Zhi-cai Ma; Jie Wu; Yong-zheng Sun

Adaptive finite-time generalized outer synchronization between two different dimensional chaotic systems with noise perturbation

Kybernetika, Vol. 53 (2017), No. 5, 838–852

Persistent URL: http://dml.cz/dmlcz/147096

K Y B E R N E T I K A — V O L U M E *53* (2 0 1 7), N U M B E R 5, P A G E S 8 3 8 – 8 5 2

ADAPTIVE FINITE-TIME GENERALIZED OUTER SYNCHRONIZATION BETWEEN TWO DIFFERENT DIMENSIONAL CHAOTIC SYSTEMS WITH NOISE PERTURBATION

Zhi-cai Ma, Jie Wu, Yong-zheng Sun

This paper is further concerned with the finite-time generalized outer synchronization between two different dimensional chaotic systems with noise perturbation via an adaptive controller. First of all, we introduce the definition of finite-time generalized outer synchronization between two different dimensional chaotic systems. Then, employing the finite-time stability theory, we design an adaptive feedback controller to realize the generalized outer synchronization between two different dimensional chaotic systems within a finite time. Moreover, we analyze the influence of control parameter on the synchronous speed. Finally, two typical examples are examined to illustrate the effectiveness and feasibility of the theoretical result.

Keywords: finite-time synchronization, different dimensional chaotic systems, adaptive control, noise perturbation

Classification: 65L99, 70K99

1. INTRODUCTION

Synchronization of oscillations is a phenomenon common to a large variety of nonlinear dynamical systems in physics, chemistry, and biology [6, 28]. Since the pioneering work of Pecocra and Carroll [27], chaos synchronization has become a hot topic and it has been applied in many fields, such as information processing, secure communication, biological system, etc. A focused problem in chaos synchronization is how to design a appropriate controller to synchronize chaotic systems. Due to different applications, a wide variety of approaches and controllers have been proposed, including feedback control [20, 21], adaptive control [2, 9, 19, 25, 42], sliding mode control [1, 10, 29, 37], the distributed impulsive control [16, 17] and sampled-data control [18].

For example, using the linear feedback control and nonlinear feedback control, the chaotic Hindmarsh-Rose Neurons have been studied and some synchronization criteria were derived in [20] and [21]. Associated with the distributed impulsive control, the quasi-synchronization of heterogeneous dynamic networks and the problem of network-based leader-following consensus of non-linear multi-agent systems were investigated

DOI: 10.14736/kyb-2017-5-0838

in [17] and [16]. By sampled-data control, the Leader-Following consensus of non-linear multi-agent systems with stochastic sampling was concerned in [18].

All of the methods mentioned above have been proposed to guarantee the asymptotic stability of the synchronization error dynamics. This means that the trajectories of the response system can not reach to the trajectories of the drive system in a finite-time. From a practical point of view, it will be more reasonable to realize chaos synchronization in a settling time. For instance, in secure communication the range of time during which the oscillators are not synchronized corresponds to the range of time during which the encoded message can not be recovered or sent [7]. Therefore, in practical engineering process, we may hope two systems achieve synchronization in a finite-time. To achieve faster convergence in control systems, finite-time control is a very useful technique [23, 31, 34, 35, 36, 38]. Moreover, the finite-time control techniques have demonstrated better disturbance rejection and robustness against uncertainties [5]. It may be noted that all above studies only focus on the outer synchronization of two identical or similar chaotic systems, but the study of generalized outer synchronization between two different dimensional chaotic systems is of critical importance. One example is the synchronization that occurs between heart and lung, where one can observe that both circulatory and respiratory systems behave in a synchronous way [30], even though their models have different orders. And the synchronization of different dimensional chaotic systems has been studied in Refs. [3, 4, 8, 12, 26, 41]. In Ref. [8], Cai, Li and Jing proposed two control strategies to realize the generalized synchronization of chaotic systems with different orders in finite-time and the synchronization problem for a drive-response chaotic system with different orders. Under the effect of both unknown model uncertainties and external disturbance are investigated by active control strategy in Ref. [3]. In Ref. [4], the authors used the nonlinear feedback control method to achieve the robust finite-time increasing order and reduced order synchronization of the chaotic systems. As is known to all, noise is ubiquitous in the real systems. Therefore, the effect of noise on the synchronization is unavoidable, it has been well studied in [11, 15, 22, 24, 32, 33]. Up to now, to the best of our knowledge, there are no published results about the finite-time generalized outer synchronization between two different dimensional chaotic systems with noise perturbation.

Inspired by the above analysis, the questions which we address in our present study are "Can finite-time generalized synchronization between two different dimensional chaotic systems be achieved with the perturbation of noise?" and "Besides the numerical evidences, are there any analytical arguments illustrating this phenomenon?" In this paper, utilizing the finite-time stability theory of stochastic differential equations, by employing a time-varying feedback gain in the linear part of the adaptive controller which automatically converge to suitable constants, we can see that two different dimensional chaotic systems can realize finite-time generalized outer synchronization with noise perturbation. Otherwise, the upper bound of the convergence time is also given. Finally, two numerical examples are examined to illustrate the effectiveness of the theoretical result.

The rest of this paper is organized as follows. In Section 2, we give the problem statement and some preliminaries. In section 3, the main result is derived based on the finite-time stability theory. In Section 4, numerical simulations are given to show the effectiveness of the theoretical results. Finally, some conclusions are drawn in Section 5.

2. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following drive system described by:

$$\dot{x}(t) = Ax(t) + f(x(t)), \tag{1}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in R^n$ is the state vector of the chaotic system, $A \in R^{n\times n}$ is a constant matrix and $f(x) \in R^n$ is a continuously differentiable nonlinear vector function.

Remark 2.1. Many chaotic and hyperchaotic systems can be transferred in the form of system (1). For instance, one may see the two chaotic systems in [30] and [8].

To realize the finite-time generalized outer synchronization between two different dimensional chaotic systems with noise perturbation, we refer to system (1) as the drive system. Likewise, the response chaotic system with circumstance noise is described as follows:

$$\dot{y}(t) = By(t) + g(y(t)) + h(y - \phi(x))\mathrm{d}w(t) + u(t), \tag{2}$$

where $y(t) = [y_1(t), y_2(t), \ldots, y_m(t)] \in R^m$ is the state vector of the response system, $B \in R^{m\times m}$ is a constant matrix and $g(y) \in R^m$ is a continuously differentiable nonlinear vector function, $h \in R^m$ is the noise intensity function and $\phi : R^n \to R^m$ is an arbitrary continuously differentiable function, for a better presentation, we set $h(y-\phi(x))\mathrm{d}w(t) = \sigma\cdot(y-\phi(x))\mathrm{d}w(t)$, σ is the noise strength and $w(t)$ is one-dimensional Brownian motion. $u(t) \in R^m$ is the control input which is yet to be determined.

Remark 2.2. In this paper, we are going to studying the finite-time generalized outer between two different dimensional chaotic systems with noise perturbation, that is in systems (1) and (2) the order $n \neq m$ and the functions $f(\cdot) \neq g(\cdot)$. The finite-time synchronization of the same order chaotic systems with noise perturbation has been researched in [36] and the finite-time generalized synchronization of chaotic systems with different order has been studied in [8]. However, there are few theoretical results about the finite-time generalized outer synchronization of two different dimensional chaotic systems with noise perturbation.

Definition 2.3. We say the chaotic systems (1) and (2) are finite-time generalized synchronization with respect to the vector map ϕ if, for any initial states $x(0) \in R^n\backslash\{0\}$ and $y(0) \in R^m\backslash\{0\}$, there exists a finite-time function T_0 such that

$$P\{|y(t, y(0)) - \phi(x(t, x(0)))| = 0\} = 1, \tag{3}$$

for all $t > T_0$ and $T_0 = \inf\{T : y(t) = \phi(x(t, x(0))), \forall t \geq T\}$ is called the stochastic settling time.

Remark 2.4. The stochastic settling time function T_0 is not only a function of $x(0)$ and $y(0)$, but a stochastic variable for fixed $x(0)$ and $y(0)$. Hence, the finite-time property of T_0 is evaluated by $0 < E(T_0) < +\infty$.

For the noise intensity function, because the speed of the environmental fluctuations is far less than the change rate of practical systems, we have the following assumption.

Assumption 2.5. The noise intensity function $\sigma(e(t))$ satisfy the Lipschitz condition and there exists a nonnegative constant q such that

$$\text{trace}(\sigma^T(e(t))\sigma(e(t))) \leq 2qe^T(t)e(t).$$

Moreover, $\sigma(0) \equiv 0$.

Consider the following d-dimensional stochastic differential equation:

$$\mathrm{d}z = \varphi(z)\mathrm{d}t + \psi(z)\mathrm{d}w(t), \tag{4}$$

on $t \geq 0$, where $z(t) \in R^d$ is an Itô process, $\varphi \in \mathcal{L}^1(R_+, R^d)$ and $\psi \in \mathcal{L}^2(R_+, R^{d\times m})$ are continuous and satisfy $\varphi(0) = 0, \psi(0) = 0$. It is assumed that Eq. (4) has a unique and global solution denoted by $z(t, z(0))(0 < t < +\infty)$, where $z(0)$ is the initial state.

Let $V \in C^{1,2}(R_+, R^d \times R_+)$. Then $V(t, z(t))$ is again an Itô process with the stochastic differential given by

$$\mathrm{d}V(t, z(t)) = \mathcal{L}V(t, z(t))\mathrm{d}t + V_z(t, z(t))\psi(t, z(t))\mathrm{d}w, \tag{5}$$

where

$$\mathcal{L}V(t, z(t)) = V_t(t, z(t)) + V_z(t, z(t))\varphi(t) + (1/2)\text{trace}[\psi^T(t)V_{zz}(t, z(t))\psi(t)]. \tag{6}$$

Lemma 2.6. (Yin et al. [39]) For system (4), define

$$T(x(0)) = \inf\{T \geq 0 : y(t; y(0)) = \phi(x(t; x(0))), \forall t \geq T\}.$$

Assume that system (4) has the unique global solution. If there exists a positive definite, twice continuously differentiable and radially unbounded Lyapunov function $V : R^d \to R^+$, K_∞ class functions μ_1 and μ_2, positive real numbers $c > 0$ and $0 < \rho < 1$, such that for all $x \in R^d$ and $t \geq 0$,

$$\mu_1(|x|) \leq V(x) \leq \mu_2(|x|),$$

$$\mathcal{L}V(x) \leq -c(V(x))^\rho,$$

where $|x|$ denotes the Euclidean norm $\sqrt{(\sum_{i=1}^n x_i^2)}$, then the trivial solution of (4) is finite-time stable in probability, and the stochastic settling time function T satisfies

$$E[T(x_0)] \leq \frac{V^{1-\rho}(x_0)}{c(1-\rho)}. \tag{7}$$

Lemma 2.7. (Hardy et al. [14]) Let $a_1, a_2, \ldots, a_n > 0$ and $0 < r < p$. Then

$$\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n a_i^r\right)^{\frac{1}{r}}.$$

3. SUFFICIENT CONDITIONS FOR FINITE-TIME GENERALIZED OUTER SYNCHRONIZATION

In this section, we will investigate the finite-time generalized outer synchronization between two different dimensional chaotic systems with noise perturbation. From Definition 2.3, we know that the study of the finite-time generalized synchronization between systems (1) and (2) can be translated into the analysis of the finite-time stability of error system (8). Next, we discuss the finite-time stability of error system (8) and obtain the following result.

