) = \int_0^z p(t)dt$, d_1, d_2 and c^* are constants such that $c_i^2(d_1d_2)^{d+i}[(d+i)c^*]^2 = -1$, if $p(z)$ is a non-zero constant, say b, then $f(z) - e = d_3 e^{c^*z}$, $g(z) - e = d_4 e^{-c^*z}$, where d_3, d_4 and c^* are constants such that $(-1)^k c_i^2(d_3d_4)^{d+i}[(d + i)c^*]^{2k} = b^2$.*

Proof. The proof of lemma follows from Lemmas 9 and 17. □

Lemma 19 ([1])**.** *Let f and g be two non-constant meromorphic functions sharing $(1, k_1)$, where $2 \leq k_1 \leq \infty$. Then*

$$\begin{aligned}&\overline{N}(r, 1; f \mid= 2) + 2\,\overline{N}(r, 1; f\mid = 3) + \cdots + (k_1 - 1)\,\overline{N}(r, 1; f \mid= k_1) + k_1\,\overline{N}_L(r, 1; f)\\ &\quad + (k_1 + 1)\,\overline{N}_L(r, 1; g) + k_1\,\overline{N}_E^{(k_1+1}(r, 1; g) \leq N(r, 1; g) - \overline{N}(r, 1; g)\,.\end{aligned}$$

Lemma 20. *Suppose that f and g be two non-constant meromorphic functions. Let $F=[P(f)]^{(k)}$, $G=[P(g)]^{(k)}$, where n, $k\in\mathbb{N}$ and $P(z)$ be defined as in* (2.1)*. Suppose $H\not\equiv 0$. If f, g share $(\infty,0)$ and F, G share $(1,k_1)$, where $0\le k_1\le\infty$ then*

$$\big(n-k-1\big)\overline{N}(r,\infty;f)\le(k+\Gamma+1)\,\{T(r,f)+T(r,g)\}\\ +\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\,.$$

Proof. If ∞ is a Picard exceptional value of f and g, then the result follows immediately.
Next we suppose ∞ is not a Picard exceptional value of f and g. Since $H\not\equiv 0$, it follows that $F\not\equiv G$. We claim that $V\not\equiv 0$. If possible suppose $V\equiv 0$. Then by integration we obtain

$$1-\frac{1}{F}=A\Big(1-\frac{1}{G}\Big)\,.$$

Note that if z_1^* is a pole of f then it is a pole of g. Hence from the definition of F and G we have $\frac{1}{F(z_1^*)}=0$ and $\frac{1}{G(z_1^*)}=0$. So $A=1$ and hence $F\equiv G$, which is a contradiction.
We suppose that z_0 is a pole of f with multiplicity q and a pole of g with multiplicity r. Clearly z_0 is a pole of F with multiplicity $nq+k$ and a pole of G with multiplicity $nr+k$. Clearly $\frac{F'(z)}{F(z)(F(z)-1)}=O((z-z_0)^{nq+k-1})$ and $\frac{G'(z)}{G(z)(G(z)-1)}=O((z-z_0)^{nr+k-1})$. Consequently, $V=O((z-z_0)^{nt+k-1})$, where $t=\min\{q,r\}$. Noting that f, g share $(\infty,0)$, from the definition of V it is clear that z_0 is a zero of V with multiplicity at least $n+k-1$. Now using the Milloux theorem [7, p. 55], and Lemma 1, we obtain from the definition of V that $m(r,V)=S(r,f)+S(r,g)$. Thus using Lemma 1 and (3.4) we get

$$\begin{aligned}\big(n+k-1\big)\overline{N}(r,\infty;f)&\le N(r,0;V)\le T(r,V)+O(1)\le N(r,\infty;V)+m(r,V)+O(1)\\ &\le \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\ &\le N_{k+1}(r,0;P(f))+N_{k+1}(r,0;P(g))+k\overline{N}(r,\infty;f)\\ &\quad+k\overline{N}(r,\infty;g)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\ &\le N_{k+1}(r,0;P(f))+N_{k+1}(r,0;P(g))+2k\overline{N}(r,\infty;f)\\ &\quad+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\ &\le (k+\Gamma+1)\;T(r,f)+(k+\Gamma+1)\;T(r,g)+2k\overline{N}(r,\infty;f)\\ &\quad+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\,.\end{aligned}$$

