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GROUP ALGEBRAS WHOSE GROUPS OF NORMALIZED UNITS HAVE EXPONENT 4

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Abstract. We give a full description of locally finite 2-groups G such that the normalized group of units of the group algebra FG over a field F of characteristic 2 has exponent 4.

Keywords: group of exponent 4; unit group; modular group algebra

MSC 2010: 16S34, 16U60

1. Introduction and result

It is well known that there does not exist an effective description of finite groups of prime square exponent p^2 (not even in the case when the exponent is 4). However, Janko (see for example [11], [12], [13]) was able to characterize these groups under certain additional restrictions on their structure. In this way he obtained interesting classes of finite p-groups.

Note also that there is no effective description of finite 2-groups with pairwise commuting involutions. On the other hand, the structure of a locally finite 2-group G is known when its normalized group of units $V(FG)$ of the group algebra FG has the property that its involutory units pairwise commute (see [4]).

There is a similar situation in the case of powerful p-groups. Despite of extensive current research of this field, their structure has been incompletely described. However, it is possible to determine [3] those cases when the normalized groups of units of the group algebras are powerful p-groups.

So it is a natural question whether it is possible to give a description of those modular group algebras whose groups of normalized units have exponent p^2. In this note we deal with the case of $p = 2$.

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In certain papers the question when the exponent of $V(FG)$ coincides with the exponent of the finite p-group G was studied. Mostly such results were obtained for the p-groups G in the case when $p \geqslant 5$ (for example, see Theorem D in [16], page 25, and Theorem 2.1 in [15], page 423). More generally this question was studied in [6]. Hence our result can be considered a complete answer to such questions in the case when the exponent of G is equal to 4.

Finally, note that the groups of exponent 4 appear in several problems in the group theory and in the theory of group ring units (for example, see [2], [7], [9], [14], [17]).

Our main reads as follows

Theorem. *Let $V(FG)$ be the normalized group of units of a group algebra FG of a locally finite 2-group G over a field F with $\mathrm{char}(F) = 2$. The group $V(FG)$ has exponent 4 if and only if G is a nilpotent group of class at most 2 with exponent 4, $|G'|$ divides 4 and the Frattini subgroup of G is central elementary abelian.*

2. Preliminaries and the proof of Theorem

An involution in a group G is an element of order 2. For any $a, b \in G$, we denote $(a, b) = a^{-1}b^{-1}ab$ and $a^b = b^{-1}ab$. Let D_8 and Q_8 be the dihedral and quaternion groups of order 8, respectively. Define the following groups:

$$\begin{aligned}
G_{16}^3 &= \langle g, h \colon\ g^4 = h^2 = 1,\ (g^2, h) = 1,\ (gh)^3 = hg^3 \rangle \\
&\cong (C_4 \times C_2) \rtimes C_2; \\
G_{16}^4 &= \langle g, h \colon\ g^4 = h^4 = 1,\ g^h = g^3 \rangle \cong C_4 \rtimes C_4; \\
G_{32}^2 &= \langle g, h \colon\ g^4 = h^4 = (gh)^2 = 1,\ (g^2, h) = (g, h^2) = 1 \rangle \\
&\cong (C_4 \times C_2) \rtimes C_4; \\
G_{32}^6 &= \langle g, h \colon\ g^4 = h^4 = (g^3h)^2 = 1,\ (g^2, h) = (g, h^2) = 1 \rangle \\
&\cong ((C_4 \times C_2) \rtimes C_2) \rtimes C_2.
\end{aligned}$$

For the designation of these groups G_m^s we use their numbers s in the Small Groups Library of order m in the computer algebra program GAP (see [8]).

All groups which arise in our proof have order at most 32. The small size groups have been well understood for a long time, and it should not be hard to find an explanation when we give their presentation and relations. The reader will be able to choose between several ways of completing these presentations or to take advantage of the computer algebra system GAP (see [8]).