In order to study the finite-time generalized synchronization between the drive system (1) and the response system (2), we define the generalized synchronization error $e(t) = y(t) - \phi(x(t))$, where $\phi : R^n \to R^m$ is a continuously differentiable map. Then the error system is described by:

$$\begin{aligned} \dot{e}(t) &= Be(t) + B\phi(x) + g(y(t)) - J_\phi(x)Ax(t) - J_\phi(x)f(x(t)) + \sigma(e(t))\mathrm{d}w(t) + u(t) \\ &= Be(t) + R(x,y) + \sigma(e(t))\mathrm{d}w(t) + u(t), \end{aligned} \tag{8}$$

where $R(x,y) = B\phi(x) + g(y(t)) - J_\phi(x)Ax(t) - J_\phi(x)f(x(t))$ and $J_\phi(x)$ denote the Jacobin matrix of the map $\phi(x)$:

$$J_\phi(x) = \begin{pmatrix} \frac{\partial\phi_1(x)}{\partial x_1} & \frac{\partial\phi_1(x)}{\partial x_2} & \cdots & \frac{\partial\phi_1(x)}{\partial x_n} \\ \frac{\partial\phi_2(x)}{\partial x_1} & \frac{\partial\phi_2(x)}{\partial x_2} & \cdots & \frac{\partial\phi_2(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial\phi_m(x)}{\partial x_1} & \frac{\partial\phi_m(x)}{\partial x_2} & \cdots & \frac{\partial\phi_m(x)}{\partial x_n} \end{pmatrix}.$$

In this paper, we designed an adaptive controller as follows:

$$u(t) = \begin{cases} -R(x,y) - r(t)e(t) - \eta\left[\mathrm{sign}(e(t)) + \frac{|r(t)-\hat{r}|e(t)}{\|e^2(t)\|}\right], & \text{if } \ e(t) \neq 0, \\ 0, & \text{if } \ e(t) = 0, \end{cases} \tag{9}$$

where arbitrary positive constant $\eta > 1$, $\mathrm{sign}(e(t)) \in R^m$ is the signum function. Thus, the denominator term $||e^2(t)|| \neq 0$ when $e(t) \neq 0$. And the control $u(t) = 0$ when the synchronization error is zero. The feedback gain $r(t) \in R^m$ is adapted according to the following update law:

$$\dot{r}(t) = k_i e^T(t)e(t), \tag{10}$$

where $k_i (i = 1, 2, \ldots, m)$ is any positive constant. For brevity, we set $k_i = 1$.

Theorem 3.1. Consider error system (8) under Assumption 2.5, if there exists a sufficiently large positive constant $\hat{r}$ such that

$$\hat{r} > q + \lambda_{\max}(\mathcal{B}), \tag{11}$$

where $\mathcal{B} = \frac{B+B^T}{2}$, then system (8) is finite-time stabilization with the controller $u(t)$ designed in the form of (9).

P r o o f . Construct the following positive-definite function:

$$V(t) = \frac{1}{2}e^T(t)e(t) + \frac{1}{2}(r(t) - \hat{r})^2. \tag{12}$$

Thus the diffusion operator $\mathcal{L}$ defined in (5) onto the function V along the error system (8) gives

$$\mathcal{L}V(t) = e^T(t)\dot{e}(t) + (r(t) - \hat{r})\dot{r}(t) + (1/2)\text{trace}(\sigma^T(e(t))\sigma(e(t))). \tag{13}$$

By introducing $\dot{e}(t)$ and $\dot{r}(t)$ (given by (8) and (10)) into the right-hand side of Eq. (13), we have

$$\begin{aligned} \mathcal{L}V(t) &= e^T(t)[Be(t) + R(x, y) + u(t)] + r(t)e^T(t)e(t) - \hat{r}e^T(t)e(t) \\ &\quad +(1/2)\text{trace}(\sigma^T(e(t))\sigma(e(t))). \end{aligned} \tag{14}$$

Substituting (9) into (14) yields

$$\begin{aligned} \mathcal{L}V(t) &= e^T(t)Be(t) - \eta\left[e^T(t)\text{sign}(e(t)) + \frac{|r(t) - \hat{r}|e^T(t)e(t)}{\|e^2(t)\|}\right] \\ &\quad -\hat{r}e^T(t)e(t) + (1/2)\text{trace}(\sigma^T(e(t))\sigma(e(t))). \end{aligned} \tag{15}$$

Since $e^T(t)\text{sign}(e(t)) = |e(t)|$ is always satisfied, one has

$$\mathcal{L}V(t) = e^T(t)Be(t) - \eta(|e(t)| + |r(t) - \hat{r}|) - \hat{r}e^T(t)e(t) + (1/2)\text{trace}(\sigma^T(e(t))\sigma(e(t))). \tag{16}$$