This gives

$$\big(n-k-1\big)\overline{N}(r,\infty;f)\le(k+\Gamma+1)\,\{T(r,f)+T(r,g)\}\\ +\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\,.$$

This completes the proof. □

4. Proofs of the Theorem

Proof of Theorem 1. Let $F = \frac{[P(f)]^{(k)}}{p}$ and $G = \frac{[P(g)]^{(k)}}{p}$. Note that since f and g are transcendental meromorphic functions, p is a small function with respect to both $[P(f)]^{(k)}$ and $[P(g)]^{(k)}$. Also F, G share $(1, k_1)$ except for the zeros of p and f, g share $(\infty, 0)$.

Case 1. Let $H \not\equiv 0$.

From (3.1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) those poles of F and G whose multiplicities are different, (iv) zeros of $F'(G')$ which are not the zeros of $F(F-1)(G(G-1))$.

Since H has only simple poles we get

$$\begin{aligned} N(r,\infty;H) &\leq \overline{N}_*(r,\infty;F,G) + \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F|\geq 2) \\ &\quad + \overline{N}(r,0;G|\geq 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g)\,, \end{aligned} \tag{4.1}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Let z_0 be a simple zero of $F(z)-1$ but $p(z_0) \neq 0$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$N(r,1;F|=1) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,f) + S(r,g)\,. \tag{4.2}$$

Using (4.1) and (4.2) we get

$$\begin{aligned} \overline{N}(r,1;F) &\leq N(r,1;F|=1) + \overline{N}(r,1;F|\geq 2) \\ &\leq \overline{N}_*(r,\infty;f,g) + \overline{N}(r,0;F|\geq 2) + \overline{N}(r,0;G|\geq 2) + \overline{N}_*(r,1;F,G) \\ &\quad + \overline{N}(r,1;F|\geq 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g) \\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;F|\geq 2) + \overline{N}(r,0;G|\geq 2) + \overline{N}_*(r,1;F,G) \\ &\quad + \overline{N}(r,1;F|\geq 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,f) + S(r,g)\,. \end{aligned} \tag{4.3}$$

Now in view of Lemmas 19 and 3 we get

$$\begin{aligned} &\overline{N}_0(r,0;G') + \overline{N}(r,1;F\,|\geq 2) + \overline{N}_*(r,1;F,G) \\ &\leq \overline{N}_0(r,0;G') + \overline{N}(r,1;F|=2) + \overline{N}(r,1;F|=3) + \cdots + \overline{N}(r,1;F|=k_1) \\ &\quad + \overline{N}_E^{(k_1+1}(r,1;F) + \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}_*(r,1;F,G) \\ &\leq \overline{N}_0(r,0;G') - \overline{N}(r,1;F|=3) - \cdots - (k_1-2)\overline{N}(r,1;F|=k_1) \\ &\quad - (k_1-1)\overline{N}_L(r,1;F) - k_1\overline{N}_L(r,1;G) - (k_1-1)\overline{N}_E^{(k_1+1}(r,1;F) \\ &\quad + N(r,1;G) - \overline{N}(r,1;G) + \overline{N}_*(r,1;F,G) \\ &\leq \overline{N}_0(r,0;G') + N(r,1;G) - \overline{N}(r,1;G) - (k_1-2)\overline{N}_L(r,1;F) \\ &\quad - (k_1-1)\overline{N}_L(r,1;G) \\ &\leq N(r,0;G' \mid G \neq 0) - (k_1-2)\overline{N}_L(r,1;F) - (k_1-1)\overline{N}_L(r,1;G) \\ &\leq \overline{N}(r,0;G) + \overline{N}(r,\infty;g) - (k_1-2)\overline{N}_*(r,1;F,G) - \overline{N}_L(r,1;G)\,. \end{aligned} \tag{4.4}$$

