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EXISTENCE OF SOLUTIONS OF IMPULSIVE BOUNDARY VALUE PROBLEMS FOR SINGULAR FRACTIONAL DIFFERENTIAL SYSTEMS

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Abstract. A class of impulsive boundary value problems of fractional differential systems is studied. Banach spaces are constructed and nonlinear operators defined on these Banach spaces. Sufficient conditions are given for the existence of solutions of this class of impulsive boundary value problems for singular fractional differential systems in which odd homeomorphism operators (Definition 2.6) are involved. Main results are Theorem 4.1 and Corollary 4.2. The analysis relies on a well known fixed point theorem: Leray-Schauder Nonlinear Alternative (Lemma 2.1). An example is given to illustrate the efficiency of the main theorems, see Example 5.1.

Keywords: singular fractional differential system; impulsive boundary value problem; fixed point theorem

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1. Introduction

Fractional differential equations are a generalization of ordinary differential equations to arbitrary non integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventieth century. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation [12], [14], [20], [19].

Many fractional ecological models have been proposed. In [2], [11], [18], [17], the following periodic type boundary value problem (BVP) for fractional differential

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equations is studied:

$$\text{BVP(1)} \qquad \begin{cases} D_{0+}^{\alpha} u(t) - \lambda u(t) = f(t, u(t)), \quad t \in (0,1],\ 0 < \alpha \leqslant 1, \\ u(1) - \lim\limits_{t \to 0} t^{1-\alpha} u(t) = 0, \end{cases}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of order α, f is continuous and $\lambda \in \mathbb{R}$.

The existence and uniqueness of solutions of BVP(2) are established under some assumptions by using Banach's contraction principle. One of the main assumptions in [2] is as follows:

(A) There exist positive numbers M and m such that $|f(t,x)| \leqslant M$, $|I(x)| \leqslant m$, $t \in [0,1]$, $x \in \mathbb{R}$ holds or (see Remark 4.4 in [2]) for each $u_0 \in C_{1-\alpha}[0,1]$ fixed, there exists $k_{u_0} > 0$ such that

$$|f(t,u) - f(t, u_0(t))| \leqslant k_{u_0} |u - u_0(t)|, \quad t \in [0,1],\ u \in \mathbb{R}.$$

The equation $D_{0+}^{\alpha} u(t) - \lambda u(t) = f(t, u(t))$ in BVP(1) can been seen as the generalized form of the ecological model $x'(t) - \lambda x(t) = f(t, x(t))$ which is a perturbation of the standard Malthus population model $y' = \lambda y$.

The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, for example, phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. For an introduction to the basic theory of impulsive differential equation see [9].

In recent years, many authors [1], [4], [6]–[8], [10], [16], [21], [24], [25] studied the existence or uniqueness of (positive or not) solutions of boundary value problems for the impulsive fractional differential equations with order $\alpha \in (1,2]$.

In [23], the authors studied the existence and uniqueness of solutions of the periodic type boundary value problem of the impulsive fractional differential equation

$$\text{BVP(2)} \quad \begin{cases} D_{0+}^{\alpha} u(t) - \lambda u(t) = f(t, u(t)), \quad t \in (0,1],\ t \neq t_1 \in (0,1),\ 0 < \alpha \leqslant 1, \\ u(1) - \lim\limits_{t \to 0} t^{1-\alpha} u(t) = 0, \\ \lim\limits_{t \to t_1^+} (t - t_1)^{1-\alpha} (u(t) - u(t_1)) = I(u(t_1)), \end{cases}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of order α, $I\colon \mathbb{R} \to \mathbb{R}$ is continuous, $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $\lambda \in \mathbb{R}$ is a constant. The existence and uniqueness of solutions of BVP(1) are established under some assumptions by using Banach's contraction principle. One of the main assumptions in [23] is as follows:

(B) There exist positive numbers M and m such that

$$|f(t,x)| \leqslant M, \quad |I(x)| \leqslant m, \quad t \in [0,1],\ x \in \mathbb{R}.$$

The solvability of boundary value problems of fractional differential systems was studied in [3], [15], [22], [24], [26]–[29]. On the contrary, the existence of solutions for impulsive boundary value problems involving Riemann-Liouville fractional differential systems has not been satisfactorily solved.

In this paper, we discuss the boundary value problem of the impulsive singular fractional differential system

$$\text{BVP(3)} \qquad \begin{cases} D_{0+}^{\alpha} x(t) - \lambda x(t) = p(t) f(t, x(t), y(t)), & t \in (0,1),\ t \neq t_1, \\ D_{0+}^{\beta} y(t) - \mu y(t) = q(t) g(t, x(t), y(t)), & t \in (0,1),\ t \neq t_1, \\ \lim\limits_{t\to 1} t^{1-\alpha} x(t) - \lim\limits_{t \to 0} t^{1-\alpha} x(t) = \int_0^1 \varphi(s) G(s, x(s), y(s))\,\mathrm{d}s, \\ \lim\limits_{t\to 1} t^{1-\beta} y(t) - \lim\limits_{t \to 0} t^{1-\beta} y(t) = \int_0^1 \psi(s) H(s, x(s), y(s))\,\mathrm{d}s, \\ \lim\limits_{t \to t_1^+} (t-t_1)^{1-\alpha} x(t) = I(t_1, x(t_1), y(t_1)), \\ \lim\limits_{t \to t_1^+} (t-t_1)^{1-\beta} y(t) = J(t_1, x(t_1), y(t_1)), \end{cases}$$

where

(a) $0 < \alpha, \beta \leqslant 1$, $\lambda, \mu \in \mathbb{R}$ with $\lambda \neq 0$, $\mu \neq 0$, D_{0+}^{α} (or D_{0+}^{β}) is the Riemann-Liouville fractional derivative of order α (or β),
(b) $0 = t_0 < t_1 < t_2 = 1$, $I, J\colon (0,1) \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions,
(c) $\Phi\colon \mathbb{R} \to \mathbb{R}$ is a sup-multiplicative-like function with supporting function ω, its inverse function is denoted by $\Phi^{-1}\colon \mathbb{R} \to \mathbb{R}$ with supporting function ν,
(d) $\varphi, \psi\colon (0,1) \to \mathbb{R}$ satisfy

$$\varphi|_{(0,t_1)}, \psi|_{(0,t_1)} \in L^1(0,t_1), \quad \varphi|_{(t_1,1)}, \psi|_{(t_1,1)} \in L^1(t_1,1),$$

(e) $p, q\colon (0,1) \to \mathbb{R}$ satisfy the growth conditions: there exist constants $l_i \geqslant 0$, $i = 1, 2$, $k_1 > -\alpha$, $k_2 > -\beta$ such that

$$|p(t)| \leqslant l_1 t^{k_1}, \quad |q(t)| \leqslant l_2 t^{k_2}, \quad t \in (0,1),$$

(f) f, g, G, H defined on $(0,1] \times \mathbb{R} \times \mathbb{R}$ are *impulsive Carathéodory functions*, I, J defined on $\{t_1\} \times \mathbb{R}^2$ are continuous functions, all of which may be singular at $t = 0$, $t = t_1$.

The functions $x, y\colon\ (0,1) \to \mathbb{R}$ are called a solution of BVP(3), if

$$x|_{(t_k,t_{k+1}]} \in C^0(t_k,t_{k+1}], \quad y|_{(t_k,t_{k+1}]} \in C^0(t_k,t_{k+1}], \quad k=0,1,$$
$$D_{0+}^{\alpha} x|_{(t_k,t_{k+1}]} \in L^1(t_k,t_{k+1}], \quad D_{0+}^{\beta} y|_{(t_k,t_{k+1}]} \in L^1(t_k,t_{k+1}], \quad k=0,1$$

and the limits

$$\lim_{t\to t_k^+} (t-t_k)^{1-\alpha} x(t), \quad \lim_{t\to t_k^+} (t-t_k)^{1-\beta} y(t), \quad k=0,1,$$

exist and x, y satisfy all equations in BVP(3).

We obtain results technically on the existence of at least one solution for BVP(3). An example is given to illustrate the efficiency of the main theorem. The results in this paper generalize the ones of [2], [18], [17], [23], in the sense that

(i) periodic type boundary value problems of coupled singular fractional differential equations with integral boundary conditions are discussed,
(ii) the assumptions (A) or (B) are replaced by weaker ones (see (C) and (D) in Theorem 4.1 in Section 4),
(iii) both p and q may be singular at $t=0$,
(iv) differently from [2], [18], [17], [23], Mawhin's fixed point theorem from [13] is used in this paper. An example is given to illustrate the applicability of the main results.

The paper is divided into five sections. Section 1 is an introduction. In Section 2, we present preliminary results. In Section 3, an important lemma is proved. The main theorems and their proofs are given in Section 4. In Section 5, an example is given to illustrate the main results.

2. Preliminary results

For the convenience of the readers, we first present the necessary definitions from the fractional calculus theory. These definitions and results can be found in the literature [7], [8].

Let the functions Gamma and Beta be

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \mathrm{e}^{-x}\,\mathrm{d}x, \quad \mathbf{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x.$$

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g\colon\ (0,\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\,\mathrm{d}s,$$

provided that the right-hand side exists.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $g\colon (0,\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}}\, \mathrm{d}s,$$

where $n-1 < \alpha < n$, provided that the right-hand side is point-wise defined on $(0,\infty)$.

Definition 2.3. Let X and Y be Banach spaces. $L\colon D(L) \subset X \to Y$ is called a Fredholm operator of index zero if $\operatorname{Im} L$ is closed in X and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$.

It is easy to see that if L is a Fredholm operator of index zero, then there exist projectors $P\colon X \to X$, and $Q\colon Y \to Y$ such that

$$\operatorname{Im} P = \operatorname{Ker} L, \quad \operatorname{Ker} Q = \operatorname{Im} L, \quad X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

If $L\colon D(L) \subset X \to Y$ is a Fredholm operator of index zero, the inverse of

$$L|_{D(L)\cap \operatorname{Ker} P}\colon D(L) \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is denoted by K_p.

Definition 2.4. Suppose that $L\colon D(L) \subset X \to Y$ is a Fredholm operator of index zero. The continuous map $N\colon X \to Y$ is called L-compact if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N(\overline{\Omega})$ is compact for each nonempty open subset Ω of X satisfying $D(L) \cap \overline{\Omega} \neq \emptyset$.

To obtain the main results, we need the abstract existence theorem.

Lemma 2.1 ([13], Leray-Schauder nonlinear alternative)**.** *Let Z_1, Z_2 be two Banach spaces and $L\colon D(L) \cap Z_1 \to Z_2$ a Fredholm operator of index zero with $\operatorname{Ker} L = \{0 \in Z_1\}$, $N\colon Z_1 \to Z_2$, L-compact. Suppose Ω is a nonempty open subset of Z_1 satisfying $D(L) \cap \overline{\Omega} \neq \emptyset$. Then either there exists $x \in \partial\Omega$ and $\theta \in (0,1)$ such that $Lx = \theta N x$ or there exists $x \in \overline{\Omega}$ such that $Lx = Nx$.*

Definition 2.5. We call $F\colon (0,1] \times \mathbb{R}^2 \to \mathbb{R}$ an *impulsive Carathéodory function* if it satisfies

(i) $t \to F(t,(t-t_k)^{\alpha-1}u,(t-t_k)^{\beta-1}v)$, are continuous on $(t_k,t_{k+1}]$, $k = 0,1$ and the limits

$$\lim_{t\to 0^+} F(t,t^{\alpha-1}u,t^{\beta-1}v), \quad \lim_{t\to t_1^+} F(t,(t-t_1)^{\alpha-1}u,(t-t_1)^{\beta-1}v)$$

exist for any $(u,v) \in \mathbb{R}^2$,

(ii) $(x,y) \to (t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)$, $k=0,1$ are continuous on $\mathbb{R}^2$ for all $t\in(0,1]$.

Definition 2.6 ([5]). An odd homeomorphism Φ of the real line $\mathbb{R}$ onto itself is called a sup-multiplicative-like function if there exists a homeomorphism ω of $[0,\infty)$ onto itself which supports Φ in the sense that for all $v_1, v_2 \geqslant 0$ we have

$$\Phi(v_1v_2) \geqslant \omega(v_1)\Phi(v_2). \tag{1}$$

Then ω is called the *supporting function* of Φ.

Remark 2.1. Note that any sup-multiplicative function is a sup-multiplicative-like function. Also any function of the form

$$\Phi(u) := \sum_{j=0}^{k} c_j|u|^j u, \quad u\in\mathbb{R}$$

is sup-multiplicative-like, provided that $c_j \geqslant 0$. Here a supporting function is defined by $\omega(u) := \min\{u^{k+1}, u\}$, $u \geqslant 0$.

Remark 2.2. It is clear that a sup-multiplicative-like function Φ and any corresponding supporting function ω are increasing functions vanishing at zero and, moreover, their inverses Φ^{-1} and ν are increasing and such that

$$\Phi^{-1}(w_1w_2) \leqslant \nu(w_1)\Phi^{-1}(w_2) \tag{2}$$

for all $w_1, w_2 \geqslant 0$ and ν is called the supporting function of Φ^{-1}.

3. Lemmas

To get solutions of BVP(3), we will use Lemma 2.1. So we define two Banach spaces Z_1 and Z_2 and define a linear operator $L\colon D(L)\cap Z_1 \to Z_2$ and a nonlinear operator $N\colon Z_1 \to Z_2$. Then we will prove technically that L is a Fredholm operator of index zero and N is L-compact. Some growth conditions are imposed on f, g, H, G, I, J to guarantee the existence of solutions of BVP(3).