Noting that

$$e^T(t)Be(t) \le \lambda_{\max}(\mathcal{B})e^T(t)e(t). \tag{17}$$

By Assumption 2.5, we obtain

$$(1/2)\text{trace}(\sigma^T(e(t))\sigma(e(t))) \le qe^T(t)e(t). \tag{18}$$

Then, from (16),(17) and (18), we have

$$\mathcal{L}V(t) \le -(\hat{r} - q - \lambda_{\max}(\mathcal{B}))e^T(t)e(t) - \eta(|e(t)| + |r(t) - \hat{r}|). \tag{19}$$

Let us choose $\hat{r} > q + \lambda_{\max}(\mathcal{B})$, then (19) implies that

$$\mathcal{L}V(t) \le -\eta(|e(t)| + |r(t) - \hat{r}|). \tag{20}$$

From Lemma 2.7, we obtain

$$\begin{aligned} \mathcal{L}V(t) &\le -\eta\left[e^T(t)e(t) + (r(t) - \hat{k})^2\right]^{\frac{1}{2}} \\ &= -\eta(2V^{\frac{1}{2}}(t)) \\ &= -\sqrt{2}\eta V^{\frac{1}{2}}(t). \end{aligned} \tag{21}$$

According to Lemma 2.6, the trivial solution of the error system (8) is finite-time stable. It implies there exists a $T_0 > 0$ such that $e(t) = 0$ if $t > T_0$. This means that the systems (1) and (2) achieved the finite-time generalized outer synchronization by the controller $u(t)$ in the form of (9). This completes the proof. □

Remark 3.2. According to Lemma 2.6, we can see that the convergent time is related to the initial states of systems and parameter ρ. The relationship can be given as follows. From (21), we can obtain that

$$c = \sqrt{2}\eta, \qquad \rho = \frac{1}{2}. \tag{22}$$

Substituting (22) into (7) yields

$$E[T_0] \leq \frac{\sqrt{2}}{\eta} V^{\frac{1}{2}}(0), \tag{23}$$

where $V(0) = [e^2(0) + (r(0) - \hat{r})^2]/2$.

Remark 3.3. From the inequality (11) we can see that, for any high level noise, there exits a sufficiently large positive constant $\hat{r}$ such that the finite-time generalized outer synchronization is realized in probability. Hence, the synchronization is robust to the noise perturbation. From Eq. (23), one can see that the convergence time of proposed algorithm is closely related to the parameter η and the initial state $V(0)$. For fixed initial state $V(0)$, the synchronization time decreases as η increases.

4. SIMULATION RESULTS

In this section, illustrative examples are given to verify the effectiveness of the theoretical criteria obtained in the preceding section. The synchronization error $e(t)$ and the total synchronization error $E(t) = \|e(t)\|$ are used to measure the evolution process. In the numerical simulations, we consider two cases $n > m$ and $n < m$, respectively.

Example 1. ($n > m$) In this example, the Chua's circuit is used to describe the drive system and the Duffing system is used to be the response system.

The Chua's circuit can be described as

$$\dot{x} = \begin{pmatrix} -p - pb & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \varphi(x_1) \\ 0 \\ 0 \end{pmatrix} \doteq Ax + f(x), \tag{24}$$

where $x = (x_1, x_2, x_3)^T \in R^3$ is the state vector, $\psi(x_1) = 0.5p(b - a)(|x_1 + 1| - |x_1 - 1|)$. In all of the simulations, we always choose the system parameters of the Chua's circuit as $p = 10, q = 14.87, a = -1.27, b = -0.68$, which causes the Chua's circuit to exhibit a double-scroll chaotic attractor.

The controlled Duffing system is described by

$$\dot{y} = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -y_1^3 + d\text{cost} \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \doteq By + g(y) + u, \tag{25}$$

where $y = (y_1, y_2)^T \in R^2$ is the state vector, when $c = 1, d = 0.8$ the system (25) is chaotic.

The error dynamic system is $\dot{e}(t) = Be(t) + B\phi(x) + g(y) + u(t) - J_\phi(x)Ax(t) - J_\phi(x)f(x)$, where the map ϕ is defined as

$$\phi(x) = (x_1, x_2 + x_3)^T.$$

Then,

$$J_\phi(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

According to (8), the error dynamic system can be written as follows:

$$\begin{cases} \dot{e}_1 = e_2 - 3.2x_1 - 9x_2 + x_3 + 0.295\psi(x_1) + u_1, \\ \dot{e}_2 = e_1 - e_2 + 14.87x_2 - 2x_3 - y_1^3 + 0.8\text{cos}t + u_2. \end{cases} \tag{26}$$

According to (9), we get the controllers

$$\begin{cases} u_1 = 3.2x_1 + 9x_2 - x_3 - 0.295\psi(x_1) - r_1 e_1 - \eta\left(\text{sign}(e_1) + \frac{|r_1 - \hat{r}|e_1}{\|e_1\|^2}\right), \\ u_2 = -14.87x_2 + 2x_3 + y_1^3 - 0.8\text{cos}t - r_2 e_2 - \eta\left(\text{sign}(e_2) + \frac{|r_2 - \hat{r}|e_2}{\|e_2\|^2}\right). \end{cases} \tag{27}$$

Next, we will illustrate the two chaotic systems achieving synchronization with adaptive controller (9) in a finite-time. We take the initial values of systems (24) and (25) as $x_1(0) = 4, x_2(0) = 2, x_3(0) = 1$ and $y_1(0) = 5, y_2(0) = 2$. It is easy to compute that $\lambda_{\max}(\mathcal{B}) = 0.6180$. Take $\hat{r} = 2$ and $q = 1.2$, obviously the condition (11) is satisfied. The value of parameter η is set as $\eta = 2$. The trajectories of synchronization error $e(t)$ and the total synchronization error $E(t)$ are shown in Figures 1(a) and (b). From Figure 1, it can be observed that the finite-time generalized outer synchronization between the chaotic systems (24) and (25) under the controller (27) is achieved about $T_0 = 0.25$. By computing (23), we get $E[T_0] \leq 0.9998$. The simulations match the theoretical result perfectly. Figure 2 shows the updated feedback strength which reach some certain constants when the systems (24) and (25) are synchronized.