In the sequel we freely use the well known equations (see [10], page 171)

$$(a, bc) = (a, b)(a, c)(a, b, c), \quad (ab, c) = (a, c)(a, c, b)(b, c). \tag{2.1}$$

Let $U(FG)$ be the group of units of the group algebra FG of a group G over the field F. It is well known that $U(FG) = U(F) \times V(FG)$, where $U(F) = F \setminus 0$ and

$$V(FG) = \left\{ \sum_{g\in G} \alpha_g g \in U(FG) \colon \sum_{g\in G} \alpha_g = 1 \right\}$$

is the normalized group of units. If G is a locally finite p-group and $\operatorname{char}(K) = p$, then

$$V(FG) = \left\{ \sum_{g\in G} \alpha_g g \in FG \colon \sum_{g\in G} \alpha_g = 1 \right\}.$$

For any $x, y \in FG$ we denote the Lie commutator by $[x, y] = xy - yx \in FG$.

Lemma 1. *Let* $\operatorname{char}(F) = 2$ *and let* H *be a non abelian* 2*-generated subgroup of a group* G. *If* $V(FG)$ *has exponent* 4*, then* $H \in \{D_8, Q_8, G_{16}^3, G_{16}^4, G_{32}^2\}$. *Moreover,* $H' \subseteq \Phi(H) \subseteq \zeta(H)$.

Proof. Assume that $V(FG)$ has exponent 4. Clearly G has exponent 4 and any two involutions in G either commute or generate a dihedral group D_8 of order 8. For any $g, h \in G$ such that $(g, h) \neq 1$ consider the 2-generated subgroup $H = \langle g, h\rangle$.

We have the following cases:

Case A. Let $|g| = 4$, $|h| = 2$ and $H \not\cong D_8$. Then $x = 1 + g + h \in V(FG)$ has order 4 (because $x^2 = g^2 + gh + hg \neq 1$) and

$$x^4 - 1 = (gh)^2 + (hg)^2 + g^3h + ghg^2 + g^2hg + hg^3 + g^2 + hg^2h = 0.$$

Comparing hg^3 with other elements, from the last equation we get $hg^3 = g^2hg$, so

$$(h, g^2) = 1 \quad \text{and} \quad (gh)^2 = (hg)^2. \tag{2.2}$$

It is easy to see that $H/\langle g^2\rangle$ is generated by two involutions $g\langle g^2\rangle$ and $h\langle g^2\rangle$. In the case when $H/\langle g^2\rangle$ is abelian we have $(g, h) = g^2$, it follows that $hgh = g^3$ and $\langle g, h\rangle \cong D_8$, a contradiction. Therefore, $H/\langle g^2\rangle \cong D_8$. Since H is not D_8, by (2.2) we obtain that $(gh)^2 = (hg)^2 \neq 1$. Consequently, $(gh)^3 = hg^3 = (gh)^{-1}$ and

$$H = \langle g, h \colon g^4 = h^2 = 1,\ (g^2, h) = 1,\ (gh)^4 = 1\rangle \cong G_{16}^3.$$

Case B. Let $|g| = |h| = 4$ and $|gh| = 2$. Clearly the unit $y = 1 + g + gh \in V(FG)$ has order 4 (because $y^2 = g^2 + g^2h + ghg \neq 1$). Since $ghg = h^3$, we have that $y^2 = g^2 + g^2h + h^3$ and

$$y^4 - 1 = gh^3gh + h^2 + h + gh^3g + g^2h^3 + h^3g^2 + g^2 + h^3g^2h = 0.$$

Comparing h with other elements, from the last equation we obtain that only $h = gh^3g$, so $(h, g^2) = 1$. Consequently

$$H = \langle g, h\colon\ g^4 = h^4 = (gh)^2 = 1,\ (g^2, h) = 1\rangle \cong G_{16}^3.$$

Case C. Let $|g| = |h| = |gh| = 4$ and $H \ncong Q_8$. Then $x = 1 + g + h \in V(FG)$ has order 4 and

$$x^4 - 1 = (gh)^2 + (hg)^2 + g^3h + ghg^2 + g^2h^2 + h^2g^2 + g^2hg + hg^3 + h^2gh + gh^3 + h^3g + hgh^2 + gh^2g + hg^2h = 0. \tag{2.3}$$

The element g^3h must coincide with one of the following elements:

Case 1. Let $g^3h = (hg)^2$. Clearly, $h = g(hg)^2$ and $h^2 = (gh)^3$, so $|gh| = 2$, a contradiction.