We use the Banach spaces

$$X = \Big\{x\colon (0,1]\to\mathbb{R}\colon x|_{(0,t_1]}\in C^0(0,t_1],\ x|_{(t_1,1]}\in C^0(t_1,1] \\ \text{there exist limits } \lim_{t\to 0^+} t^{1-\alpha}x(t),\ \lim_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t)\Big\}$$

with the norm

$$\|x\| = \|x\|_\infty = \max\Big\{ \sup_{t\in(0,t_1]} t^{1-\alpha}|x(t)|, \sup_{t\in(t_1,1]} (t-t_1)^{1-\alpha}|x(t)| \Big\},$$

$$Y = \Big\{ y\colon (0,1] \to R\colon\ y|_{(0,t_1]} \in C^0(0,t_1],\ y|_{(t_1,1]} \in C^0(t_1,1]$$
$$\text{there exist the limits } \lim_{t\to 0^+} t^{1-\beta}y(t),\ \lim_{t\to t_1^+} (t-t_1)^{1-\beta}y(t) \Big\}$$

with the norm

$$\|y\| = \|y\|_\infty = \max\Big\{ \sup_{t\in(0,t_1]} t^{1-\beta}|y(t)|, \sup_{t\in(t_1,1]} (t-t_1)^{1-\beta}|y(t)| \Big\},$$

and $L^1[0,1]$ with the norm

$$\|u\|_1 = \int_0^1 |u(s)|\,\mathrm{d}s.$$

Choose $Z_1 = X \times Y$ with the norm

$$\|(x,y)\| = \max\{\|x\|_\infty, \|y\|_\infty\}.$$

Choose $Z_2 = L^1(0,1) \times L^1(0,1) \times \mathbb{R}^4$ with the norm

$$\|(u,v,a,b,c,d)\| = \max\{\|u\|_1, \|v\|_1, |a|, |b|, |c|, |d|\}.$$

Define L to be the linear operator from $D(L) \cap Z_1$ to Z_2 with

$$D(L) = \{(x,y) \in E\colon\ D_{0^+}^{\alpha}x, D_{0^+}^{\beta}y \in L^1(0,1)\}$$

and

$$L(x,y)(t) = \begin{pmatrix} D_{0^+}^{\alpha}x(t) - \lambda x(t) \\ D_{0^+}^{\beta}y(t) - \mu y(t) \\ \lim\limits_{t\to 1} t^{1-\alpha}x(t) - \lim\limits_{t\to 0} t^{1-\alpha}x(t) \\ \lim\limits_{t\to 1} t^{1-\beta}y(t) - \lim\limits_{t\to 0} t^{1-\beta}y(t) \\ \lim\limits_{t\to t_1^+} (t-t_1)^{1-\alpha}x(t) \\ \lim\limits_{t\to t_1^+} (t-t_1)^{1-\beta}y(t) \end{pmatrix}^{\mathrm{T}}$$

for $(x, y) \in E$. Define $N\colon\ E \to Z$ by

$$N(x,y)(t) = \begin{pmatrix} p(t)f(t,x(t),y(t)) \\ q(t)g(t,x(t),y(t)) \\ \displaystyle\int_0^1 \varphi(s)G(t,x(t),y(t))\,\mathrm{d}t \\ \displaystyle\int_0^1 \psi(s)H(t,x(t),y(t))\,\mathrm{d}t \\ I(t_1,x(t_1),y(t_1)) \\ J(t_1,x(t_1),y(t_1)) \end{pmatrix}^{\mathrm{T}} \quad \text{for } (x,y) \in E.$$

Then BVP(3) can be written as

$$L(x,y) = N(x,y), \quad (x,y) \in E.$$

Lemma 3.1. *Suppose that* (a)–(f) *hold and f, g, G, H are impulsive Carathéodory functions, I and J continuous functions. Then L is a Fredholm operator with index zero and $N\colon\ Z_1 \to Z_2$ is L-compact.*

P r o o f. To prove that L is a Fredholm operator with index zero, we should do the following three steps.

Step 1. Prove that $\operatorname{Ker} L = \{(0,0) \in E\}$.

We know that $(x, y) \in \operatorname{Ker} L$ if and only if

$$(3) \qquad \begin{cases} D_{0^+}^{\alpha} x(t) - \lambda x(t) = 0, \\ D_{0^+}^{\beta} y(t) - \mu y(t) = 0, \\ \lim\limits_{t\to 1} t^{1-\alpha} x(t) - \lim\limits_{t\to 0} t^{1-\alpha} x(t) = 0, \\ \lim\limits_{t\to 1} t^{1-\beta} y(t) - \lim\limits_{t\to 0} t^{1-\beta} y(t) = 0, \\ \lim\limits_{t\to t_1^+} (t-t_1)^{1-\alpha} x(t) = 0, \\ \lim\limits_{t\to t_1^+} (t-t_1)^{1-\beta} y(t) = 0. \end{cases}$$

From Lemma 2.1 in [3], $(x, y) \in \operatorname{Ker} L$ if and only if $x(t) = 0$ and $y(t) = 0$. Thus $\operatorname{Ker} L = \{(0,0) \in E\}$.

Step 2. Prove that $\operatorname{Im} L = Z$.

For $(u, v, a, b, c, d) \in Z$, we know that $(u, v, a, b, c, d) \in \operatorname{Im} L$ if and only if there exist $(x, y) \in E$ such that

$$(4) \qquad \begin{cases} D_{0^+}^{\alpha} x(t) - \lambda x(t) = u(t), \\ D_{0^+}^{\beta} y(t) - \mu y(t) = v(t), \\ \lim\limits_{t\to 1} t^{1-\alpha} x(t) - \lim\limits_{t\to 0} t^{1-\alpha} x(t) = a, \\ \lim\limits_{t\to 1} t^{1-\beta} y(t) - \lim\limits_{t\to 0} t^{1-\beta} y(t) = b, \\ \lim\limits_{t\to t_1^+} (t - t_1)^{1-\alpha} x(t) = c, \\ \lim\limits_{t\to t_1^+} (t - t_1)^{1-\beta} y(t) = d. \end{cases}$$

From Lemma 2.1 in [3], we know that

$$D_{0^+}^{\alpha} x(t) - \lambda x(t) = u(t), \quad \lim_{t\to 1} t^{1-\alpha} x(t) - \lim_{t\to 0} t^{1-\alpha} x(t) = a,$$

has a unique solution

$$\begin{aligned} x(t) = {} & \frac{\Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha})}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(\lambda)} \int_0^t E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})(1-s)^{\alpha-1} u(s)\,\mathrm{d}s \\ & + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^{\alpha}) u(s)\,\mathrm{d}s \\ & + \frac{\Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha})}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(\lambda)} \int_t^1 E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})(1-s)^{\alpha-1} u(s)\,\mathrm{d}s \\ & - \frac{a\Gamma(\alpha)}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(\lambda)} t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha}) \\ = {} & \int_0^1 G_{\lambda,\alpha}(t,s) u(s)\,\mathrm{d}s - \frac{a\Gamma(\alpha)}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(\lambda)} t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha}), \end{aligned}$$

where

$$G_{\lambda,\alpha}(t,s) = \begin{cases} \dfrac{\Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha})}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(\lambda)} E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})(1-s)^{\alpha-1} \\ \qquad\qquad + (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^{\alpha}), & 0 \leqslant s \leqslant t, \\ \dfrac{\Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha})}{1 - \Gamma(\alpha) E_{\alpha,\alpha}(\lambda)} E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})(1-s)^{\alpha-1}, & t \leqslant s \leqslant 1, \end{cases}$$

and

$$E_{\alpha,\alpha}(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma((k+1)\alpha)}$$

is the Mitag-Leffler function [8].

From Lemma 2.1 in [24], we know that

$$D_{0^+}^{\alpha}x(t)-\lambda x(t)=0,\quad \lim_{t\to 1}t^{1-\alpha}x(t)-\lim_{t\to 0}t^{1-\alpha}x(t)=0,\\ \lim_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t)=c, \tag{5}$$

has a unique solution

$$x(t)=\Gamma(\alpha)G_{\lambda,\alpha}(t,t_1)c.$$

Hence

$$\begin{cases} D_{0^+}^{\alpha}x(t)-\lambda x(t)=u(t),\\ \lim\limits_{t\to 1}t^{1-\alpha}x(t)-\lim\limits_{t\to 0}t^{1-\alpha}x(t)=a,\\ \lim\limits_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t)=c \end{cases}$$

has a solution

$$\begin{aligned} x(t)=&\frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^t E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})(1-s)^{\alpha-1}u(s)\,\mathrm{d}s\\ &+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})u(s)\,\mathrm{d}s\\ &+\frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_t^1 E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})(1-s)^{\alpha-1}u(s)\,\mathrm{d}s\\ &-\frac{a\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})+\Gamma(\alpha)G_{\lambda,\alpha}(t,t_1)c. \end{aligned} \tag{6}$$

Similarly we can prove that the system

$$D_{0^+}^{\beta}y(t)-\mu y(t)=v(t),\quad \lim_{t\to 1}t^{1-\beta}y(t)-\lim_{t\to 0}t^{1-\beta}y(t)=b,\quad \lim_{t\to t_1^+}(t-t_1)^{1-\beta}y(t)=d$$

has a unique solution

$$\begin{aligned} y(t)=&\frac{\Gamma(\beta)t^{\beta-1}E_{\beta,\beta}(\mu t^{\beta})}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}\int_0^t E_{\beta,\beta}(\mu(1-s)^{\beta})(1-s)^{\beta-1}v(s)\,\mathrm{d}s\\ &+\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(\mu(t-s)^{\beta})v(s)\,\mathrm{d}s\\ &+\frac{\Gamma(\beta)t^{\beta-1}E_{\beta,\beta}(\mu t^{\beta})}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}\int_t^1 E_{\beta,\beta}(\mu(1-s)^{\beta})(1-s)^{\beta-1}v(s)\,\mathrm{d}s\\ &-\frac{b\Gamma(\beta)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}t^{\beta-1}E_{\beta,\beta}(\mu t^{\beta})+\Gamma(\beta)G_{\mu,\beta}(t,t_1)d, \end{aligned} \tag{7}$$

where

$$G_{\mu,\beta}(t,s) = \begin{cases} \dfrac{\Gamma(\beta)t^{\beta-1}E_{\beta,\beta}(\mu t^{\beta})}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}E_{\beta,\beta}(\mu(1-s)^{\beta})(1-s)^{\beta-1} \\ \qquad\qquad\qquad +(t-s)^{\beta-1}E_{\beta,\beta}(\mu(t-s)^{\beta}), \quad 0\leqslant s\leqslant t, \\ \dfrac{\Gamma(\beta)t^{\beta-1}E_{\beta,\beta}(\mu t^{\beta})}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}E_{\beta,\beta}(\mu(1-s)^{\beta})(1-s)^{\beta-1}, \quad t\leqslant s\leqslant 1, \end{cases}$$

and

$$E_{\beta,\beta}(\mu) = \sum_{k=0}^{\infty}\frac{\mu^k}{\Gamma((k+1)\beta)}$$

is the Mitag-Leffler function (see [8]).

It is easy to show that $(x,y)\in E\cap D(L)$. Then $\operatorname{Im} L = Z$.

Step 3. Prove that $\operatorname{Im} L$ is closed in X and $\dim\operatorname{Ker} L = \operatorname{codim}\operatorname{Im} L < \infty$.

From Step 2 $\operatorname{Im} L = Z$ is closed in Z. It follows from $\operatorname{Ker} L = \{(0,0)\in E\}$ that $\dim\operatorname{Ker} L = 0$. Define the projector $P\colon E\to E$ by

$$P(x,y)(t) = (0,0) \quad \text{for } (x,y)\in E. \tag{8}$$

It is easy to prove that

$$\operatorname{Im} P = \operatorname{Ker} L, \quad X = \operatorname{Ker} L\oplus\operatorname{Ker} P. \tag{9}$$

Define the projector $Q\colon Z\to Z$ by

$$Q(u,v,a,b,c,d)(t) = (0,0,0,0,0,0) \tag{10}$$

for $(u,v,a,b,c,d)\in Z$.

It is easy to show that

$$\operatorname{Im} L = \operatorname{Ker} Q, \quad Y = \operatorname{Im} Q\oplus\operatorname{Im} L. \tag{11}$$

From the above discussion, we see that $\dim\operatorname{Ker} L = \operatorname{codim}\operatorname{Im} L = 0 < \infty$. So L is a Fredholm operator of index zero.

Now, we prove that N is L-compact. The proof is standard and is divided into three steps.

Step 1. We prove that N is continuous.

Let $(x_n,y_n)\in E$ with $(x_n,y_n)\to(x_0,y_0)$ as $n\to\infty$. We will show that $N(x_n,y_n)\to N(x_0,y_0)$ as $n\to\infty$.

In fact, we have

$$\|(x_n, y_n)\| = \max\Big\{ \sup_{t\in(0,t_1]} t^{1-\alpha}|x_n(t)|, \sup_{t\in(t_1,1]} (t-t_1)^{1-\alpha}|x_n(t)|, \sup_{t\in(0,t_1]} t^{1-\beta}|y_n(t)|, \sup_{t\in(t_1,1]} (t-t_1)^{1-\beta}|y_n(t)|\Big\} = r < \infty$$

and

(12)
$$\max\Big\{\sup_{t\in(t_k,t_{k+1}]} (t-t_k)^{1-\alpha}|x_n(t)-x_0(t)|,\ k=0,1\Big\} \to 0, \quad n\to\infty,$$
$$\max\Big\{\sup_{t\in(t_k,t_{k+1}]} (t-t_k)^{1-\beta}|y_n(t)-y_0(t)|,\ k=0,1\Big\} \to 0, \quad n\to\infty.$$

By definition,

$$N(x_n,y_n)(t) = \begin{pmatrix} p(t)f(t,x_n(t),y_n(t)) \\ q(t)g(t,x_n(t),y_n(t)) \\ \int_0^1 \varphi(t)G(t,x_n(t),y_n(t))\,\mathrm{d}t \\ \int_0^1 \psi(t)H(t,x_n(t),y_n(t))\,\mathrm{d}t \\ I(t_1,x_n(t_1),y_n(t_1)) \\ J(t_1,x_n(t_1),y_n(t_1)) \end{pmatrix}^{\mathrm{T}} \quad \text{for } (x,y)\in E.$$

For any $\varepsilon > 0$, since f, g, G, H are *impulsive Carathéodory functions* and I, J are continuous functions, we know that $f(t,(t-t_k)^{\alpha-1}u,(t-t_k)^{\beta-1}v)$ is continuous on $[t_k,t_{k+1}]\times[-r,r]^2$, $k=0,1$, so $f(t,(t-t_k)^{\alpha-1}u,(t-t_k)^{\beta-1}v)$ is uniformly continuous on $[t_k,t_{k+1}]\times[-r,r]^2$, $k=0,1$.