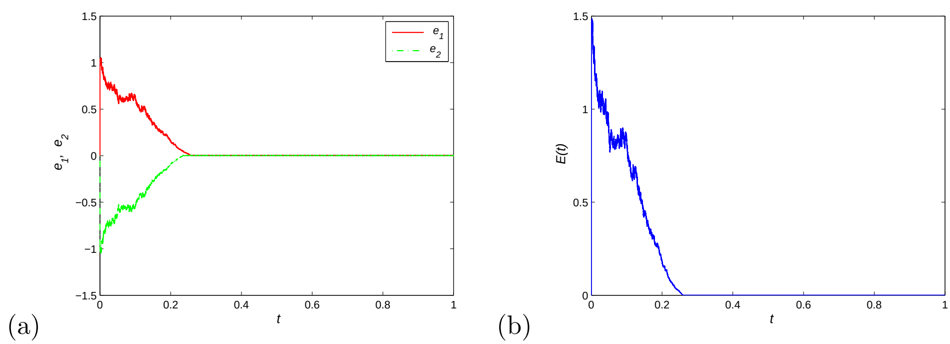


Fig. 1. Trajectories of the synchronization error (a) and the total synchronization error (b) between systems (24) and (25) with $\eta = 2$ and $\sigma = 1$.

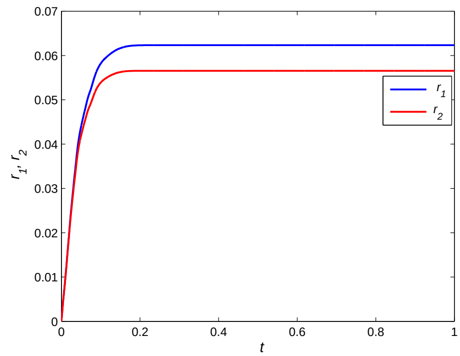


Fig. 2. Feedback strength r_i of adaptive controller (9) for systems (24) and (25) with $\eta = 2$ and $\sigma = 1$.

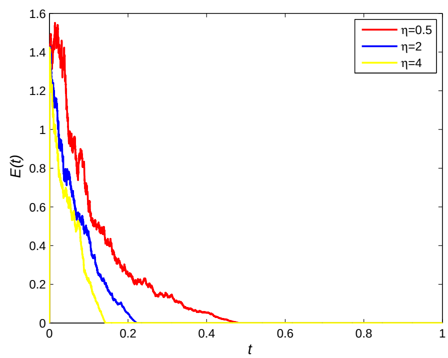


Fig. 3. Time evolutions of total synchronization error $E(t)$ with control parameter $\eta = 0.5, 2, 4$ and $\sigma = 1$.

To study the effect of the control parameter η on the settling time, we simulate the trajectories of two chaotic systems with the controller defined in Eq. (9) through taking different values of η. Figure 3 shows that the synchronization time decreases when parameter η increases, which is consistent with the analysis of Remark 3.3.

Example 2. $(n < m)$ In this example, the Duffing system is used to the drive system and the Chua's circuit is used to be the response system. They can be describe as follows:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -x_1^3 + d\mathrm{cos}t \end{pmatrix} \doteq Ax + f(x), \tag{28}$$

and

$$\dot{y} = \begin{pmatrix} -p - pb & p & 0 \\ 1 & -1 & 1 \\ 0 & -q & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} \psi(y_1) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \doteq By + f(y) + u. \tag{29}$$

The error dynamic system is $\dot{e}(t) = Be(t) + B\phi(x) + g(y(t)) + u(t) - J_\phi(x)Ax(t) - J_\phi(x)f(x(t))$, where the map ϕ is defined as

$$\phi(x) = (x_1^2, x_1 + x_2, x_2^2)^T.$$

Then,

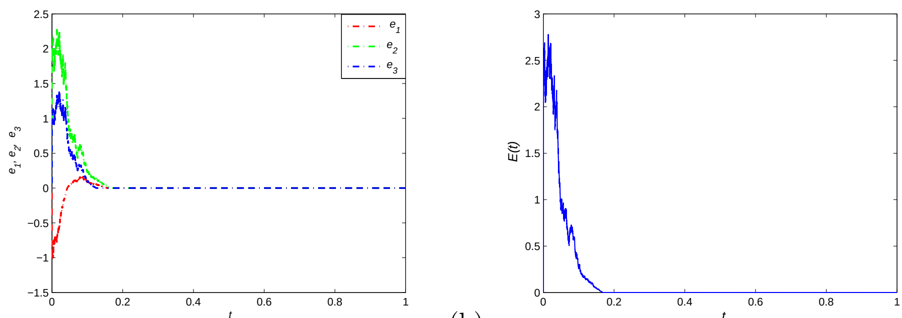
$$J_\phi(x) = \begin{pmatrix} 2x_1 & 0 \\ 1 & 1 \\ 0 & 2x_2 \end{pmatrix}.$$