Hence using (4.3), (4.4), Lemmas 2 and 20 we get from the second fundamental theorem that

$$\begin{aligned}
n\,T(r,f) &\le T(r,F) + N_{k+2}\big(r,0;P(f)\big) - N_2(r,0;F) + S(r,f)\\
&\le \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,1;F) + N_{k+2}\big(r,0;P(f)\big)\\
&\quad - N_2(r,0;F) - N_0(r,0;F') + S(r,f)\\
&\le \overline{N}(r,\infty,f) + \overline{N}(r,\infty;g) + \overline{N}(r,0;F) + N_{k+2}\big(r,0;P(f)\big)\\
&\quad + \overline{N}(r,0;F|\ge 2) + \overline{N}(r,0;G|\ge 2) + \overline{N}(r,1;F|\ge 2)\\
&\quad + \overline{N}_*(r,1;F,G) + \overline{N}_0(r,0;G^{'}) - N_2(r,0;F) + S(r,f) + S(r,g)\\
&\le 3\,\overline{N}(r,\infty;f) + N_{k+2}\big(r,0;P(f)\big) + N_2(r,0;G) - (k_1-2)\,\overline{N}_*(r,1;F,G)\\
&\quad - \overline{N}_L(r,1;G) + S(r,f) + S(r,g)\\
&\le 3\,\overline{N}(r,\infty;f) + N_{k+2}\big(r,0;P(f)\big) + k\,\overline{N}(r,\infty;g) + N_{k+2}\big(r,0;P(g)\big)\\
&\quad - (k_1-2)\,\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
&\le (3+k)\,\overline{N}(r,\infty;f) + (k+\Gamma+2)\,T(r,f) + (k+\Gamma+2)\,T(r,g)\\
&\quad - (k_1-2)\,\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
&\le (k+\Gamma+2)\,\{T(r,f)+T(r,g)\} + (3+k)\overline{N}(r,\infty;f)\\
&\quad - (k_1-2)\,\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
&\le (k+\Gamma+2)\,\{T(r,f)+T(r,g)\} + \frac{(3+k)(k+\Gamma+1)}{n-k-1}\,\{T(r,f)+T(r,g)\}\\
&\quad + \frac{3+k}{n-k-1}\,\overline{N}_*(r,1;F,G) - (k_1-2)\,\overline{N}_*(r,1;F,G) + S(r,f) + S(r,g)\\
&\le \Big[k+\Gamma+2+\frac{(3+k)(k+\Gamma+1)}{n-k-1}\Big]\,\{T(r,f)+T(r,g)\}\\
&\quad + S(r,f) + S(r,g)\,. \qquad (4.5)
\end{aligned}$$

In a similar way we can obtain

$$\begin{aligned}
n\,T(r,g) &\le \Big[k+\Gamma+2+\frac{(3+k)(k+\Gamma+1)}{n-k-1}\Big]\,\{T(r,f)+T(r,g)\}\\
&\quad + S(r,f) + S(r,g)\,. \qquad (4.6)
\end{aligned}$$

Adding (4.5) and (4.6) we get

$$\Big[n-2\Gamma-2k-4-\frac{(6+2k)(k+\Gamma+1)}{n-k-1}\Big]\,\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g)\,,$$

i.e.,

$$\Big[\frac{n^2-n(3k+2\Gamma+5)-(2k+2)}{n-k-1}\Big]\,\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g)\,. \qquad (4.7)$$

Note that

$$2\Gamma+3k+6 > \frac{2\Gamma+3k+5+\sqrt{(2\Gamma+3k+5)^2+4(2k+2)}}{2}\,.$$

Consequently when $n > 2\Gamma + 3k + 6$, we obtain a contradiction from (4.7).
Case 2. Let $H \equiv 0$. Then by Lemma 10 we have

$$[P(f)]^{(k)}[P(g)]^{(k)} \equiv p^2 \tag{4.8}$$

or

$$P(f) \equiv P(g)\,. \tag{4.9}$$

From (4.9) we get

$$f_1^d(c_m f_1^m + c_{m-1} f_1^{m-1} + \cdots + c_0) \equiv g_1^d(c_m g_1^m + c_{m-1} g_1^{m-1} + \cdots + c_0)\,. \tag{4.10}$$

Let $h = \frac{f_1}{g_1}$. If h is a constant, then substituting $f_1 = g_1 h$ into (4.10) we deduce that

$$c_m g_1^{d+m}(h^{d+m} - 1) + c_{m-1} g_1^{d+m-1}(h^{d+m-1} - 1) + \cdots + c_0 g_1^d(h^d - 1) \equiv 0\,,$$

which implies $h^{d_0} = 1$, where $d_0 = GCD(d+m, \ldots, d+m-i, \ldots, d)$, $c_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$. Thus $f_1 \equiv t g_1$, i.e., $f(z) - e \equiv t(g(z) - e)$ for a constant t such that $t^{d_0} = 1$, where $d_0 = GCD(d+m, \ldots, d+m-i, \ldots, d)$, $c_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$.