Case 2. Let $g^3h = ghg^2$. Then $(h, g^2) = 1$ and (2.3) can be rewritten as

$$(gh)^2 + (hg)^2 + h^2gh + gh^3 + h^3g + hgh^2 + gh^2g + g^2h^2 = 0.$$

Comparing gh^3 with other elements, from the last equation we obtain that only $gh^3 = h^2gh$, so $(g, h^2) = 1$. After substitution into the last equality we get that $(gh)^2 = (hg)^2$. It follows that

$$H = \langle g, h\colon\ g^4 = h^4 = (g, h^2) = (g^2, h) = 1,\ (gh)^2 = (hg)^2\rangle \cong G_{32}^2.$$

Case 3. Let $g^3h = h^2g^2$. Then $h = gh^2g^2$ and $hg = gh^2g^3$, so $(hg)^2 = 1$ which is impossible.

Case 4. Let $g^3h = h^2gh$ or $g^3h = gh^3$. Then $g^2 = h^2$ and $(gh)^2 = (hg)^2$ by (2.3). Hence $H = \langle g, h\colon\ g^4 = h^4 = 1,\ g^2 = h^2,\ (gh)^2 = (hg)^2\rangle \cong G_{16}^4$.

Case 5. Let $g^3h = hgh^2$. Then $|gh| = 2$, a contradiction.

Case 6. Let $g^3h = gh^2g$. Then $h = g^2h^2g$, so $gh = g^{-1}(h^2)g$ and $2 = |h^2| = |gh|$, a contradiction.

Case 7. Let $g^3h = h^3g$. Then $gh^3 = hg^3$ and by (2.3) we get that

$$ghgh + hghg + ghg^2 + g^2h^2 + h^2g^2 + g^2hg + h^2gh + gh^2g + hgh^2 + hg^2h = 0. \tag{2.4}$$

It is easy to check that $ghg^2 \in \{(hg)^2, h^2gh, hgh^2, hg^2h\}$.

Consider each case separately.

Case 7.1. Let $ghg^2 = hghg$. Then from (2.4) it follows that

$$g^2h^2 + h^2g^2 + hgh^2 + h^2gh + gh^2g + hg^2h = 0. \tag{2.5}$$

It is easy to check that the only possible cases are $h^2gh \in \{g^2h^2, gh^2g\}$.

If $h^2gh = g^2h^2$ then $g^2h = h^2g$ and $h^3g = g^3h = g(g^2h) = g(h^2g)$, so $h = g$, a contradiction.

If $h^2gh = gh^2g$ then $h(hgh) = (ghg)g^3hg$. Using the fact that $hgh = ghg$ (see Case 7.1), we have $h(ghg) = (hgh)g^3hg$, so $g = h$, a contradiction.

Case 7.2. Let $ghg^2 = h^2gh$. Multiplying it on the left side by g^2 and on the right side by gh we obtain that $1 = (g^3h)^2 = g^2h^2ghgh$. Since $|gh| = 4$, this yields that $g^2h^2 = ghgh$ and $(g, h) = 1$, a contradiction.

Case 7.3. Let $ghg^2 = hg^2h$. Multiplying it on the left side by g^2 and on the right side by gh we obtain that $1 = (g^3h)^2 = g^2hg^2hgh$. Since $|gh| = 4$, this yields that $g^2hg = ghgh$ and $ghg = hgh$. Clearly $hghg = ghg^2 = hg^2h$, so $(g, h) = 1$, a contradiction.