According to (8), the error dynamic system can be written as follows:

$$\begin{cases} \dot{e}_1 = -3.2e_1 + 10e_2 - 3.2x_1^2 + 10x_1 + 10x_2 - 2x_1x_2 + \psi(y_1) + u_1, \\ \dot{e}_2 = e_1 - e_2 + e_3 + x_1^2 - x_1^3 + x_2^2 - 2x_1 - x_2 - 0.8\text{cos}t + u_2, \\ \dot{e}_3 = -14.87(e_2 + x_1 + x_2) - 2x_1x_2 + 2x_2^2 - 2x_1^3x_2 - 1.6x_2\text{cos}t + u_3. \end{cases} \tag{30}$$

According to (9), we get the controllers

$$\begin{cases} u_1 = 3.2x_1^2 - 10x_1 - 10x_2 + 2x_1x_2 - \psi(y_1) - r_1e_1 - \eta\left(\text{sign}(e_1) + \frac{|r_1 - \hat{r}|e_1}{\|e_1\|^2}\right), \\ u_2 = -x_1^2 - x_2^2 + x_1^3 + 2x_1 + x_2 + 0.8\text{cos}t - r_2e_2 - \eta\left(\text{sign}(e_2) + \frac{|r_2 - \hat{r}|e_2}{\|e_2\|^2}\right), \\ u_2 = 14.87x_1 + 14.87x_2 + 2x_1x_2 - 2x_2^2 + 2x_1^3 - r_3e_3 - \eta\left(\text{sign}(e_3) + \frac{|r_3 - \hat{r}|e_3}{\|e_3\|^2}\right). \end{cases} \tag{31}$$



(a) (b)

Fig. 4. Trajectories of the synchronization error (a) and the total synchronization error (b) between systems (28) and (29) with $\eta = 2$ and $\sigma = 2$.

Next, we will illustrate the two chaotic systems achieving synchronization with adaptive controller (9) in a finite time. We take the initial values of systems (28) and (29) as $x_1(0) = 1, x_2(0) = 1$ and $y_1(0) = 0, y_2(0) = 4, y_3(0) = 2$. It is easy to compute that $\lambda_{\max}(\mathcal{B}) = 7.8570$. Take $\hat{r} = 10$ and $q = 1.2$, obviously the condition (11) is satisfied. The value of parameter η is set as $\eta = 2$ and the noise strength $\sigma = 2$. The trajectories of synchronization error $e(t)$ and total synchronization $E(t)$ are shown in Figures 4 (a) and (b). From Figure 4, it can be observed that the finite-time generalized synchronization between the chaotic systems (28) and (29) under the controller (31) is achieved

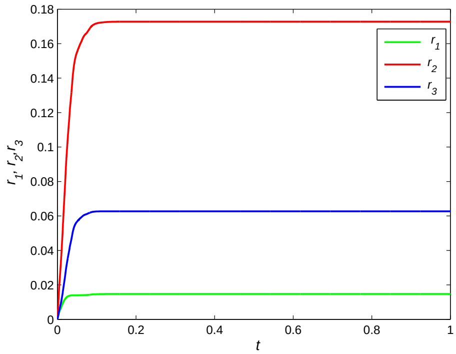


Fig. 5. Feedback strength r_i of adaptive controller (9) for systems (28) and (29) with $\eta = 2$ and $\sigma = 2$.

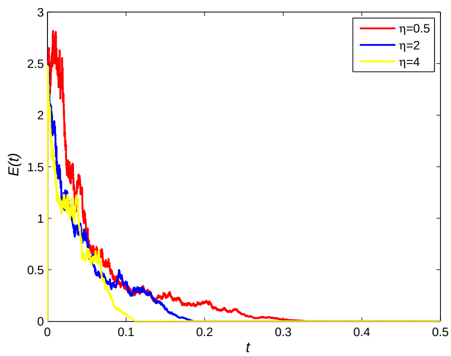


Fig. 6. Time evolutions of total synchronization error $E(t)$ with control parameter $\eta = 0.5, 2, 4$ and $\sigma = 2$.

about $T_0 = 0.18$. By computing (23), we get $E[T_0] \leq 1.7318$. The simulations match the theoretical result perfectly. The feedback strength is shown in Figure 5. In Figure 6, we simulated the different value of parameter η, from the figure we can see that the synchronization time decreases when parameter η increases, which is consistent with the analysis of Remark 3.3.

5. CONCLUSIONS

In this paper, we have investigated the finite-time generalized outer synchronization between two different dimensional chaotic systems with noise perturbation via adaptive control strategy. First of all, the sufficient condition for finite-time generalized outer synchronization is obtained based on the finite-time stochastical stability theory. The theoretical result show that two chaotic systems with noise perturbation can achieve finite-time generalized outer synchronization even if the two chaotic systems have different dimensional. Numerical simulations fully verify our main result. Then, we considered the effect of control parameter on the synchronization time. From the simulation results we can see that the synchronization time decreases along with the control parameter increases. Besides, we simulated the trajectories of the adaptive feedback strengths.

However, in the proposed synchronization framework, the drive system and the response system are connected point-to-point as the controller uses continual states of these systems. In most practical synchronization applications, networking the drive and response systems is more preferable. For instance, the distributed networked control systems and the remote surgery master slave system [13, 40]. In this case, the controller should be designed in the presence of some network-induced constraints, such as communication delays, data losses, limited resources. This is our future research direction.