If h is not a constant, then from (4.10) we see that f_1 and g_1 satisfying the algebraic equation $R(f_1, g_1) = 0$, where $R(\omega_1, \omega_2) = \omega_1^d(c_m\omega_1^m + c_{m-1}\omega_1^{m-1} + \cdots + c_0) - \omega_2^d(c_m\omega_2^m + c_{m-1}\omega_2^{m-1} + \cdots + c_0)$.

Remaining part of the theorem follows from (4.8) and Lemma 18. This completes the proof. □

References

[1] Banerjee, A., *On a question of Gross*, J. Math. Anal. Appl. **327** (2) (2007), 1273–1283.

[2] Banerjee, A., *Uniqueness of certain nonlinear differential polynomials sharing 1-points*, Kyungpook Math. J. **51** (2011), 43–58.

[3] Chang, J.M., Zalcman, L., *Meromorphic functions that share a set with their derivatives*, J. Math. Anal. Appl. **338** (2008), 1191–1205.

[4] Fang, M.L., Qiu, H.L., *Meromorphic functions that share fixed points*, J. Math. Anal. Appl. **268** (2002), 426–439.

[5] Frank, G., *Eine Vermutung Von Hayman über Nullslellen meromorphic Funktion*, Math. Z. **149** (1976), 29–36.

[6] Hayman, W.K., *Picard values of meromorphic functions and their derivatives*, Ann. of Math. (2) **70** (1959).

[7] Hayman, W.K., *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.

[8] Lahir, I., Banerjee, A., *Weighted sharing of two sets*, Kyungpook Math. J. **46** (2006).

[9] Lahiri, I., *Value distribution of certain differential polynomials*, Internat. J. Math. Math. Sci. **28** (2001), 83–91.

[10] Lahiri, I., *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J. **161** (2001), 193–206.

[11] Lahiri, I., *Weighted value sharing and uniqueness of meromorphic functions*, Complex Variables Theory Appl. **46** (2001), 241–253.

[12] Lahiri, I., Dewan, S., *Value distribution of the product of a meromorphic function and its derivative*, Kodai Math. J. **26** (2003), 95–100.

[13] Majumder, S., *On an open problem of Xiao-Bin Zhang and Jun-Feng Xu*, Demonstratio Math. **49** (2) (2016), 161–182.

[14] Pang, X.C., *Normality conditions for differential polynomials*, Kexue Tongbao (Chinese) **33** (1988), 1690–1693.

[15] Schiff, J., *Normal families*, Berlin, 1993.

[16] Xu, J.F., Lü, F., Yi, H.X., *Fixed points and uniqueness of meromorphic functions*, Comput. Math. Appl. **59** (2010), 9–17.

[17] Yamanoi, K., *The second main theorem for small functions and related problems*, Acta Math. **192** (2004), 225–294.

[18] Yang, C.C., *On deficiencies of differential polynomials II*, Math. Z. **125** (1972), 107–112.

[19] Yang, C.C., Hua, X.H., *Uniqueness and value sharing of meromorphic functions*, Ann. Acad. Sci. Fenn. Math. **22** (1997), 395–406.

[20] Yang, C.C., Yi, H.X., *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.

[21] Zalcman, L., *A heuristic principle in complex theory*, Amer. Math. Monthly **82** (1975), 813–817.

[22] Zalcman, L., *Normal families, new perspectives*, Bull. Amer. Math. Soc. **35** (1998), 215–230.

[23] Zhang, Q.C., *Meromorphic function that shares one small function with its derivative*, J. Inequal. Pure Appl. Math. **6** (4) (2005), Art.116 online `http://jipam.vu.edu.au/`.

[24] Zhang, X.B., Xu, J.F., *Uniqueness of meromorphic functions sharing a small function and its applications*, Comput. Math. Appl. **61** (2011), 722–730.

Department of Mathematics, Raiganj University,
Raiganj, West Bengal 733134, India
E-mail: `sujoy.katwa@gmail.com`, `sm05math@gmail.com`, `smajumder05@yahoo.in`
E-mail: `rajibmathresearch@gmail.com`, `rajib_math@yahoo.in`