Case 7.4. Let $ghg^2 = hgh^2$. Since $g^3h = h^3g$ (see Case 7) and $|g| = |h| = 4$,

$$\begin{aligned} H &= \langle g, h\colon\ g^4 = h^4 = 1,\ (g^3h)^2 = 1,\ ghg^2 = hgh^2\rangle \\ &\cong ((C_4 \times C_2) \rtimes C_2) \rtimes C_2 \cong G_{32}^6. \end{aligned}$$

Put $w = 1 + g(1 + h) \in V(FH)$. Using the package LAGUNA (see [5]) of the computational algebra system GAP (see [8]) we get that $w^4 \neq 1$. However, we assume $w^4 = 1$. By a straightforward calculation $w^2 = 1 + g^2 + (gh)^2 + g^2h + ghg$ and

$$\begin{aligned} w^4 &= h + g^2 + (gh)^4 + (g^3h)g + g(ghg^2) + (ghg^2)hg \\ &\qquad + g^2(hg)^2 + g(hg^3) + (g^2h)^2 \\ &= h + g^2 + (gh)^4 + h^3g^2 + g(hgh^2) + (hgh^2)hg \\ &\qquad + g^2(hg)^2 + g^2h^3 + (g^2h)^2 = 1. \end{aligned}$$

Comparing the element h with other elements we have $h = g^2(hg)^2$. This yields that $hg = g^2h(ghg^2) = g^2h(hgh^2)$ and $g^{-1}hg = (gh^2)^2$. However, $4 = |h| > |(gh^2)^2| \leqslant 2$ because $\exp(G) = 4$, a contradiction. Consequently $\exp(V(FH)) > 4$, which is impossible.

Since $\Phi(H) = H^2$, it is easy to see that $H' \subseteq \Phi(H) \subseteq \zeta(H)$ for each 2-generated non abelian H (also it can be easily checked by GAP, see [8]). □

Corollary 1. *If* $\exp(V(FG)) = 4$ *and* G *is non abelian, then* $G' \leqslant \Phi(G) \leqslant \zeta(G)$, $\Phi(G)$ *is elementary abelian and* G *has nilpotency class* 2.

Proof. Let $H = \langle a, b \in G\colon (a,b) \neq 1\rangle$. Clearly $(a,b) = a^{-2}(ab^{-1})^2 b^2 \in G^2 = \Phi(G)$.

Using induction on $n \geqslant 1$, let us prove that $G' \leqslant \Phi(G) \leqslant \zeta(G)$, i.e., $(c,x) = 1$ for any $x \in G$ and $c = g_1^2 \dots g_n^2 \in G'$.

Base of induction: $n = 1$. Obviously $\langle g_1, x\rangle$ and $\langle g_1^2, x\rangle$ are 2-generated groups, so

$$(g_1^2, x) = (g_1, x)(g_1, x, g_1)(g_1, x) = (g_1, x)^2 = 1$$

by (2.1) and Lemma 1. Put $w = g_1^2 g_2^2 \dots g_{n-1}^2$. Using (2.1), the induction assumption and Lemma 1

$$(wg_n^2, x) = (w, x)(w, x, g_n^2)(g_n^2, x) = (g_n^2, x) = (g_n, x)^2 = 1.$$

□

Lemma 2. *Let* G *be a finite* 2*-group such that its Frattini subgroup* $\Phi(G)$ *is central elementary abelian,* $G' \leqslant \Phi(G)$ *and* $|G'| \leqslant 4$. *If* $\mathrm{char}(F) = 2$ *then the exponent of* $V(FG)$ *is equal to* 4.

Proof. Let $G = g_1\Phi(G) \cup \ldots \cup g_m\Phi(G)$, where $g_1 = 1$. Then any $u \in V(FG)$ can be written as $u = \sum\limits_{i=1}^{m} g_i u_i$, where $u_1, \dots, u_m \in F\Phi(G)$. Obviously

$$u^2 = \sum_{i=1}^{m} g_i^2 u_i^2 + \sum_{1<i<j}^{m} [g_i, g_j] u_i u_j$$

by Brauer's lemma (see [1], Proposition 3.1, page 17), where $[g_i, g_j] = g_i g_j - g_j g_i \in FG$. The element $\sum\limits_{i=1}^{m} g_i^2 u_i^2$ is a central involution, so

$$u^4 = 1 + \Bigg(\sum_{1<i<j}^{m} [g_i, g_j] u_i u_j\Bigg)^2.$$

Since $[g_i, g_j] = g_i g_j (1 - (g_j, g_i))$ and $1 - (g_j, g_i)$ is a central nilpotent element of index 2, by Brauer's lemma ([1], Proposition 3.1, page 17) we have that

$$\begin{aligned} z &= \Bigg(\sum_{1<i<j}^{m} g_i g_j (1 - (g_j, g_i)) u_i u_j\Bigg)^2 \\ &= \sum_{1<i<j, 1<k<l}^{m} g_i g_j g_k g_l (1 - (g_k g_l, g_i g_j))(1 - (g_j, g_i))(1 - (g_l, g_k)) u_i u_j u_k u_l. \end{aligned}$$