ACKNOWLEDGEMENT

The authors would like to thank anonymous reviewers and editors for their valuable comments and suggestions to improve the presentation and theoretical results of this paper. This work was supported by the National Natural Science Foundation of China (grant nos. 61403393).

(Received November 14, 2016)

REFERENCES

[1] M. P. Aghababa: Finite-time chaos control and synchronization of fractional-order nonautonomous chaotic (hyperchaotic) systems using fractional nonsingular terminal sliding mode technique. Nonlinear Dynam. *69* (2012), 247–261. DOI:10.1007/s11071-011-0261-6

[2] M. P. Aghababa and H. P. Aghababa: A general nonlinear adaptive control scheme for finite-time synchronization of chaotic systems with uncertain parameters and nonlinear inputs. Nonlinear Dynam. *69* (2012), 1903–1914. DOI:10.1007/s11071-012-0395-1

[3] I. Ahmad, A. B. Saaban, A. B. Ibrahima, M. Shahzadb, and N. Naveedca: The synchronization of chaotic systems with different dimensionals by a robust generalized active control. Optik *127* (2016), 4859–4871. DOI:10.1016/j.ijleo.2015.12.134

[4] I. Ahmada, M. Shafiq, A. B. Saaban, A. B. Ibrahim, and M. Shahzad: Robust finite-time global synchronization of chaotic systemswith different orders. Optik *127* (2016), 8172–8185. DOI:10.1016/j.ijleo.2016.05.065

[5] S. P. Bhat and D. S. Bernstein: Finite-time stability of continuous autonomous systems. SIAM J. Control Optim. *38* (2000), 751–766. DOI:10.1137/s0363012997321358

[6] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou: The synchronization of chaotic systems. Phys. Rep. *366* (2002), 1–101. DOI:10.1016/s0370-1573(02)00137-0

[7] S. Bowong, M. Kakmeni, and R. Koina: Chaos synchronization and duration time of a class of uncertain chaotic systems. Math. Comput. Simulat. *71* (2006), 212–228. DOI:10.1016/j.matcom.2006.01.006

[8] N. Cai, W. Q. Li, and Y. W. Jing: Finite-time generalized synchronization of chaotic systems with different order. Nonlinear Dynam. *64* (2011), 385–393. DOI:10.1007/s11071-010-9869-1

[9] X. Y. Chen and J. F. Lu: Adaptive synchronization of different chaotic systems with fully unknown parameters. Phys. Lett. A *364* (2007), 123–128. DOI:10.1016/j.physleta.2006.11.092

[10] D. Y. Chen, R. F. Zhang, X. Y. Ma, and S. Liu: Chaotic synchronization and anti-synchronization for a novel class of multiple chaotic systems via a sliding mode control scheme. Nonlinear Dynam. *69* (2012), 35–55. DOI:10.1007/s11071-011-0244-7

[11] R. Erban, J. Haskovec, and Y. Sun: A Cucker-Smale model with noise and delay. SIAM J. Appl. Math. *76* (2016), 1535–1557. DOI:10.1137/15m1030467

[12] Z. M. Ge and C. H. Yang: The generalized synchronization of a Quantum-CNN chaotic oscillator with different order systems. Chaos Solitons and Fractals *35* (2008), 980–990. DOI:10.1016/j.chaos.2006.05.090

[13] X. H. Ge ,F. W. Yang, and Q. L. Han: Distributed networked control systems: A brief overview. Inform. Sciences *380* (2017), 117–131. DOI:10.1016/j.ins.2015.07.047

[14] G. H. Hardy, J. E. Littlewood, and G. Pólya: Inequalities. Cambridge At The University Press, Cambridge 1934.

[15] B. Hauschildt, N. B. Jason, A. Balanov, and A. Scholl: Noise-induced cooperative dynamics and its control in coupled neuron models. Phys. Rev. E *74* (2006), 051906. DOI:10.1103/physreve.74.051906

[16] W. L. He, G. R. Chen, Q. L. Han, and F. Qian: Network-based leader-following consensus of nonlinear multi-agent systems via distributed impulsive control. Inform. Sciences *380* (2017), 145–158. DOI:10.1016/j.ins.2015.06.005

[17] W. L. He, F. Qian, J. Lam, G. R. Chen, Q. L. Han, and J. Kurths: Quasi-synchronization of heterogeneous dynamic networks via distributed impulsive control: Error estimation, optimization and design. Automatica *62* (2015), 249–262. DOI:10.1016/j.automatica.2015.09.028

[18] W. L. He, B. Zhang, Q. L. Han, F. Qian, J. Kurths, and J. D. Cao: Leader-following consensus of nonlinear multiagent systems with stochastic sampling. IEEE Trans. Cybern. *47* (2017), 327–338.