Similarly, g, G, H are uniformly continuous on $[t_k,t_{k+1}]\times[-r,r]^2$, $k=0,1$. Then there exists $\delta>0$ for $t\in(t_k,t_{k+1}]$ such that

$$|f(t,(t-t_k)^{\alpha-1}u_1,(t-t_k)^{\beta-1}v_1) - f(t,(t-t_k)^{\alpha-1}u_2,(t-t_k)^{\beta-1}v_2)| < \varepsilon,$$
$$|g(t,(t-t_k)^{\alpha-1}u_1,(t-t_k)^{\beta-1}v_1) - g(t,(t-t_k)^{\alpha-1}u_2,(t-t_k)^{\beta-1}v_2)| < \varepsilon,$$
$$|G(t,(t-t_k)^{\alpha-1}u_1,(t-t_k)^{\beta-1}v_1) - G(t,(t-t_k)^{\alpha-1}u_2,(t-t_k)^{\beta-1}v_2)| < \varepsilon,$$
$$|H(t,(t-t_k)^{\alpha-1}u_1,(t-t_k)^{\beta-1}v_1) - H(t,(t-t_k)^{\alpha-1}u_2,(t-t_k)^{\beta-1}v_2)| < \varepsilon,$$

and

$$|I(t_1,(1-t_1)^{\alpha-1}u_1,(1-t_1)^{\beta-1}v_1) - I(t_1,(1-t_1)^{\alpha-1}u_2,(1-t_1)^{\beta-1}v_2)| < \varepsilon,$$
$$|J(t_1,(1-t_1)^{\alpha-1}u_1,(1-t_1)^{\beta-1}v_1) - J(t_1,(1-t_1)^{\alpha-1}u_2,(1-t_1)^{\beta-1}v_2)| < \varepsilon$$

for all $k=0,1$, $|u_1-u_2|<\delta$ and $|v_1-v_2|<\delta$ with $u_1,u_2,v_1,v_2\in[-,r,r]$.

From (12), there exists $N_0(\delta)$ such that

$$(13)\qquad \begin{aligned}&(t-t_k)^{1-\alpha}|x_n(t)-x_0(t)|<\delta, \quad t\in(t_k,t_{k+1}],\ k=0,1,\ n>N_0(\delta),\\ &(t-t_k)^{1-\beta}|y_n(t)-y_0(t)|<\delta, \quad t\in(t_k,t_{k+1}],\ k=0,1,\ n>N_0(\delta).\end{aligned}$$

Hence, we get

$$\begin{aligned}&\int_0^1 |p(t)f(t,x_n(t),y_n(t))-p(t)f(t,x_0(t),y_0(t))|\,\mathrm{d}t\\ &\quad=\sum_{k=0}^{1}\int_{t_k}^{t_{k+1}}\big\{|p(t)f(t,(t-t_k)^{\alpha-1}(t-t_k)^{1-\alpha}x_n(t),(t-t_k)^{\beta-1}(t-t_k)^{1-\beta}y_n(t))\\ &\quad\quad-p(t)f(t,(t-t_k)^{\alpha-1}(t-t_k)^{1-\alpha}x_0(t),(t-t_k)^{\beta-1}(t-t_k)^{1-\beta}y_0(t))|\big\}\,\mathrm{d}t\\ &\quad<\sum_{k=0}^{1}\int_{t_k}^{t_{k+1}}\varepsilon p(t)\,\mathrm{d}t=\varepsilon\int_0^1 p(t)\,\mathrm{d}t, \quad n>N_0(\delta).\end{aligned}$$

It follows that

$$(14)\qquad \left|\int_0^1 p(t)f(t,x_n(t),y_n(t))\,\mathrm{d}t-\int_0^1 p(t)f(t,x_0(t),y_0(t))\,\mathrm{d}t\right|<\varepsilon\int_0^1 p(t)\,\mathrm{d}t, \quad n>N_0(\delta).$$

Similarly for $n>N_0(\delta)$ we get

$$(15)\qquad \left|\int_0^1 q(t)g(t,x_n(t),y_n(t))\,\mathrm{d}t-\int_0^1 q(t)g(t,x_0(t),y_0(t))\,\mathrm{d}t\right|<\varepsilon\int_0^1 q(t)\,\mathrm{d}t,$$

$$(16)\qquad \left|\int_0^1 \varphi(t)G(t,x_n(t),y_n(t))\,\mathrm{d}t-\int_0^1 \varphi(t)G(t,x_0(t),y_0(t))\,\mathrm{d}t\right|<\varepsilon\int_0^1 \varphi(t)\,\mathrm{d}t,$$

$$(17)\qquad \left|\int_0^1 \psi(t)H(t,x_n(t),y_n(t))\,\mathrm{d}t-\int_0^1 \psi(t)H(t,x_0(t),y_0(t))\,\mathrm{d}t\right|<\varepsilon\int_0^1 \psi(t)\,\mathrm{d}t,$$

and

$$(18)\qquad |I(t_1,x_n(t_1),y_n(t_1))-I(t_1,x_0(t_1),y_0(t_1))|<\varepsilon, \quad n>N_0(\delta),$$

$$(19)\qquad |J(t_1,x_n(t_1),y_n(t_1))-J(t_1,x_0(t_1),y_0(t_1))|<\varepsilon, \quad n>N_0(\delta).$$

Then (14)–(19) imply that

$$\|N(x_n,y_n)-N(x_0,y_0)\|\to 0, \quad n\to\infty.$$

It follows that N is continuous.

Let $P\colon X \to X$ and $Q\colon Y \to Y$ be defined, respectively, by (8) and (10). For $(u, v, a, b, c, d) \in \operatorname{Im} L = Z$, let

$$K_P(u, v, a, b, c, d)(t) = (x(t), y(t)), \tag{20}$$

where $x(t)$ and $y(t)$ are defined by (6) and (7), respectively.

One can see that $K_P(u, v, a, b, c, d) \in D(L) \cap E$ and $K_P\colon \operatorname{Im} L \to D(L) \cap \operatorname{Ker} P$ is the inverse of $L\colon D(L) \cap \operatorname{Ker} P \to \operatorname{Im} L$. The isomorphism $\wedge\colon \operatorname{Ker} L \to Y/\operatorname{Im} L$ is given by

$$\wedge(0, 0) = (0, 0, 0, 0, 0, 0).$$

Furthermore, one has

$$QN(x, y)(t) = (0, 0, 0, 0, 0, \ldots, 0, 0, \ldots, 0), \tag{21}$$

and

$$K_p(I - Q)N(x, y)(t) = K_pN(x, y)(t) = (x_1(t), y_1(t)),$$

where

$$\begin{aligned} (22)\ x_1(t) &\\ = \frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1 - \Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} &\int_0^t E_{\alpha,\alpha}(\lambda(1 - s)^\alpha)(1 - s)^{\alpha-1}p(s)f(s, x(s), y(s))\,\mathrm{d}s \\ &+ \int_0^t (t - s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t - s)^\alpha)p(s)f(s, x(s), y(s))\,\mathrm{d}s \\ &+ \frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1 - \Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \int_t^1 E_{\alpha,\alpha}(\lambda(1 - s)^\alpha)(1 - s)^{\alpha-1}p(s)f(s, x(s), y(s))\,\mathrm{d}s \\ &- \int_0^1 \varphi(s)G(s, x(s), y(s))\,\mathrm{d}s\ \frac{\Gamma(\alpha)}{1 - \Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha) \\ &+ \Gamma(\alpha)G_{\lambda,\alpha}(t, t_1)I(t_1, x(t_1), y(t_1)), \end{aligned}$$

and

$$\begin{aligned} (23)\ y_1(t) &\\ = \frac{\Gamma(\beta)t^{\beta-1}E_{\beta,\beta}(\mu t^\beta)}{1 - \Gamma(\beta)E_{\beta,\beta}(\mu)} &\int_0^t E_{\beta,\beta}(\mu(1 - s)^\beta)(1 - s)^{\beta-1}q(s)g(s, x(s), y(s))\,\mathrm{d}s \\ &+ \int_0^t (t - s)^{\beta-1}E_{\beta,\beta}(\mu(t - s)^\beta)q(s)g(s, x(s), y(s))\,\mathrm{d}s \\ &+ \frac{\Gamma(\beta)t^{\beta-1}E_{\beta,\beta}(\mu t^\beta)}{1 - \Gamma(\beta)E_{\beta,\beta}(\mu)} \int_t^1 E_{\beta,\beta}(\mu(1 - s)^\beta)(1 - s)^{\beta-1}q(s)g(s, x(s), y(s))\,\mathrm{d}s \end{aligned}$$

$$- \int_0^1 \psi(s)H(s,x(s),y(s))\,\mathrm{d}s\;\frac{\Gamma(\beta)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}t^{\beta-1}E_{\beta,\beta}(\mu t^{\beta})$$
$$+\Gamma(\beta)G_{\mu,\beta}(t,t_1)J(t_1,x(t_1),y(t_1)).$$

Let Ω be a bounded open subset of E satisfying $\overline{\Omega}\cap D(L)\neq\emptyset$. We have

$$\|(x,y)\| = \max\Big\{\sup_{t\in(0,t_1]} t^{1-\alpha}|x(t)|,\ \sup_{t\in(t_1,1]}(t-t_1)^{1-\alpha}|x(t)|,\ \sup_{t\in(0,t_1]} t^{1-\beta}|y(t)|,\ \sup_{t\in(t_1,1]}(t-t_1)^{1-\beta}|y(t)|\Big\} = r<\infty,\quad (x,y)\in\Omega. \tag{24}$$

Since f is an *impulsive Carathéodory function*, $f(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)$ is continuous on $(t_k,t_{k+1}]\times[-r,r]^2$, $k=0,1$ and the limits

$$\lim_{t\to0^+} f(t,t^{\alpha-1}x,t^{\beta-1}y),\qquad \lim_{t\to t_1^+} f(t,(t-t_1)^{\alpha-1}x,(t-t_1)^{\beta-1}y)$$

exist for every $(x,y)\in[-r,r]^2$. Then $f(t,(t-t_1)^{\alpha-1}x,(t-t_1)^{\beta-1}y)$ is bounded on $[t_k,t_{k+1}]\times[-r,r]^2$, $k=0,1$.

Similarly, f, g, G, H and I, J are *impulsive Carathéodory functions*, I, J are continuous functions, which together with (24) implies that there exists a constant $M>0$ such that

$$\begin{aligned} &|f(t,x(t),y(t))| \\ &\quad= \big|f\big(t,(t-t_k)^{\alpha-1}(t-t_k)^{1-\alpha}x(t),(t-t_k)^{\beta-1}(t-t_k)^{1-\beta}y(t)\big)\big|\leqslant M,\\ &|g(t,x(t),y(t))|\leqslant M,\quad |G(t,x(t),y(t))|\leqslant M,\quad\text{and}\quad |H(t,x(t),y(t))|\leqslant M \\ &\hspace{20em}\text{hold for all } t\in(0,1),\\ &|I(t_1,x(t_1),y(t_1))|\leqslant M,\quad |J(t_1,x(t_1),y(t_1))|\leqslant M. \end{aligned} \tag{25}$$

Step 2. Prove that $QN(\overline{\Omega})$ is bounded.

It follows from (21) that $QN(\overline{\Omega})$ is bounded.

Step 3. Prove that $K_P(I-Q)N\colon\overline{\Omega}\to E$ is compact, i.e., prove that $K_P\times(I-Q)N(\overline{\Omega})$ is relatively compact. This is divided into three substeps:

Substep 3a. Prove that $K_P(I-Q)N(\overline{\Omega})$ is uniformly bounded.

We have for $t\in(0,t_1]$ that

$$\begin{aligned} &t^{1-\alpha}|x_1(t)| \\ &\quad= t^{1-\alpha}\bigg|\frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^t E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})(1-s)^{\alpha-1}p(s)f(s,x(s),y(s))\,\mathrm{d}s \\ &\qquad+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})p(s)f(s,x(s),y(s))\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \int_t^1 E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}p(s)f(s,x(s),y(s))\,\mathrm{d}s \\
&\quad - \int_0^1 \varphi(s)G(s,x(s),y(s))\,\mathrm{d}s\ \frac{\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha) \\
&\qquad\qquad\qquad\qquad + \Gamma(\alpha)G_{\lambda,\alpha}(t,t_1)I(t_1,x(t_1),y(t_1))\Bigg| \\
&\leqslant \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \int_0^t E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}l_1 s^{k_1}M\,\mathrm{d}s \\
&\quad + t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)l_1 s^{k_1}M\,\mathrm{d}s \\
&\quad + \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \int_t^1 E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}l_1 s^{k_1}M\,\mathrm{d}s \\
&\quad + \int_0^1 \varphi(s)M\,\mathrm{d}s\ \frac{\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda t^\alpha) \\
&\quad + \Gamma(\alpha)\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1}M \\
&\leqslant Ml_1\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \int_0^t E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}s^{k_1}\,\mathrm{d}s \\
&\quad + Ml_1t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)s^{k_1}\,\mathrm{d}s \\
&\quad + Ml_1\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \int_t^1 E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}s^{k_1}\,\mathrm{d}s \\
&\quad + \frac{M\|\varphi\|_1\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda t^\alpha) \\
&\quad + \Gamma(\alpha)\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1}M \\
&\leqslant Ml_1\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \int_0^t \sum_{k=0}^\infty \frac{\lambda^k(1-s)^{k\alpha}}{\Gamma((k+1)\alpha)}(1-s)^{\alpha-1}s^{k_1}\,\mathrm{d}s \\
&\quad + Ml_1t^{1-\alpha}\int_0^t \sum_{k=0}^\infty \frac{\lambda^k(t-s)^{k\alpha}}{\Gamma((k+1)\alpha)}(t-s)^{\alpha-1}s^{k_1}\,\mathrm{d}s \\
&\quad + Ml_1\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \int_t^1 \sum_{k=0}^\infty \frac{\lambda^k(1-s)^{k\alpha}}{\Gamma((k+1)\alpha)}(1-s)^{\alpha-1}s^{k_1}\,\mathrm{d}s \\
&\quad + \frac{M\|\varphi\|_1\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(|\lambda|) \\
&\quad + \Gamma(\alpha)\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1}M
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant Ml_1\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\sum_{k=0}^{\infty}\frac{\lambda^k\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}\\
&\quad+Ml_1t^{1-\alpha}\sum_{k=0}^{\infty}\frac{\lambda^k t^{\alpha(k+1)+k_1}\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}\\
&\quad+Ml_1\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\sum_{k=0}^{\infty}\frac{\lambda^k\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}\\
&\quad+\frac{M\|\varphi\|_1\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(|\lambda|)\\
&\quad+\Gamma(\alpha)\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1}M<\infty.
\end{aligned}
$$