Suppose that $G' \cong C_2 \times C_2$. It is easy to see that $1-(g_j,g_i) \in \omega(FG')$ and $\omega(FG')^3=0$. Consequently, $z=0$ and $\exp(V(FG))=4$. $\square$

Lemma 3. *Let* $G = H \times \langle a \rangle$ *and let* $|a|$ *divide* 4. *If* $\exp(V(FH)) = 4$, *then* $\exp(V(FG)) = 4$, *too.*

Proof. First assume that $|a|=2$. Any $x \in V(FG)$ has the form

$$x = \sum_{g\in H} \alpha_g g + a \sum_{h\in H} \beta_h h$$

and $g^2, h^2 \in \Phi(G)$ are central by Corollary 1. By hypothesis the order of the unit $y = \sum_{g\in H} \alpha_g g + \sum_{h\in H} \beta_h h$ divides 4 and

$$y^2 = \sum_{g^2\in H} \alpha_g^2 g^2 + \sum_{h^2\in H} \beta_h^2 h^2 + \sum_{g,h\in H} \alpha_g \beta_h [g,h].$$

The unit $\sum_{g^2\in H} \alpha_g^2 g^2 + \sum_{h^2\in H} \beta_h^2 h^2$ is central and its order divides 2, so

$$y^4 = 1 + \Big(\Big[\sum_{g\in H} \alpha_g g, \sum_{h\in H} \beta_h h\Big]\Big)^2 = 1$$

and $\Big(\Big[\sum_{g\in H} \alpha_g g, \sum_{h\in H} \beta_h h\Big]\Big)^2 = 0$. It follows that $|x|$ also divides 4.

Finally let $|a|=4$ and $L = H \times \langle a^2 \rangle$. Then $\exp(V(FL))=4$ and any $x \in V(FG)$ has the form $x = \sum_{g\in L} \alpha_g g + a \sum_{h\in L} \beta_h h$. By repeating the previous argument, it is easily checked that $|x|$ divides 4. $\square$

Lemma 4. *Let* $\operatorname{char}(F)=2$ *and let* G *be a finite* 2*-group, such that* G' *is central elementary abelian. If* $|G'| \geqslant 8$, *then* $\exp(V(FG)) > 4$.

Proof. If $x = \sum_{g\in G} \alpha_g g \in V(FG)$, then $x^2 = \sum_{g\in G} \alpha_g^2 g^2 + w_2$, where $w_2 \in \langle g_i g_j (1+(g_i,g_j))\colon\ i,j \in \mathbb{N}\rangle_F$. Since $g^2 \in \Phi(G) \subseteq \zeta(G)$, we have

$$x^4 = \sum_{g\in G} \alpha_g^4 g^4 + w_2^2 = \sum_{g\in G} \alpha_g^4 + w_2^2.$$

Using the equalities (2.1) and the fact that $|G'| \geqslant 8$, we have that

$$w_2^2 \in \langle g_i g_j g_k g_l (1+(g_ig_j, g_kg_l))(1+(g_i,g_j))(1+(g_l,g_k))\colon\ i,j,k,l\rangle_F \neq 0,$$

so $x^4 \neq 1$. $\square$

P r o o f of Theorem. Let G be a locally finite 2-group and let $u \in V(FG)$. Clearly $\mathrm{supp}(u)$ and $H = \langle \mathrm{supp}(u) \rangle$ are a finite set and a finite group, respectively. Hence $u \in V(FH)$. Now the proof follows from Lemmas 1–4. □

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