[19] D. B. Huang: Simple adaptive-feedback controller for identical chaos synchronization. Phys. Rev. E. *71* (2005), 037203. DOI:10.1103/physreve.71.037203

[20] D. Ke and Q. L. Han: Synchronization of two coupled Hindmarsh-Rose neurons. Kybernetika *51* (2015), 784–799. DOI:10.14736/kyb-2015-5-0784

[21] D. Ke and Q. L. Han: Master-Slave synchronization criteria for chaotic Hindmarsh-Rose neurons using linear feedback control. Complexity *21* (2016), 319–327. DOI:10.1002/cplx.21658

[22] G. Korniss: Synchronization in weighted unccorrelated complex networks in a noisy environment: optimization and connections with transport efficiency. Phys. Rev. E *75* (2007), 051121. DOI:10.1103/physreve.75.051121

[23] S. H. Li, and Y. P. Tian: Finite-time synchronization of chaotic systems. Chaos Solitons and Fractals *15* (2003), 303–310. DOI:10.1016/s0960-0779(02)00100-5

[24] W. Lin and G. R. Chen: Using white noise to enhance synchronization of coupled chaotic systems. Chaos *16* (2006), 013134. DOI:10.1063/1.2183734

[25] J. S. Lin and J. J. Yan: Adaptive synchronization for two identical generalized Lorenz chaotic systems via a single controller. Nonlinear Anal. RWA *10* (2009), 1151–1159. DOI:10.1016/j.nonrwa.2007.12.005

[26] A. Ouannas, and Z. Odibat: Generalized synchronization of different dimensional chaotic dynamical systems in discrete time. Nonlinear Dynam. *81* (2015), 765–771. DOI:10.1007/s11071-015-2026-0

[27] L. M. Pecora and T. L. Carroll: Synchronization in chaotic systems. Phys. Rev. Lett. *64* (1990), 821–824. DOI:10.1103/physrevlett.64.821

[28] A. Pikovsky, M. Rosenblum, and J. Kurths: Synchronization: A Universal Concept in Nonlinear Sciences. Cambridge University Press, Cambridge 2001. DOI:10.1017/cbo9780511755743

[29] M. Pourmahmood, S. Khanmohammadi, and G. Alizadeh: Synchronization of two different uncertain chaotic systems with unknown parameters using a robust adaptive sliding mode controller. Commun. Nonlinear Sci. Numer. Simulat. *16* (2011), 2853–2868. DOI:10.1016/j.cnsns.2010.09.038

[30] A. Stefanovska, H. Haken, E. McClintock, M. Hožič, F. Bajrović, and S. Ribarič: Reversible transitions between synchronization states of the cardiorespiratory system. Phys. Rev. Lett. *85* (2000), 4831–4834. DOI:10.1103/physrevlett.85.4831

[31] Y. Z. Sun, W. Li, and D. H. Zhao: Finite-time stochastic outer synchronization between two complex dynamical networks with different topologies. Chaos *22* (2012), 023152. DOI:10.1063/1.4731265

[32] Y. Z. Sun and J. Ruan: Synchronization in coupled time-delayed systems with parameter mismath and noise perturbation. Chaos *19* (2009), 043113. DOI:10.1063/1.3262488

[33] Y. Z. Sun, H. J. Shi, E. A. Bakare, and Q. X. Meng: Noise-induced outer synchronization between two different complex dynamical networks. Nonlinear Dynam. *76* (2014), 519–528. DOI:10.1007/s11071-013-1145-8

[34] U. E. Vincent and R. Guo: Finite-time synchronization for a class of chaotic and hyperchaotic systems via adaptive feedback controller. Phys. Lett. A *375* (2011), 2322–2326. DOI:10.1016/j.physleta.2011.04.041

[35] H. Wang, Z. Z. Han, and W. Zhang: Finite-time chaos control of unified chaotic systems with uncertain parameters. Nonlinear Dynam. *55* (2009), 323–328. DOI:10.1007/s11071-008-9364-0

[36] J. Wu, Z. C. Ma, Y. Z. Sun, and F. Liu: Finite-time synchronization of chaotic systems with noise perturbation. Kybernetika *54* (2015), 137–149. DOI:10.14736/kyb-2015-1-0137

[37] J. J. Yan, M. L. Hung, T. Y. Chang, and Y. S. Yang: Robust synchronization of chaotic systems via adaptive sliding mode control. Phys. Lett. A *356* (2006), 220–225. DOI:10.1016/j.physleta.2006.03.047

[38] Y. Q. Yang, and X. F. Wu: Global finite-time synchronization of a class of the nonautonomous chaotic systems. Nonlinear Dynam. *70* (2012), 197–208. DOI:10.1007/s11071-012-0442-y

[39] J. L. Yin, S. Khoo, Z. H. Man, and X. H. Yu: Finite-time stability and instability of stochastic nonlinear systems. Automatica *47* (2011), 2671–2677. DOI:10.1016/j.automatica.2011.08.050

[40] X. M. Zhang, Q. L. Han, and X. H. Yu: Survey on recent advances in networked control systems. IEEE Trans. Ind. Informat. *12* (2016), 1740–1752. DOI:10.1109/tii.2015.2506545

[41] G. Zhang, Z. R. Liu, and Z. J. Ma: Generalized synchronization of different dimensional chaotic dynamical systems. Chaos Solitons and Fractals *32* (2007), 773–779. DOI:10.1016/j.chaos.2005.11.099

[42] X. B. Zhou, M. R. Jiang, and Y. Q. Huang: Switched modified function projective synchronization between two complex nonlinear hyperchaotic systems based on adaptive control and parameter identification. Kybernetika *50* (2014), 632–642. DOI:10.14736/kyb-2014-4-0632

Zhi-cai Ma, Corresponding author. Department of Mechanics and Engineering Science, College of Civil Engineering and Mechanics, Lanzhou University, Lanzhou, Gansu 730000. P. R. China.
e-mail: zhicai_ma@hotmail.com

Jie Wu, Department of Electronic Engineering, School of Information Science and Engineering, Fudan University, 200433, Shanghai. P. R. China.
e-mail: wujiecumt@hotmail.com

Yong-zheng Sun, School of Mathematics, China University of Mining and Technology, Xuzhou, 221008. P R. China.
e-mail: yzsung@gmail.com