For $t\in(t_1,1]$, we have similarly that

$$
\begin{aligned}
&(t-t_1)^{1-\alpha}|x_1(t)|\\
&=(t-t_1)^{1-\alpha}\Bigg|\frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^t E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}p(s)f(s,x(s),y(s))\,\mathrm{d}s\\
&\quad+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)p(s)f(s,x(s),y(s))\,\mathrm{d}s\\
&\quad+\frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_t^1 E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}p(s)f(s,x(s),y(s))\,\mathrm{d}s\\
&\quad-\int_0^1\varphi(s)G(s,x(s),y(s))\,\mathrm{d}s\ \frac{\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)\\
&\qquad\qquad+\Gamma(\alpha)G_{\lambda,\alpha}(t,t_1)I(t_1,x(t_1),y(t_1))\Bigg|\\
&\leqslant(1-t_1)^{1-\alpha}\frac{\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^t E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}l_1s^{k_1}M\,\mathrm{d}s\\
&\quad+(t-t_1)^{1-\alpha}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)l_1s^{k_1}M\,\mathrm{d}s\\
&\quad+(t-t_1)^{1-\alpha}\frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_t^1 E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}l_1s^{k_1}M\,\mathrm{d}s\\
&\quad+(t-t_1)^{1-\alpha}\int_0^1\varphi(s)M\,\mathrm{d}s\ \frac{\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)\\
&\quad+(t-t_1)^{1-\alpha}\Gamma(\alpha)G_{\lambda,\alpha}(t,t_1)M\\
&\leqslant(1-t_1)^{1-\alpha}\frac{\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^t E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}l_1s^{k_1}M\,\mathrm{d}s\\
&\quad+(t-t_1)^{1-\alpha}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)l_1s^{k_1}M\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+ (1-t_1)^{1-\alpha}\frac{\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_t^1 E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}l_1 s^{k_1}M\,\mathrm{d}s\\
&+ (1-t_1)^{1-\alpha}\frac{M\|\varphi\|_1\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\\
&+ (t-t_1)^{1-\alpha}\Gamma(\alpha)\Big(\frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1}\\
&\qquad\qquad + (t-t_1)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-t_1)^\alpha)\Big)M\\
\leqslant{}& Ml_1(1-t_1)^{1-\alpha}\frac{\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^t E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}s^{k_1}\,\mathrm{d}s\\
&+ Ml_1(t-t_1)^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)s^{k_1}\,\mathrm{d}s\\
&+ Ml_1(1-t_1)^{1-\alpha}\frac{\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_t^1 E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}s^{k_1}\,\mathrm{d}s\\
&+ (1-t_1)^{1-\alpha}\frac{M\|\varphi\|_1\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)\\
&+ M(1-t_1)^{1-\alpha}\Gamma(\alpha)\frac{\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1}\\
&+ E_{\alpha,\alpha}(|\lambda|)M<\infty.
\end{aligned}
$$

Due to the above discussion, there exists $M_1>0$ such that

$$
\|x_1\|_\infty=\max\Big\{\sup_{t\in(0,t_1]}t^{1-\alpha}|x_1(t)|,\ \sup_{t\in(t_1,1]}(t-t_1)^{1-\alpha}|x_1(t)|\Big\}\leqslant M_1<\infty.
$$

Similarly, we can show that there exists $M_2>0$ such that

$$
\|y_1\|_\infty=\max\Big\{\sup_{t\in(0,t_1]}t^{1-\beta}|y_1(t)|,\ \sup_{t\in(t_1,1]}(t-t_1)^{1-\beta}|y_1(t)|\Big\}\leqslant M_2<\infty.
$$

Hence $K_P(I-Q)N(\overline{\Omega})$ is uniformly bounded.

Substep 3b. Prove that $K_P(I-Q)N(\overline{\Omega})$ is equi-continuous on each subinterval $[e,f]$ of either $(0,t_1]$ or $(t_1,1]$.

Let $s_2\leqslant s_1$ and $s_1,s_2\in[e,f]$. Since

$$
\begin{aligned}
&\bigg|s_1^{1-\alpha}\int_0^{s_1}(s_1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(s_1-s)^\alpha)p(s)f(s,x(s),y(s))\,\mathrm{d}s\\
&\qquad - s_2^{1-\alpha}\int_0^{s_2}(s_2-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(s_2-s)^\alpha)p(s)f(s,x(s),y(s))\,\mathrm{d}s\bigg|\\
&\leqslant |s_1^{1-\alpha}-s_2^{1-\alpha}|\int_0^{s_1}(s_1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(s_1-s)^\alpha)|p(s)f(s,x(s),y(s))|\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+ s_2^{1-\alpha} \int_{s_2}^{s_1} (s_1 - s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(s_1 - s)^\alpha) |p(s) f(s, x(s), y(s))| \,\mathrm{d}s \\
&+ s_2^{1-\alpha} \int_0^{s_2} \big|(s_1 - s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(s_1 - s)^\alpha) \\
&\qquad\qquad - (s_2 - s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(s_2 - s)^\alpha)\big| |p(s) f(s, x(s), y(s))| \,\mathrm{d}s \\
\leqslant{}& M l_1 |s_1^{1-\alpha} - s_2^{1-\alpha}| \int_0^{s_1} (s_1 - s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(s_1 - s)^\alpha) s^{k_1} \,\mathrm{d}s \\
&+ M l_1 s_2^{1-\alpha} \int_{s_2}^{s_1} (s_1 - s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(s_1 - s)^\alpha) s^{k_1} \,\mathrm{d}s \\
&+ M l_1 s_2^{1-\alpha} \int_0^{s_2} \big|(s_1 - s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(s_1 - s)^\alpha) \\
&\qquad\qquad - (s_2 - s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(s_2 - s)^\alpha)\big| s^{k_1} \,\mathrm{d}s \\
={}& M l_1 |s_1^{1-\alpha} - s_2^{1-\alpha}| \int_0^{s_1} (s_1 - s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{\lambda^k (s_1 - s)^{\alpha k}}{\Gamma(\alpha(k+1))} s^{k_1} \,\mathrm{d}s \\
&+ M l_1 s_2^{1-\alpha} \int_{s_2}^{s_1} (s_1 - s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{\lambda^k (s_1 - s)^{\alpha k}}{\Gamma(\alpha(k+1))} s^{k_1} \,\mathrm{d}s \\
&+ M l_1 s_2^{1-\alpha} \int_0^{s_2} \Bigg|(s_1 - s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{\lambda^k (s_1 - s)^{\alpha k}}{\Gamma(\alpha(k+1))} \\
&\qquad\qquad - (s_2 - s)^{\alpha-1} \sum_{k=0}^{\infty} \frac{\lambda^k (s_2 - s)^{\alpha k}}{\Gamma(\alpha(k+1))}\Bigg| s^{k_1} \,\mathrm{d}s \\
={}& M l_1 |s_1^{1-\alpha} - s_2^{1-\alpha}| \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha(k+1))} \int_0^{s_1} (s_1 - s)^{\alpha(k+1)-1} s^{k_1} \,\mathrm{d}s \\
&+ M l_1 s_2^{1-\alpha} \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha(k+1))} \int_{s_2}^{s_1} (s_1 - s)^{\alpha(k+1)-1} s^{k_1} \,\mathrm{d}s \\
&+ M l_1 s_2^{1-\alpha} \int_0^{s_2} \Bigg|\sum_{k=0}^{\infty} \frac{\lambda^k (s_1 - s)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))} - \sum_{k=0}^{\infty} \frac{\lambda^k (s_2 - s)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))}\Bigg| s^{k_1} \,\mathrm{d}s \\
\leqslant{}& M l_1 |s_1^{1-\alpha} - s_2^{1-\alpha}| \sum_{k=0}^{\infty} \frac{\lambda^k s_1^{\alpha(k+1)+k_1}}{\Gamma(\alpha(k+1))} \int_0^1 (1 - w)^{\alpha(k+1)-1} w^{k_1} \,\mathrm{d}w \\
&+ M l_1 s_2^{1-\alpha} \sum_{k=0}^{\infty} \frac{\lambda^k s_1^{\alpha(k+1)+k_1}}{\Gamma(\alpha(k+1))} \int_{s_2/s_1}^1 (1 - w)^{\alpha(k+1)-1} w^{k_1} \,\mathrm{d}w \\
&+ M l_1 s_2^{1-\alpha} \int_0^{s_2} \Bigg|\sum_{k=0}^{\infty} \frac{\lambda^k (s_1 - s)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))} - \sum_{k=0}^{\infty} \frac{\lambda^k (s_2 - s)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))}\Bigg| s^{k_1} \,\mathrm{d}s
\end{aligned}
$$

$$\leqslant Ml_1|s_1^{1-\alpha}-s_2^{1-\alpha}|\sum_{k=0}^{\infty}\frac{\lambda^k s_1^{\alpha(k+1)+k_1}}{\Gamma(\alpha(k+1))}\int_0^1(1-w)^{\alpha(k+1)-1}w^{k_1}\,\mathrm{d}w$$
$$+Ml_1\sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha(k+1))}\int_{s_2/s_1}^1(1-w)^{\alpha(k+1)-1}w^{k_1}\,\mathrm{d}w$$
$$+Ml_1 s_2^{1-\alpha}\int_0^{s_2}\left|\sum_{k=0}^{\infty}\frac{\lambda^k(s_1-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))}-\sum_{k=0}^{\infty}\frac{\lambda^k(s_2-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))}\right|s^{k_1}\,\mathrm{d}s.$$

It is easy to see that

$$0<\int_0^1(1-w)^{\alpha(k+1)-1}w^{k_1}\,\mathrm{d}w\leqslant\int_0^1(1-w)^{\alpha-1}w^{k_1}\,\mathrm{d}w,$$
$$0<\int_{s_2/s_1}^1(1-w)^{\alpha(k+1)-1}w^{k_1}\,\mathrm{d}w\leqslant\int_0^1(1-w)^{\alpha-1}w^{k_1}\,\mathrm{d}w.$$

Then

$$|s_1^{1-\alpha}-s_2^{1-\alpha}|\sum_{k=0}^{\infty}\frac{\lambda^k s_1^{\alpha(k+1)+k_1}}{\Gamma(\alpha(k+1))}\int_0^1(1-w)^{\alpha(k+1)-1}w^{k_1}\,\mathrm{d}w\to 0$$

uniformly as $t_1\to t_2$,

and

$$\sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(\alpha(k+1))}\int_{s_2/s_1}^1(1-w)^{\alpha(k+1)-1}w^{k_1}\,\mathrm{d}w\to 0\quad\text{uniformly as } t_1\to t_2.$$

For estimating the third term, we denote

$$H_k=s_2^{1-\alpha}\int_0^{s_2}\left|\frac{\lambda^k(s_1-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))}-\frac{\lambda^k(s_2-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))}\right|s^{k_1}\,\mathrm{d}s.$$

We distinguish two cases:

Case 1. $\alpha(k+1)-1\geqslant 0$. We get

$$\begin{aligned}
H_k&=s_2^{1-\alpha}\int_0^{s_2}\left(\frac{\lambda^k(s_1-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))}-\frac{\lambda^k(s_2-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))}\right)s^{k_1}\,\mathrm{d}s\\
&=s_2^{1-\alpha}\left(\frac{\lambda^k s_1^{\alpha(k+1)+k_1}}{\Gamma(\alpha(k+1))}\int_0^{s_2/s_1}(1-w)^{\alpha k+\alpha-1}w^{k_1}\,\mathrm{d}w\right.\\
&\qquad\left.-\frac{\lambda^k s_2^{\alpha(k+1)+k_1}}{\Gamma(\alpha(k+1))}\int_0^1(1-w)^{\alpha k+\alpha-1}w^{k_1}\,\mathrm{d}w\right)\\
&\leqslant\lambda^k\left(\frac{s_1^{\alpha(k+1)+k_1}}{\Gamma(\alpha(k+1))}-\frac{s_2^{\alpha(k+1)+k_1}}{\Gamma(\alpha(k+1))}\right)\int_0^1(1-w)^{\alpha k+\alpha-1}w^{k_1}\,\mathrm{d}w\\
&\leqslant\lambda^k\left(\frac{s_1^{\alpha(k+1)+k_1}}{\Gamma(\alpha(k+1))}-\frac{s_2^{\alpha(k+1)+k_1}}{\Gamma(\alpha(k+1))}\right)\int_0^1(1-w)^{\alpha-1}w^{k_1}\,\mathrm{d}w.
\end{aligned}$$

Case 2. $\alpha(k+1)-1<0$. Then

$$\begin{aligned}
H_k &= s_2^{1-\alpha}\Big(\frac{\lambda^k s_2^{\alpha(k+1)+k_1}\int_0^1(1-w)^{\alpha k+\alpha-1}w^{k_1}\,\mathrm{d}w}{\Gamma(\alpha(k+1))}\\
&\quad-\frac{\lambda^k s_1^{\alpha(k+1)+k_1}\int_0^{s_2/s_1}(1-w)^{\alpha k+\alpha-1}w^{k_1}\,\mathrm{d}w}{\Gamma(\alpha(k+1))}\Big)\\
&\leqslant \frac{\lambda^k}{\Gamma(\alpha(k+1))}\Big(s_2^{\alpha(k+1)+k_1}\int_0^1(1-w)^{\alpha k+\alpha-1}w^{k_1}\,\mathrm{d}w\\
&\quad-s_1^{\alpha(k+1)+k_1}\int_0^{s_2/s_1}(1-w)^{\alpha k+\alpha-1}w^{k_1}\,\mathrm{d}w\Big)\\
&= \frac{\lambda^k}{\Gamma(\alpha(k+1))}\Big((s_2^{\alpha(k+1)+k_1}-s_1^{\alpha(k+1)+k_1})\int_0^1(1-w)^{\alpha k+\alpha-1}w^{k_1}\,\mathrm{d}w\\
&\quad+s_1^{\alpha(k+1)+k_1}\int_{s_2/s_1}^1(1-w)^{\alpha k+\alpha-1}w^{k_1}\,\mathrm{d}w\Big).
\end{aligned}$$

Denoting by $[\cdot]$ the integer part function, we have

$$\begin{aligned}
&s_2^{1-\alpha}\int_0^{s_2}\Big|\sum_{k=0}^{\infty}\frac{\lambda^k(s_1-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))}-\sum_{k=0}^{\infty}\frac{\lambda(s_2-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))}\Big|s^{k_1}\,\mathrm{d}s\\
&\leqslant\sum_{k=0}^{[1/\alpha]-1}\int_0^{s_2}\Big(-\sum_{k=0}^{\infty}\frac{\lambda^k(s_1-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))}+\sum_{k=0}^{\infty}\frac{\lambda(s_2-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))}\Big)s^{k_1}\,\mathrm{d}s\\
&\quad+\sum_{k=[1/\alpha]}^{\infty}\int_0^{s_2}\Big(\sum_{k=0}^{\infty}\frac{\lambda^k(s_1-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))}-\sum_{k=0}^{\infty}\frac{\lambda(s_2-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))}\Big)s^{k_1}\,\mathrm{d}s\\
&\leqslant\sum_{k=0}^{[1/\alpha]-1}\frac{\lambda^k}{\Gamma(\alpha(k+1))}\Big((s_2^{\alpha(k+1)+k_1}-s_1^{\alpha(k+1)+k_1})\int_0^1(1-w)^{\alpha k+\alpha-1}w^{k_1}\,\mathrm{d}w\\
&\qquad\qquad+s_1^{\alpha(k+1)+k_1}\int_{s_2/s_1}^1(1-w)^{\alpha k+\alpha-1}w^{k_1}\,\mathrm{d}w\Big)\\
&\quad+\sum_{k=[1/\alpha]}^{\infty}\lambda^k\Big(\frac{s_1^{\alpha(k+1)+k_1}}{\Gamma(\alpha(k+1))}-\frac{s_2^{\alpha(k+1)+k_1}}{\Gamma(\alpha(k+1))}\Big)\int_0^1(1-w)^{\alpha-1}w^{k_1}\,\mathrm{d}w.
\end{aligned}$$

The finite sum obviously has limit zero as $s_1\to s_2$. The infinite sum is equal to

$$\sum_{k=[1/\alpha]}^{\infty}\lambda^k\Big(\frac{s_1^{\alpha(k+1)+k_1}}{\Gamma(\alpha(k+1))}-\frac{s_2^{\alpha(k+1)+k_1}}{\Gamma(\alpha(k+1))}\Big)\int_0^1(1-w)^{\alpha-1}w^{k_1}\,\mathrm{d}w$$

and its limit as $s_1\to s_2$ is zero as well.

From Cases 1 and 2, we get

$$\Big|s_1^{1-\alpha}\int_0^{s_1}(s_1-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(s_1-s)^\alpha)p(s)f(s,x(s),y(s))\,\mathrm{d}s - s_2^{1-\alpha}\int_0^{s_2}(s_2-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(s_2-s)^\alpha)p(s)f(s,x(s),y(s))\,\mathrm{d}s\Big| \to 0$$

uniformly as $s_1 \to s_2$.

It is easy to see that

$$\begin{aligned}
&\Big|\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda s_1^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_{s_1}^1 E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}p(s)f(s,x(s),y(s))\,\mathrm{d}s \\
&\quad - \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda s_2^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_{s_2}^1 E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}p(s)f(s,x(s),y(s))\,\mathrm{d}s\Big| \\
&\quad \leqslant \frac{\Gamma(\alpha)|E_{\alpha,\alpha}(\lambda s_1^\alpha)-E_{\alpha,\alpha}(\lambda s_2^\alpha)|}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_{s_1}^1 E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}l_1 s^{k_1}M\,\mathrm{d}s \\
&\quad + \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda s_2^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_{s_2}^{s_1} E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}l_1 s^{k_1}M\,\mathrm{d}s \\
&\quad \leqslant Ml_1\frac{\Gamma(\alpha)|E_{\alpha,\alpha}(\lambda s_1^\alpha)-E_{\alpha,\alpha}(\lambda s_2^\alpha)|}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_{s_1}^1\sum_{k=0}^\infty\frac{\lambda^k(1-s)^{k\alpha}}{\Gamma((k+1)\alpha)}(1-s)^{\alpha-1}s^{k_1}\,\mathrm{d}s \\
&\quad + Ml_1\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_{s_2}^{s_1}\sum_{k=0}^\infty\frac{\lambda^k(1-s)^{k\alpha}}{\Gamma((k+1)\alpha)}(1-s)^{\alpha-1}s^{k_1}\,\mathrm{d}s \\
&\quad \leqslant Ml_1\frac{\Gamma(\alpha)|E_{\alpha,\alpha}(\lambda s_1^\alpha)-E_{\alpha,\alpha}(\lambda s_2^\alpha)|}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\sum_{k=0}^\infty\frac{\lambda^k\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)} \\
&\quad + Ml_1\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\sum_{k=0}^\infty\frac{\lambda^k}{\Gamma((k+1)\alpha)}\int_{s_2}^{s_1}(1-s)^{(k+1)\alpha-1}s^{k_1}\,\mathrm{d}s \to 0
\end{aligned}$$

uniformly as $s_1 \to s_2$.

Similarly we can show that

$$\Big|\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda s_1^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^{s_1} E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}p(s)f(s,x(s),y(s))\,\mathrm{d}s - \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda s_2^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^{s_2} E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}p(s)f(s,x(s),y(s))\,\mathrm{d}s\Big| \to 0$$

uniformly as $s_1 \to s_2$

and

$$\Big|-\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda s_1^{\alpha})}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^1\varphi(s)G(s,x(s),y(s))\,\mathrm{d}s + s_1^{1-\alpha}\Gamma(\alpha)G_{\lambda,\alpha}(s_1,t_1)I(t_1,x(t_1),y(t_1)) - \Big(-\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda s_2^{\alpha})}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^1\varphi(s)G(s,x(s),y(s))\,\mathrm{d}s + s_2^{1-\alpha}\Gamma(\alpha)G_{\lambda,\alpha}(s_2,t_1)I(t_1,x(t_1),y(t_1))\Big)\Big|\to 0$$

uniformly as $s_1\to s_2$.

For each $[e,f]\subseteq(0,t_1]$, and $s_1,s_2\in[e,f]$ with $s_2\geqslant s_1$, using (22), we have

$$|s_1^{1-\alpha}x_1(s_1)-s_2^{1-\alpha}x_1(s_2)|\to 0 \tag{26}$$

uniformly as $s_1\to s_2$, $s_1,s_2\in[e,f]\subseteq(0,t_1]$.

For each $[e,f]\subseteq(t_1,1]$, and $s_1,s_2\in[e,f]$ with $s_2\geqslant s_1$, using (22), we have

$$|(s_1-t_1)^{1-\alpha}x_1(s_1)-(s_2-t_1)^{1-\alpha}x_1(s_2)|\to 0$$

uniformly as $s_1\to s_2$.

Similarly, we can prove that

$$|(s_1-t_k)^{1-\beta}y_2(s_1)-(s_2-t_k)^{1-\beta}y_2(s_2)|\to 0 \tag{27}$$

uniformly as $s_1\to s_2$, $s_1,s_2\in[e,f]\subseteq(t_k,t_{k+1}]$, $k=0,1$.

Hence $K_P(I-Q)N(\overline{\Omega})$ is equi-continuous on each subinterval $[e,f]$ of either $(0,t_1]$ or $(t_1,1]$.

Substep 3c. Prove that $K_P(I-Q)N(\overline{\Omega})$ is equi-convergent both at $t=0$ and $t=t_1$.

Since

$$\Big|t^{1-\alpha}x_1(t)-\Big(\frac{1}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^1 E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})(1-s)^{\alpha-1}p(s)f(s,x(s),y(s))\,\mathrm{d}s - \frac{1}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^1\varphi(s)G(s,x(s),y(s))\,\mathrm{d}s + \frac{1}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})(1-t_1)^{\alpha-1}\Big)\Big|$$

$$
\begin{aligned}
&\leqslant \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^{\alpha})}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \int_0^t E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})(1-s)^{\alpha-1} l_1 s^{k_1} M \,\mathrm{d}s \\
&+ t^{1-\alpha} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^{\alpha}) l_1 s^{k_1} M \,\mathrm{d}s \\
&+ \Bigg| \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^{\alpha})}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \int_t^1 E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})(1-s)^{\alpha-1} p(s) f(s,x(s),y(s)) \,\mathrm{d}s \\
&- \frac{1}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \int_0^1 E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})(1-s)^{\alpha-1} p(s) f(s,x(s),y(s)) \,\mathrm{d}s \Bigg| \\
&+ \frac{M\|\varphi\|_1}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} |\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^{\alpha}) - 1| \\
&+ \Gamma(\alpha) \frac{|\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^{\alpha}) - 1|}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})(1-t_1)^{\alpha-1} M \to 0 \\
&\hspace{20em} \text{uniformly as } t \to t_k.
\end{aligned}
$$

Similarly we can show that

$$(t-t_1)^{1-\alpha} x_1(t) \text{ is convergent uniformly as } t \to t_1, \tag{28}$$

$$t^{1-\beta} y_1(t) \text{ is convergent uniformly as } t \to 0, \tag{29}$$

and

$$(t-t_1)^{1-\beta} y_1(t) \text{ is convergent uniformly as } t \to t_1. \tag{30}$$

So $K_P(I-Q)N(\overline{\Omega})$ is relatively compact. Then N is L-compact. The proofs are completed. □

4. Main results

Now, we prove the main theorem in this paper. Suppose that (a)–(f) hold. Let a_i, b_i, c_i, $i=1,2$, A_i, B_i, C_i, $i=1,2$ and $\overline{A}_i$, $\overline{B}_i$, $\overline{C}_i$, $i=1,2$ be nonnegative numbers. Denote

$$
\begin{aligned}
\sigma_1 =: \max \Bigg\{ & l_1 b_1 \Big(\frac{2\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} + 1 \Big) \sum_{k=0}^{\infty} \frac{\lambda^k \mathbf{B}((k+1)\alpha, k_1+1)}{\Gamma((k+1)\alpha)} \\
&+ \frac{\|\varphi\|_1 B_1 \Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \\
&+ \overline{B}_1 \Gamma(\alpha) \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})(1-t_1)^{\alpha-1},
\end{aligned}
$$

$$
\begin{aligned}
& l_1 b_1 (1-t_1)^{1-\alpha}\Big(\frac{\Gamma(\alpha) t_1^{\alpha-1} E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \sum_{k=0}^{\infty} \frac{\lambda^k \mathbf{B}((k+1)\alpha, k_1+1)}{\Gamma((k+1)\alpha)} \\
& + \sum_{k=0}^{\infty} \frac{\lambda^k t_1^{(k+1)\alpha+k_1} \mathbf{B}((k+1)\alpha, k_1+1)}{\Gamma((k+1)\alpha)}\Big) \\
& + B_1 (1-t_1)^{1-\alpha} \frac{\|\varphi\|_1 \Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} t_1^{\alpha-1} E_{\alpha,\alpha}(|\lambda|) \\
& + \overline{B}_1 \Big(\Gamma(\alpha) \frac{\Gamma(\alpha) t_1^{\alpha-1} E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha}) + E_{\alpha,\alpha}(|\lambda|)\Big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\delta_1 =: \max\Big\{ & l_1 a_1 \Big(\frac{2\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} + 1\Big) \sum_{k=0}^{\infty} \frac{\lambda^k \mathbf{B}((k+1)\alpha, k_1+1)}{\Gamma((k+1)\alpha)} \\
& + \frac{\|\varphi\|_1 A_1 \Gamma(\alpha) E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \\
& + \overline{A}_1 \Gamma(\alpha) \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})(1-t_1)^{\alpha-1}, \\
& l_1 a_1 (1-t_1)^{1-\alpha}\Big(\frac{2\Gamma(\alpha) t_1^{\alpha-1} E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \sum_{k=0}^{\infty} \frac{\lambda^k \mathbf{B}((k+1)\alpha, k_1+1)}{\Gamma((k+1)\alpha)} \\
& + \sum_{k=0}^{\infty} \frac{\lambda^k t_1^{(k+1)\alpha+k_1} \mathbf{B}((k+1)\alpha, k_1+1)}{\Gamma((k+1)\alpha)}\Big) \\
& + A_1 (1-t_1)^{1-\alpha} \frac{\|\varphi\|_1 \Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} t_1^{\alpha-1} E_{\alpha,\alpha}(|\lambda|) \\
& + \overline{A}_1 \Big(\Gamma(\alpha) \frac{\Gamma(\alpha) t_1^{\alpha-1} E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha}) + E_{\alpha,\alpha}(|\lambda|)\Big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\sigma_2 =: \max\Big\{ & l_2 b_2 \Big(\frac{2\Gamma(\beta)E_{\beta,\beta}(|\mu|)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)} + 1\Big) \sum_{k=0}^{\infty} \frac{\mu^k \mathbf{B}((k+1)\beta, k_2+1)}{\Gamma((k+1)\beta)} \\
& + \frac{\|\psi\|_1 B_2 \Gamma(\beta) E_{\beta,\beta}(|\mu|)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)} \\
& + \overline{B}_2 \Gamma(\beta) \frac{\Gamma(\beta)E_{\beta,\beta}(|\mu|)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)} E_{\beta,\beta}(\mu(1-t_1)^{\beta})(1-t_1)^{\beta-1}, \\
& l_2 b_2 (1-t_1)^{1-\beta}\Big(\frac{2\Gamma(\beta) t_1^{\beta-1} E_{\beta,\beta}(|\mu|)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)} \sum_{k=0}^{\infty} \frac{\mu^k \mathbf{B}((k+1)\beta, k_2+1)}{\Gamma((k+1)\beta)} \\
& + \sum_{k=0}^{\infty} \frac{\mu^k t_1^{(k+1)\beta+k_2} \mathbf{B}((k+1)\beta, k_2+1)}{\Gamma((k+1)\beta)}\Big) \\
& + B_2 (1-t_1)^{1-\beta} \frac{\|\psi\|_1 \Gamma(\beta)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)} t_1^{\beta-1} E_{\beta,\beta}(|\mu|) \\
& + \overline{B}_2 \Big(\Gamma(\beta) \frac{\Gamma(\beta) t_1^{\beta-1} E_{\beta,\beta}(|\mu|)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)} E_{\beta,\beta}(\mu(1-t_1)^{\beta}) + E_{\beta,\beta}(|\mu|)\Big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\delta_2 =: \max\Big\{& l_2 a_2\Big(\frac{2\Gamma(\beta)E_{\beta,\beta}(|\mu|)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}+1\Big)\sum_{k=0}^{\infty}\frac{\mu^k\mathbf{B}((k+1)\beta,k_2+1)}{\Gamma((k+1)\beta)}\\
&+\frac{\|\psi\|_1 A_2\Gamma(\beta)E_{\beta,\beta}(|\mu|)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}\\
&+\overline{A}_2\Gamma(\beta)\frac{\Gamma(\beta)E_{\beta,\beta}(|\mu|)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}E_{\beta,\beta}(\mu(1-t_1)^{\beta})(1-t_1)^{\beta-1},\\
& l_2 a_2(1-t_1)^{1-\beta}\Big(\frac{2\Gamma(\beta)t_1^{\beta-1}E_{\beta,\beta}(|\mu|)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}\sum_{k=0}^{\infty}\frac{\mu^k\mathbf{B}((k+1)\beta,k_2+1)}{\Gamma((k+1)\beta)}\\
&+\sum_{k=0}^{\infty}\frac{\mu^k t_1^{(k+1)\beta+k_2}\mathbf{B}((k+1)\beta,k_2+1)}{\Gamma((k+1)\beta)}\Big)\\
&+A_2(1-t_1)^{1-\beta}\frac{\|\psi\|_1\Gamma(\beta)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}t_1^{\beta-1}E_{\beta,\beta}(|\mu|)\\
&+\overline{A}_2\Big(\Gamma(\beta)\frac{\Gamma(\beta)t_1^{\beta-1}E_{\beta,\beta}(|\mu|)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}E_{\beta,\beta}(\mu(1-t_1)^{\beta})+E_{\beta,\beta}(|\mu|)\Big)\Big\}.
\end{aligned}
$$

Theorem 4.1. *Suppose that* (a)–(f) *(see Introduction) hold and*

(C) *Φ is a sup-multiplicative-like function with its supporting function w, the inverse function of Φ is Φ^{-1} with supporting function ν.*

(D) *f, g, H, G are impulsive Carathéodory functions, I, J are continuous functions and satisfy that there exist nonnegative constants c_i, b_i, a_i, $i=1,2$, C_i, B_i, A_i, $i=1,2$ and $\overline{C}_i$, $\overline{B}_i$, $\overline{A}_i$, $i=1,2$ for $t\in(t_k,t_{k+1}]$, $k=0,1$, such that*

$$
\begin{aligned}
|f(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)| &\leqslant c_1+b_1|x|+a_1\Phi^{-1}(|y|),\\
|g(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)| &\leqslant c_2+b_2\Phi(|x|)+a_2|y|,\\
|G(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)| &\leqslant C_1+B_1|x|+A_1\Phi^{-1}(|y|),\\
|H(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)| &\leqslant C_2+B_2\Phi(|x|)+A_2|y|,
\end{aligned}
$$

and

$$
\begin{aligned}
|I(t_1,(1-t_1)^{\alpha-1}x,(1-t_1)^{\beta-1}y)| &\leqslant \overline{C}_1+\overline{B}_1|x|+\overline{A}_1\Phi^{-1}(|y|),\\
|J(t_1,(1-t_1)^{\alpha-1}x,(1-t_1)^{\beta-1}y)| &\leqslant \overline{C}_2+\overline{B}_2\Phi(|x|)+\overline{A}_2|y|.
\end{aligned}
$$

Then BVP(3) *has at least one solution if*

$$
\sigma_1<1,\ \delta_2<1,\quad \frac{\sigma_2}{1-\delta_2}\Big/w\Big(\frac{1-\sigma_1}{2\delta_1}\Big)<1 \tag{31}
$$

or

$$
\sigma_1<1,\ \delta_2<1,\quad \frac{\delta_1}{1-\sigma_1}\nu\Big(\frac{2\sigma_2}{1-\delta_2}\Big)<1.
$$

P r o o f. For reader's convenient, denote

$$\mu_1 =: \max\Big\{l_1c_1\Big(\frac{2\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}+1\Big)\sum_{k=0}^{\infty}\frac{\lambda^k\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}$$
$$+\frac{\|\varphi\|_1C_1\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}$$
$$+\overline{C}_1\Gamma(\alpha)\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})(1-t_1)^{\alpha-1},$$
$$l_1c_1(1-t_1)^{1-\alpha}\Big(\frac{2\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\sum_{k=0}^{\infty}\frac{\lambda^k\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}$$
$$+\sum_{k=0}^{\infty}\frac{\lambda^k t_1^{(k+1)\alpha+k_1}\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}\Big)$$
$$+C_1(1-t_1)^{1-\alpha}\frac{\|\varphi\|_1\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)$$
$$+\overline{C}_1\Big(\Gamma(\alpha)\frac{\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})+E_{\alpha,\alpha}(|\lambda|)\Big)\Big\},$$

and

$$\mu_2 =: \max\Big\{l_2c_2\Big(\frac{2\Gamma(\beta)E_{\beta,\beta}(|\mu|)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}+1\Big)\sum_{k=0}^{\infty}\frac{\mu^k\mathbf{B}((k+1)\beta,k_2+1)}{\Gamma((k+1)\beta)}$$
$$+\frac{\|\psi\|_1C_2\Gamma(\beta)E_{\beta,\beta}(|\mu|)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}$$
$$+\overline{C}_2\Gamma(\beta)\frac{\Gamma(\beta)E_{\beta,\beta}(|\mu|)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}E_{\beta,\beta}(\mu(1-t_1)^{\beta})(1-t_1)^{\beta-1},$$
$$l_2c_2(1-t_1)^{1-\beta}\Big(\frac{2\Gamma(\beta)t_1^{\beta-1}E_{\beta,\beta}(|\mu|)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}\sum_{k=0}^{\infty}\frac{\mu^k\mathbf{B}((k+1)\beta,k_2+1)}{\Gamma((k+1)\beta)}$$
$$+\sum_{k=0}^{\infty}\frac{\mu^k t_1^{(k+1)\beta+k_2}\mathbf{B}((k+1)\beta,k_2+1)}{\Gamma((k+1)\beta)}\Big)$$
$$+C_2(1-t_1)^{1-\beta}\frac{\|\psi\|_1\Gamma(\beta)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}t_1^{\beta-1}E_{\beta,\beta}(|\mu|)$$
$$+\overline{C}_2\Big(\Gamma(\beta)\frac{\Gamma(\beta)t_1^{\beta-1}E_{\beta,\beta}(|\mu|)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}E_{\beta,\beta}(\mu(1-t_1)^{\beta})+E_{\beta,\beta}(|\mu|)\Big)\Big\}.$$

To apply Lemma 2.1, we should define an open bounded subset Ω of E centered at zero such that assumptions in Lemma 2.1 hold.

Let $\Omega_1=\{(x,y)\in E\cap D(L)\setminus\operatorname{Ker}L, L(x,y)=\theta N(x,y)$ for some $\theta\in(0,1)\}$. We prove that Ω_1 is bounded.

For $(x,y)\in\Omega_1$, we get $L(x,y)=\theta N(x,y)$ and $N(x,y)\in \operatorname{Im} L$. Then

$$\tag{32}
\begin{cases}
D^{\alpha}_{0+}x(t)-\lambda x(t)=\theta p(t)f(t,x(t),y(t)), & t\in(0,1),\ t\neq t_1,\\
D^{\beta}_{0+}y(t)-\mu y(t)=\theta q(t)g(t,x(t),y(t)), & t\in(0,1),\ t\neq t_1,\\
\lim\limits_{t\to1}t^{1-\alpha}x(t)-\lim\limits_{t\to0}t^{1-\alpha}x(t)=\theta\int_0^1\varphi(s)G(s,x(s),y(s))\,\mathrm{d}s,\\
\lim\limits_{t\to1}t^{1-\beta}y(t)-\lim\limits_{t\to0}t^{1-\beta}y(t)=\theta\int_0^1\psi(s)H(s,x(s),y(s))\,\mathrm{d}s,\\
\lim\limits_{t\to t_1^+}(t-t_1)^{1-\alpha}x(t)=\theta I(t_1,x(t_1),y(t_1)),\\
\lim\limits_{t\to t_1^+}(t-t_1)^{1-\beta}y(t)=\theta J(t_1,u(t_1),D^{\alpha}_{0+}u(t_1)).
\end{cases}$$

So

$$\tag{33}
\begin{aligned}
x(t)&\\
&=\theta\frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^t E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})(1-s)^{\alpha-1}p(s)f(s,x(s),y(s))\,\mathrm{d}s\\
&\quad+\theta\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})p(s)f(s,x(s),y(s))\,\mathrm{d}s\\
&\quad+\theta\frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_t^1 E_{\alpha,\alpha}(\lambda(1-s)^{\alpha})(1-s)^{\alpha-1}p(s)f(s,x(s),y(s))\,\mathrm{d}s\\
&\quad-\theta\frac{\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})\int_0^1\varphi(s)G(s,x(s),y(s))\,\mathrm{d}s\\
&\quad+\theta\Gamma(\alpha)G_{\lambda,\alpha}(t,t_1)I(t_1,x(t_1),y(t_1)),
\end{aligned}$$

and

$$\tag{34}
\begin{aligned}
y(t)&\\
&=\theta\frac{\Gamma(\beta)t^{\beta-1}E_{\beta,\beta}(\mu t^{\beta})}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}\int_0^t E_{\beta,\beta}(\mu(1-s)^{\beta})(1-s)^{\beta-1}q(s)g(s,x(s),y(s))\,\mathrm{d}s\\
&\quad+\theta\int_0^t(t-s)^{\beta-1}E_{\beta,\beta}(\mu(t-s)^{\beta})q(s)g(s,x(s),y(s))\,\mathrm{d}s\\
&\quad+\theta\frac{\Gamma(\beta)t^{\beta-1}E_{\beta,\beta}(\mu t^{\beta})}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}\int_t^1 E_{\beta,\beta}(\mu(1-s)^{\beta})(1-s)^{\beta-1}q(s)g(s,x(s),y(s))\,\mathrm{d}s\\
&\quad-\theta\frac{\Gamma(\beta)}{1-\Gamma(\beta)E_{\beta,\beta}(\mu)}t^{\beta-1}E_{\beta,\beta}(\mu t^{\beta})\int_0^1\psi(s)H(s,x(s),y(s))\,\mathrm{d}s\\
&\quad+\theta\Gamma(\beta)G_{\mu,\beta}(t,t_1)J(t_1,x(t_1),y(t_1)).
\end{aligned}$$

Using (D), we get

$$\begin{aligned}|f(t,x(t),y(t))| &= f(t,t^{\alpha-1}t^{1-\alpha}x(t),t^{\beta-1}t^{1-\beta}y(t))| \\ &\leqslant c_1+b_1t^{1-\alpha}|x(t)|+a_1\Phi^{-1}(t^{1-\beta}|y(t)|) \\ &\leqslant c_1+b_1\|x\|+a_1\Phi^{-1}(\|y\|), \\ |G(t,x(t),y(t))| &\leqslant C_1+B_1\|x\|+A_1\Phi^{-1}(\|y\|), \\ |I(t_1,x(t_1),y(t_1))| &= |I(t_1,(1-t_1)^{\alpha-1}(1-t_1)^{1-\alpha}x(t_1),(1-t_1)^{\beta-1}(1-t_1)^{1-\beta}y(t_1))| \\ &\leqslant \overline{C}_1+\overline{B}_1(1-t_1)^{1-\alpha}|x(t_1)|+\overline{A}_1\Phi^{-1}((1-t_1)^{1-\beta}|y(t_1)|) \\ &\leqslant \overline{C}_1+\overline{B}_1\|x\|+\overline{A}_1\Phi^{-1}(\|y\|).\end{aligned}$$

Then, for $t\in(0,t_1]$, we have

$$\begin{aligned}&t^{1-\alpha}|x(t)| \\ &\quad\leqslant \frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^t E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}l_1s^{k_1}|f(s,x(s),y(s))|\,\mathrm{d}s \\ &\qquad+t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)l_1s^{k_1}|f(s,x(s),y(s))|\,\mathrm{d}s \\ &\qquad+\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_t^1 E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}l_1s^{k_1}|f(s,x(s),y(s))|\,\mathrm{d}s \\ &\qquad+\int_0^1\varphi(s)|G(s,x(s),y(s))|\,\mathrm{d}s\ \frac{\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda t^\alpha) \\ &\qquad+\Gamma(\alpha)\frac{\Gamma(\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1}|I(t_1,x(t_1),y(t_1))| \\ &\quad\leqslant l_1\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^t E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}s^{k_1}\,\mathrm{d}s \\ &\qquad\qquad\times(c_1+b_1\|x\|+a_1\Phi^{-1}(\|y\|)) \\ &\qquad+l_1t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)s^{k_1}\,\mathrm{d}s\ (c_1+b_1\|x\|+a_1\Phi^{-1}(\|y\|)) \\ &\qquad+l_1\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_t^1 E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}s^{k_1}\,\mathrm{d}s \\ &\qquad\qquad\times(c_1+b_1\|x\|+a_1\Phi^{-1}(\|y\|)) \\ &\qquad+\frac{\|\varphi\|_1(C_1+B_1\|x\|+A_1\Phi^{-1}(\|y\|))\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(|\lambda|) \\ &\qquad+\Gamma(\alpha)\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1} \\ &\qquad\qquad\times(\overline{C}_1+\overline{B}_1\|x\|+\overline{A}_1\Phi^{-1}(\|y\|))\end{aligned}$$

$$
\begin{aligned}
&= l_1 \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \int_0^t \sum_{k=0}^{\infty} \frac{\lambda^k (1-s)^{k\alpha}}{\Gamma((k+1)\alpha)} (1-s)^{\alpha-1} s^{k_1}\,\mathrm{d}s \\
&\qquad\qquad \times \left(c_1 + b_1\|x\| + a_1\Phi^{-1}(\|y\|)\right) \\
&\quad + l_1 t^{1-\alpha} \int_0^t \sum_{k=0}^{\infty} \frac{\lambda^k (t-s)^{k\alpha}}{\Gamma((k+1)\alpha)} (t-s)^{\alpha-1} s^{k_1}\,\mathrm{d}s\ \left(c_1 + b_1\|x\| + a_1\Phi^{-1}(\|y\|)\right) \\
&\quad + l_1 \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \int_t^1 \sum_{k=0}^{\infty} \frac{\lambda^k (1-s)^{k\alpha}}{\Gamma((k+1)\alpha)} (1-s)^{\alpha-1} s^{k_1}\,\mathrm{d}s \\
&\qquad\qquad \times \left(c_1 + b_1\|x\| + a_1\Phi^{-1}(\|y\|)\right) \\
&\quad + \frac{\|\varphi\|_1 (C_1 + B_1\|x\| + A_1\Phi^{-1}(\|y\|))\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} E_{\alpha,\alpha}(|\lambda|) \\
&\quad + \Gamma(\alpha) \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})(1-t_1)^{\alpha-1} \\
&\qquad\qquad \times \left(\overline{C}_1 + \overline{B}_1\|x\| + \overline{A}_1\Phi^{-1}(\|y\|)\right) \\
&\leqslant l_1 \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \sum_{k=0}^{\infty} \frac{\lambda^k \mathbf{B}((k+1)\alpha, k_1+1)}{\Gamma((k+1)\alpha)} \left(c_1 + b_1\|x\| + a_1\Phi^{-1}(\|y\|)\right) \\
&\quad + l_1 t^{1-\alpha} \sum_{k=0}^{\infty} \frac{\lambda^k t^{(k+1)\alpha+k_1} \mathbf{B}((k+1)\alpha, k_1+1)}{\Gamma((k+1)\alpha)} \left(c_1 + b_1\|x\| + a_1\Phi^{-1}(\|y\|)\right) \\
&\quad + l_1 \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \sum_{k=0}^{\infty} \frac{\lambda^k \mathbf{B}((k+1)\alpha, k_1+1)}{\Gamma((k+1)\alpha)} \\
&\qquad\qquad \times \left(c_1 + b_1\|x\| + a_1\Phi^{-1}(\|y\|)\right) \\
&\quad + \frac{\|\varphi\|_1 (C_1 + B_1\|x\| + A_1\Phi^{-1}(\|y\|))\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} E_{\alpha,\alpha}(|\lambda|) \\
&\quad + \Gamma(\alpha) \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})(1-t_1)^{\alpha-1} \\
&\qquad\qquad \times \left(\overline{C}_1 + \overline{B}_1\|x\| + \overline{A}_1\Phi^{-1}(\|y\|)\right) \\
&\leqslant l_1 \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \sum_{k=0}^{\infty} \frac{\lambda^k \mathbf{B}((k+1)\alpha, k_1+1)}{\Gamma((k+1)\alpha)} \left(c_1 + b_1\|x\| + a_1\Phi^{-1}(\|y\|)\right) \\
&\quad + l_1 \sum_{k=0}^{\infty} \frac{\lambda^k \mathbf{B}((k+1)\alpha, k_1+1)}{\Gamma((k+1)\alpha)} \left(c_1 + b_1\|x\| + a_1\Phi^{-1}(\|y\|)\right) \\
&\quad + l_1 \frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} \sum_{k=0}^{\infty} \frac{\lambda^k \mathbf{B}((k+1)\alpha, k_1+1)}{\Gamma((k+1)\alpha)} \\
&\qquad\qquad \times \left(c_1 + b_1\|x\| + a_1\Phi^{-1}(\|y\|)\right) \\
&\quad + \frac{\|\varphi\|_1 (C_1 + B_1\|x\| + A_1\Phi^{-1}(\|y\|))\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)} E_{\alpha,\alpha}(|\lambda|)
\end{aligned}
$$

$$
\begin{aligned}
&+\Gamma(\alpha)\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})(1-t_1)^{\alpha-1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times(\overline{C}_1+\overline{B}_1\|x\|+\overline{A}_1\Phi^{-1}(\|y\|))\\
=\ & l_1c_1\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\sum_{k=0}^{\infty}\frac{\lambda^k\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}\\
&+l_1c_1\sum_{k=0}^{\infty}\frac{\lambda^k\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}\\
&+l_1c_1\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\sum_{k=0}^{\infty}\frac{\lambda^k\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}\\
&+\frac{\|\varphi\|_1C_1\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\\
&+\overline{C}_1\Gamma(\alpha)\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})(1-t_1)^{\alpha-1}\\
&+\Bigg(l_1b_1\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\sum_{k=0}^{\infty}\frac{\lambda^k\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}\\
&+l_1b_1\sum_{k=0}^{\infty}\frac{\lambda^k\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}\\
&+l_1b_1\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\sum_{k=0}^{\infty}\frac{\lambda^k\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}\\
&+\frac{\|\varphi\|_1B_1\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\\
&+\overline{B}_1\Gamma(\alpha)\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})(1-t_1)^{\alpha-1}\Bigg)\|x\|\\
&+\Bigg(l_1a_1\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\sum_{k=0}^{\infty}\frac{\lambda^k\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}\\
&+l_1a_1\sum_{k=0}^{\infty}\frac{\lambda^k\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}\\
&+l_1a_1\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\sum_{k=0}^{\infty}\frac{\lambda^k\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}\\
&+\frac{\|\varphi\|_1A_1\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\\
&+\overline{A}_1\Gamma(\alpha)\frac{\Gamma(\alpha)E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})(1-t_1)^{\alpha-1}\Bigg)\Phi^{-1}(\|y\|).
\end{aligned}
$$

For $t \in (t_1, 1]$, we have

$$\begin{aligned}
&(t-t_1)^{1-\alpha}|x(t)| \\
&= (t-t_1)^{1-\alpha}\bigg|\frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^t E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times p(s)f(s,x(s),y(s))\,\mathrm{d}s \\
&\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)p(s)f(s,x(s),y(s))\,\mathrm{d}s \\
&\quad + \frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_t^1 E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}p(s)f(s,x(s),y(s))\,\mathrm{d}s \\
&\quad - \frac{\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)\int_0^1 \varphi(s)G(s,x(s),y(s))\,\mathrm{d}s \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \Gamma(\alpha)G_{\lambda,\alpha}(t,t_1)I(t_1,x(t_1),y(t_1))\bigg| \\
&\leqslant (t-t_1)^{1-\alpha}\frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^t E_{\alpha,\alpha}(\lambda(1-s)^\alpha) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times (1-s)^{\alpha-1}l_1 s^{k_1}|f(s,x(s),y(s))|\,\mathrm{d}s \\
&\quad + (t-t_1)^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)l_1 s^{k_1}|f(s,x(s),y(s))|\,\mathrm{d}s \\
&\quad + (t-t_1)^{1-\alpha}\frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_t^1 E_{\alpha,\alpha}(\lambda(1-s)^\alpha) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times (1-s)^{\alpha-1}l_1 s^{k_1}|f(s,x(s),y(s))|\,\mathrm{d}s \\
&\quad + (t-t_1)^{1-\alpha}\frac{\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)\int_0^1 |\varphi(s)||G(s,x(s),y(s))|\,\mathrm{d}s \\
&\quad + \Big((t-t_1)^{1-\alpha}\Gamma(\alpha)\frac{\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^\alpha)(1-t_1)^{\alpha-1} \\
&\quad + E_{\alpha,\alpha}(\lambda(t-t_1)^\alpha)\Big)|I(t_1,x(t_1),y(t_1))| \\
&\leqslant (1-t_1)^{1-\alpha}\frac{\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^t E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}l_1 s^{k_1}\,\mathrm{d}s \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times (c_1 + b_1\|x\| + a_1\Phi^{-1}(\|y\|)) \\
&\quad + (1-t_1)^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)l_1 s^{k_1}\,\mathrm{d}s\ (c_1 + b_1\|x\| + a_1\Phi^{-1}(\|y\|)) \\
&\quad + (1-t_1)^{1-\alpha}\frac{\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_t^1 E_{\alpha,\alpha}(\lambda(1-s)^\alpha)(1-s)^{\alpha-1}l_1 s^{k_1}\,\mathrm{d}s \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times (c_1 + b_1\|x\| + a_1\Phi^{-1}(\|y\|)) \\
&\quad + (1-t_1)^{1-\alpha}\frac{\|\varphi\|_1\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)(C_1 + B_1\|x\| + A_1\Phi^{-1}(\|y\|))
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\Gamma(\alpha)\frac{\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})+E_{\alpha,\alpha}(|\lambda|)\Big)\\
&\qquad\times(\overline{C}_1+\overline{B}_1\|x\|+\overline{A}_1\Phi^{-1}(\|y\|))\\
&=(1-t_1)^{1-\alpha}\frac{\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_0^t\sum_{k=0}^{\infty}\frac{\lambda^k(1-s)^{k\alpha}}{\Gamma((k+1)\alpha)}(1-s)^{\alpha-1}l_1s^{k_1}\,\mathrm{d}s\\
&\qquad\times(c_1+b_1\|x\|+a_1\Phi^{-1}(\|y\|))\\
&+(1-t_1)^{1-\alpha}\int_0^t\sum_{k=0}^{\infty}\frac{\lambda^k(t-s)^{k\alpha}}{\Gamma((k+1)\alpha)}(t-s)^{\alpha-1}l_1s^{k_1}\,\mathrm{d}s(c_1+b_1\|x\|+a_1\Phi^{-1}(\|y\|))\\
&+(1-t_1)^{1-\alpha}\frac{\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\int_t^1\sum_{k=0}^{\infty}\frac{\lambda^k(1-s)^{k\alpha}}{\Gamma((k+1)\alpha)}(1-s)^{\alpha-1}l_1s^{k_1}\,\mathrm{d}s\\
&\qquad\times(c_1+b_1\|x\|+a_1\Phi^{-1}(\|y\|))\\
&+(1-t_1)^{1-\alpha}\frac{\|\varphi\|_1\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)(C_1+B_1\|x\|+A_1\Phi^{-1}(\|y\|))\\
&+\Big(\Gamma(\alpha)\frac{\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})+E_{\alpha,\alpha}(|\lambda|)\Big)\\
&\qquad\times(\overline{C}_1+\overline{B}_1\|x\|+\overline{A}_1\Phi^{-1}(\|y\|))\\
&\leqslant l_1(1-t_1)^{1-\alpha}\frac{\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\sum_{k=0}^{\infty}\frac{\lambda^k\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}\\
&\qquad\times(c_1+b_1\|x\|+a_1\Phi^{-1}(\|y\|))\\
&+l_1(1-t_1)^{1-\alpha}\sum_{k=0}^{\infty}\frac{\lambda^kt^{(k+1)\alpha+k_1}\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}(c_1+b_1\|x\|+a_1\Phi^{-1}(\|y\|))\\
&+l_1(1-t_1)^{1-\alpha}\frac{\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}\sum_{k=0}^{\infty}\frac{\lambda^k\mathbf{B}((k+1)\alpha,k_1+1)}{\Gamma((k+1)\alpha)}\\
&\qquad\times(c_1+b_1\|x\|+a_1\Phi^{-1}(\|y\|))\\
&+(1-t_1)^{1-\alpha}\frac{\|\varphi\|_1\Gamma(\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)(C_1+B_1\|x\|+A_1\Phi^{-1}(\|y\|))\\
&+\Big(\Gamma(\alpha)\frac{\Gamma(\alpha)t_1^{\alpha-1}E_{\alpha,\alpha}(|\lambda|)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(\lambda)}E_{\alpha,\alpha}(\lambda(1-t_1)^{\alpha})+E_{\alpha,\alpha}(|\lambda|)\Big)\\
&\qquad\times(\overline{C}_1+\overline{B}_1\|x\|+\overline{A}_1\Phi^{-1}(\|y\|)).
\end{aligned}
$$

Then

$$\|x\|\leqslant\mu_1+\sigma_1\|x\|+\delta_1\Phi^{-1}(\|y\|).$$

Since $\sigma_1<1$, we get

$$\|x\|\leqslant\frac{\mu_1}{1-\sigma_1}+\frac{\delta_1}{1-\sigma_1}\Phi^{-1}(\|y\|). \tag{35}$$

Similarly to the above discussion, we can prove that

$$\|y\| \leqslant \frac{\mu_2}{1-\delta_2} + \frac{\sigma_2}{1-\delta_2}\Phi(\|x\|). \tag{36}$$

Case 1. $\dfrac{\sigma_2}{1-\delta_2}\dfrac{2\delta_1}{w(1-\sigma_1)} < 1$

Without loss of generality, suppose that $\|y\| > \Phi(\mu_1/\delta_1)$. Using BVP(4), we get

$$\begin{aligned}
\|y\| &\leqslant \frac{\mu_2}{1-\delta_2} + \frac{\sigma_2}{1-\delta_2}\Phi\Big(\frac{\mu_1}{1-\sigma_1} + \frac{\delta_1}{1-\sigma_1}\Phi^{-1}(\|y\|)\Big)\\
&\leqslant \frac{\mu_2}{1-\delta_2} + \frac{\sigma_2}{1-\delta_2}\Phi\Big(\frac{2\delta_1}{1-\sigma_1}\Phi^{-1}(\|y\|)\Big)\\
&\leqslant \frac{\mu_2}{1-\delta_2} + \frac{\sigma_2}{1-\delta_2}\frac{\Phi(\Phi^{-1}(\|y\|))}{w\Big(\dfrac{1-\sigma_1}{2\delta_1}\Big)}\\
&= \frac{\mu_2}{1-\delta_2} + \frac{\sigma_2}{1-\delta_2}\frac{1}{w\Big(\dfrac{1-\sigma_1}{2\delta_1}\Big)}\|y\|.
\end{aligned}$$

Due to (31), there exists a constant $M_1 > \Phi(\mu_1/\delta_1)$ such that $\|y\| \leqslant M_1$. Hence (35) implies that

$$\|x\| \leqslant \frac{\mu_1}{1-\sigma_1} + \frac{\delta_1}{1-\sigma_1}\Phi^{-1}(M_1).$$

It follows that Ω_1 is bounded.

Case 2. $\dfrac{\delta_1}{1-\sigma_1}\nu\Big(\dfrac{2\sigma_2}{1-\delta_2}\Big) < 1.$

Without loss of generality, suppose that $\|x\| > \Phi^{-1}(\mu_2/\sigma_2)$. Using (1), we get

$$\begin{aligned}
\|x\| &\leqslant \frac{\mu_1}{1-\sigma_1} + \frac{\delta_1}{1-\sigma_1}\Phi^{-1}(\|y\|)\\
&\leqslant \frac{\mu_1}{1-\sigma_1} + \frac{\delta_1}{1-\sigma_1}\Phi^{-1}\Big(\frac{\mu_2}{1-\delta_2} + \frac{\sigma_2}{1-\delta_2}\Phi(\|x\|)\Big)\\
&\leqslant \frac{\mu_1}{1-\sigma_1} + \frac{\delta_1}{1-\sigma_1}\Phi^{-1}\Big(\frac{2\sigma_2}{1-\delta_2}\Phi(\|x\|)\Big)\\
&\leqslant \frac{\mu_1}{1-\sigma_1} + \frac{\delta_1}{1-\sigma_1}\Phi^{-1}(\Phi(\|x\|))\nu\Big(\frac{2\sigma_2}{1-\delta_2}\Big)\\
&= \frac{\mu_1}{1-\sigma_1} + \frac{\delta_1}{1-\sigma_1}\nu\Big(\frac{2\sigma_2}{1-\delta_2}\Big)\|x\|.
\end{aligned}$$

By virtue of (31), there exists a constant $M_2 > \Phi^{-1}(\mu_2/\sigma_2)$ such that $\|x\| \leqslant M_2$. Hence (33) implies that

$$\|y\| \leqslant \frac{\mu_2}{1-\delta_2} + \frac{\sigma_2}{1-\delta_2}\Phi(M_2).$$

It follows that Ω_1 is bounded.

To apply Lemma 2.1, let Ω be a nonempty open bounded subset of X such that $\Omega \supset \overline{\Omega_1}$ centered at zero.

It is easy to see from Lemma 2.2 that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. One can see that

$$L(x,y) \neq \theta N(x,y) \quad \text{for all } (x,y) \in E \cap \partial\Omega \text{ and } \theta \in (0,1).$$

Thus, by Lemma 2.1,

$$L(x,y) = N(x,y)$$

has at least one solution $(x,y) \in E \cap \overline{\Omega}$. So (x,y) is a solution of BVP(3). The proof of Theorem 4.1 is complete. □

Remark 4.1. It is easy to see that BVP(3) has at least one solution for sufficiently small numbers $b_i \geqslant 0$, $a_i \geqslant 0$, $B_i \geqslant 0$, $A_i \geqslant 0$, $i = 1,2$ and any c_i, C_i, $i = 1,2$.

Corollary 4.1. *Suppose that* (C) *in Theorem* 4.1 *holds and*

(D1) f, g, H, G *are impulsive Carathéodory functions*, I, J *are continuous functions and satisfy that there exists a nonnegative constant* $M > 0$ *such that*

$$\begin{gathered}
|f(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)| \leqslant M, \quad k=0,1,\\
|g(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)| \leqslant M, \quad k=0,1,\\
|G(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)| \leqslant M, \quad k=0,1,\\
|H(t,(t-t_k)^{\alpha-1}x,(t-t_k)^{\beta-1}y)| \leqslant M, \quad k=0,1,\\
|I(t_1,(1-t_1)^{\alpha-1}x,(1-t_1)^{\beta-1}y)| \leqslant M,\\
|J(t_1,(1-t_1)^{\alpha-1}x,(1-t_1)^{\beta-1}y)| \leqslant M.
\end{gathered}$$

Then BVP(3) *has at least one solution.*

Proof. From (D1), we see that (D) in Theorem 4.1 holds with $c_i = C_i = \overline{C_i} = M$, $b_i = a_i = B_i = A_i = \overline{B_i} = \overline{A_i} = 0$, $i = 1,2$. It is easy to see that (34) holds. The result comes from Theorem 4.1 and its proof is omitted. □

5. An example

Now, we present an example which cannot be covered by the known results, to illustrate Theorem 4.1.

E x a m p l e 5.1. Consider the boundary value problem for fractional differential equation

$$
\text{BVP}(4) \qquad \begin{cases} D_{0+}^{2/3}x(t) - x(t) = t^{-1/4}f(t,x(t),y(t)), \qquad t\in(0,1),\ t\neq \dfrac{1}{2},\\ D_{0+}^{1/2}y(t) - y(t) = t^{-1/4}g(t,x(t),y(t)), \qquad t\in(0,1),\ t\neq \dfrac{1}{2},\\ \lim\limits_{t\to 1} t^{1/3}x(t) - \lim\limits_{t\to 0} t^{1/3}x(t) = \dfrac{1}{2}\displaystyle\int_0^1 s^{-1/2}G(s,x(s),y(s))\,\mathrm{d}s,\\ \lim\limits_{t\to 1} t^{1/2}y(t) - \lim\limits_{t\to 0} t^{1/2}y(t) = \dfrac{1}{2}\displaystyle\int_0^1 s^{-1/2}H(s,x(s),y(s))\,\mathrm{d}s,\\ \lim\limits_{t\to 1/2^+}\Big(t-\dfrac{1}{2}\Big)^{1/3}x(t) = 1,\\ \lim\limits_{t\to 1/2^+}\Big(t-\dfrac{1}{2}\Big)^{1/2}y(t) = 1, \end{cases}
$$

where

$$
f(t,x,y) = \begin{cases} c_1 + b_1 t^{1/3}x + a_1 t^{1/6}y^{1/3}, \quad t\in\big(0,\frac{1}{2}\big],\\ c_1 + b_1\big(t-\frac{1}{2}\big)^{1/3}x + a_1\big(t-\frac{1}{2}\big)^{1/6}y^{1/3}, \quad t\in\big(\frac{1}{2},1\big], \end{cases}
$$

$$
g(t,x,y) = \begin{cases} c_2 + b_2 t x^3 + a_2 t^{1/2}y, \quad t\in\big(0,\frac{1}{2}\big],\\ c_2 + b_2\big(t-\frac{1}{2}\big)x^3 + a_2\big(t-\frac{1}{2}\big)^{1/2}y, \quad t\in\big(\frac{1}{2},1\big], \end{cases}
$$

$$
G(t,x,y) = \begin{cases} C_1 + B_1 t^{1/3}x + A_1 t^{1/6}y^{1/3}, \quad t\in\big(0,\frac{1}{2}\big],\\ C_1 + B_1\big(t-\frac{1}{2}\big)^{1/3}x + A_1\big(t-\frac{1}{2}\big)^{1/6}y^{1/3}, \quad t\in\big(\frac{1}{2},1\big], \end{cases}
$$

$$
H(t,x,y) = \begin{cases} C_2 + B_2 t x^3 + A_2 t^{1/2}y, \quad t\in\big(0,\frac{1}{2}\big],\\ C_2 + B_2\big(t-\frac{1}{2}\big)x^3 + A_2\big(t-\frac{1}{2}\big)^{1/2}y, \quad t\in\big(\frac{1}{2},1\big], \end{cases}
$$

with c_i, b_i, a_i, C_i, B_i, A_i, $i=1,2$ being nonnegative numbers. Then there exists at least one solution of BVP(4).

P r o o f. Corresponding to BVP(3), let $\alpha=\frac{2}{3}$, $\beta=\frac{1}{2}$, $\lambda=\mu=1$, $t_1=\frac{1}{2}$, $p(t)=q(t)=t^{-1/4}$, $\varphi(t)=\psi(t)=\frac{1}{2}t^{-1/2}$, $\Phi(x)=x^3$ with $\Phi^{-1}(x)=x^{1/3}$; the supporting function of Φ is $\omega(x)=x^3$ and the supporting function of Φ^{-1} is $\nu(x)=x^{1/3}$, $I(t,x,y)=J(t,x,y)=1$.

It is easy to see that $p(t) \leqslant l_1 t^{k_1}$ and $q(t) \leqslant l_2 t^{k_2}$ with $l_1 = l_2 = 1$ and $k_1 = k_2 = -\frac{1}{4}$, and

$$\begin{aligned} f(t,(t-t_k)^{-1/3}x,(t-t_k)^{-1/2}y) &= c_1 + b_1 x + a_1 \Phi^{-1}(y), \\ g(t,(t-t_k)^{-1/3}x,(t-t_k)^{-1/2}y) &= c_2 + b_2 \Phi(x) + a_1 y, \\ G(t,(t-t_k)^{-1/3}x,(t-t_k)^{-1/2}y) &= C_1 + B_1 x + A_1 \Phi^{-1}(y), \\ H(t,(t-t_k)^{-1/3}x,(t-t_k)^{-1/2}y) &= C_2 + B_2 \Phi(x) + A_2 y. \end{aligned}$$

It is easy to see that $\overline{C}_1 = \overline{C}_2 = 1$ and $\overline{B}_1 = \overline{B}_2 = \overline{A}_1 = \overline{A}_2 = 0$ with

$$\begin{aligned} \Big| I\Big(\frac{1}{2}, \Big(\frac{1}{2}\Big)^{-1/3} x, \Big(\frac{1}{2}\Big)^{-1/2} y\Big)\Big| &\leqslant \overline{C}_1 + \overline{B}_1 |x| + \overline{A}_1 \Phi^{-1}(|y|), \\ \Big| J\Big(\frac{1}{2}, \Big(\frac{1}{2}\Big)^{-1/3} x, \Big(\frac{1}{2}\Big)^{-1/2} y\Big)\Big| &\leqslant \overline{C}_2 + \overline{B}_2 \Phi(|x|) + \overline{A}_2 |y|. \end{aligned}$$

One sees that (C) and (D) hold. By computation, we get

$$\begin{aligned} \sigma_1 =: \max\Big\{ & b_1\Big(\frac{2\Gamma(\frac{2}{3})E_{2/3,2/3}(1)}{1-\Gamma(\frac{2}{3})E_{2/3,2/3}(1)} + 1\Big) \sum_{k=0}^{\infty} \frac{\mathbf{B}(\frac{2}{3}(k+1),\frac{3}{4})}{\Gamma(\frac{2}{3}(k+1))} \\ & + B_1 \frac{\Gamma(\frac{2}{3})E_{2/3,2/3}(1)}{1-\Gamma(\frac{2}{3})E_{2/3,2/3}(1)}, \\ & b_1\Big(\frac{2\Gamma(\frac{2}{3})E_{2/3,2/3}(1)}{1-\Gamma(\frac{2}{3})E_{2/3,2/3}(1)} \sum_{k=0}^{\infty} \frac{\mathbf{B}(\frac{2}{3}(k+1),\frac{3}{4})}{\Gamma(\frac{2}{3}(k+1))} \\ & + \sqrt[3]{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})^{2(k+1)/3-1/4}\mathbf{B}(\frac{2}{3}(k+1),\frac{3}{4})}{\Gamma(\frac{2}{3}(k+1))}\Big) + B_1 \frac{\Gamma(\frac{2}{3})E_{2/3,2/3}(1)}{1-\Gamma(\frac{2}{3})E_{2/3,2/3}(1)}\Big\}, \\ \delta_1 =: \max\Big\{ & a_1\Big(\frac{2\Gamma(\frac{2}{3})E_{2/3,2/3}(1)}{1-\Gamma(\frac{2}{3})E_{2/3,2/3}(1)} + 1\Big) \sum_{k=0}^{\infty} \frac{\mathbf{B}(\frac{2}{3}(k+1),\frac{3}{4})}{\Gamma(\frac{2}{3}(k+1))} \\ & + \frac{A_1\Gamma(\frac{2}{3})E_{2/3,2/3}(1)}{1-\Gamma(\frac{2}{3})E_{2/3,2/3}(1)}, \\ & a_1\Big(\frac{2\Gamma(\frac{2}{3})E_{2/3,2/3}(1)}{1-\Gamma(\frac{2}{3})E_{2/3,2/3}(1)} \sum_{k=0}^{\infty} \frac{\mathbf{B}(\frac{2}{3}(k+1),\frac{3}{4})}{\Gamma(\frac{2}{3}(k+1))} \\ & + \sqrt[3]{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(\frac{1}{2})^{2(k+1)/3-1/4}\mathbf{B}(\frac{2}{3}(k+1),\frac{3}{4})}{\Gamma(\frac{2}{3}(k+1))}\Big) \\ & + A_1 \frac{\Gamma(\frac{2}{3})}{1-\Gamma(\frac{2}{3})E_{2/3,2/3}(1)} E_{2/3,2/3}(1)\Big\}, \end{aligned}$$

and

$$\sigma_2 =: \max\Big\{b_2\Big(\frac{2\Gamma(\frac{1}{2})E_{1/2,1/2}(1)}{1-\Gamma(\frac{1}{2})E_{1/2,1/2}(1)}+1\Big)\sum_{k=0}^{\infty}\frac{\mathbf{B}(\frac{1}{2}(k+1),\frac{3}{4})}{\Gamma(\frac{1}{2}(k+1))}$$
$$+\frac{B_2\Gamma(\frac{1}{2})E_{1/2,1/2}(1)}{1-\Gamma(\frac{1}{2})E_{1/2,1/2}(1)},$$
$$b_2\Big(\frac{2\Gamma(\frac{1}{2})E_{1/2,1/2}(1)}{1-\Gamma(\frac{1}{2})E_{11/2,1/2}(1)}\sum_{k=0}^{\infty}\frac{\mathbf{B}(\frac{1}{2}(k+1),\frac{3}{4})}{\Gamma(\frac{1}{2}(k+1))}$$
$$+\sqrt{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})^{(k+1)/2-1/4}\mathbf{B}(\frac{1}{2}(k+1),\frac{3}{4})}{\Gamma(\frac{1}{2}(k+1))}\Big)$$
$$+B_2\frac{\Gamma(\frac{1}{2})}{1-\Gamma(\frac{1}{2})E_{1/2,1/2}(1)}E_{1/2,1/2}(1)\Big\},$$
$$\delta_2 =: \max\Big\{a_2\Big(\frac{2\Gamma(\frac{1}{2})E_{1/2,1/2}(1)}{1-\Gamma(\frac{1}{2})E_{1/2,1/2}(1)}+1\Big)\sum_{k=0}^{\infty}\frac{\mathbf{B}(\frac{1}{2}(k+1),\frac{3}{4})}{\Gamma(\frac{1}{2}(k+1))}$$
$$+\frac{A_2\Gamma(\frac{1}{2})E_{1/2,1/2}(1)}{1-\Gamma(\frac{1}{2})E_{1/2,1/2}(1)},$$
$$a_2\Big(\frac{2\Gamma(\frac{1}{2})E_{1/2,1/2}(1)}{1-\Gamma(\frac{1}{2})E_{1/2,1/2}(1)}\sum_{k=0}^{\infty}\frac{\mathbf{B}(\frac{1}{2}(k+1),\frac{3}{4})}{\Gamma(\frac{1}{2}(k+1))}$$
$$+\sqrt{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})^{(k+1)/2-1/4}\mathbf{B}(\frac{1}{2}(k+1),\frac{3}{4})}{\Gamma(\frac{1}{2}(k+1))}\Big)$$
$$+A_2\frac{\Gamma(\frac{1}{2})}{1-\Gamma(\frac{1}{2})E_{1/2,1/2}(1)}E_{1/2,1/2}(1)\Big\}.$$

Then Theorem 4.1 implies that BVP(4) has at least one solution if

$$\sigma_1<1,\ \delta_2<1,\quad \frac{\sigma_2}{1-\delta_2}\Big(\frac{2\delta_1}{1-\sigma_1}\Big)^3<1$$

or

$$\sigma_1<1,\ \delta_2<1,\quad \frac{\delta_1}{1-\sigma_1}\Big(\frac{2\sigma_2}{1-\delta_2}\Big)^{1/3}<1.$$

□

R e m a r k 5.1. It is easy to see that BVP(4) has at least one solution for sufficiently small numbers $|b_i|$, $|a_i|$, $|B_i|$, $|A_i|$, $i=1,2$ and any c_i, C_i, $i=1,2$.

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