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Absolute continuity with respect to a subset of an interval

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Abstract. The aim of this paper is to introduce a generalization of the classical absolute continuity to a relative case, with respect to a subset M of an interval I. This generalization is based on adding more requirements to disjoint systems $\{(a_k, b_k)\}_K$ from the classical definition of absolute continuity – these systems should be not too far from M and should be small relative to some covers of M. We discuss basic properties of relative absolutely continuous functions and compare this class with other classes of generalized absolutely continuous functions.

Keywords: absolute continuity; quasi-uniformity; acceptable mapping

Classification: 26A46, 26A36

1. Introduction

In the descriptive definitions of integrals at all generality levels, two things play the crucial role: a type of (absolute) continuity of the function which should be primitive and a kind of the derivation used. This article focuses on the first of the conditions.

It is well-known that for Newton primitive function continuity of this function suffices, but the example of Cantor function shows that for a definition of primitive function in spirit of Lebesgue we need to use the classical absolute continuity ([2], p. 337):

Definition 1.1 (Classical absolute continuity)**.** A real-valued function f is said to be *absolutely continuous* on an interval $I = [a, b]$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervals $\{(a_k, b_k)\}_{k=1}^n$ of I satisfies $\sum_{k=1}^n (b_k - a_k) < \delta$, then $\sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon$.

For descriptive definitions of other integrals it was necessary to define several generalized absolute continuities. In one-dimensional case, the articles are mainly devoted to definitions that are used for descriptive definition of Henstock-Kurzweil integral. There are several attempts to this notion; let us mention Luzin's and Denjoy's definitions of AC, ACG, AC* and ACG* or for continuous functions equivalent approach by Khintchine (see [7], [13] for definitions and [15] for the comparison of these approaches). In agreement with classical constructive definition of Henstock-Kurzweil integral, Gordon ([6]) introduced the notion of AC$_\delta$

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and ACG_δ functions and showed that the classes of ACG^* and ACG_δ are equivalent on compact intervals. There are many other approaches to the notion of absolute continuity on real line, see the articles of Ene ([3], [4]), Gong ([5]), Lee ([10]) or Sworowski ([14]) for instance.

In this paper, we introduce the notion of generalized absolute continuity that is based on adding more requirements to disjoint intervals from the classical definition (Definition 1.1) in the sense that they are not too far from a given subset M of the interval I and are small relative to some covers of M. These requirements will be represented by a multivalued mapping that is defined on some class of open covers of the set M. Subsequently, we define absolute continuity relative to this mapping. We will also be concerned with the properties of such absolutely continuous functions and relations to other approaches to a definition of absolute continuity.

2. Definitions

In the sequel, assume that I is a non-empty interval in $\mathbb{R}$, M is a nonempty subset of I and, unless stated otherwise, all topological notions are related to I (e.g. "a set is open" means that this set is open in I).

First, we define the notion of a cover of a set M.

Definition 2.1. We say that a system $\mathcal{U} = \{U_k\}_{k\in K}$ of sets in $\mathbb{R}$ is *an open cover of a set M* if

(1) $M \subset \bigcup_{k\in K} U_k$;
(2) for every $k \in K$, U_k is an open set;
(3) for every $k \in K$, $U_k \cap M \neq \emptyset$.

Throughout this article, we will work with the notion of the refinement of a given system of sets (often of covers). Let us define it now.

Definition 2.2. A system $\mathcal{V}$ of open sets is said to refine a system $\mathcal{U}$ of open sets (briefly $\mathcal{V} \prec \mathcal{U}$) if every $V \in \mathcal{V}$ is contained in some element of $\mathcal{U}$.

As mentioned before, the multivalued mapping used in the definition of absolute continuity will be defined on a class of open covers of the set M. We require this class to be a filter of open covers of M (relative to refinements) — we use the term introduced by Isbell (cf. [9, p. 125], ; this term is also used for other concepts, for instance see [8, p. 1]:

Definition 2.3. A non-empty system $\mathfrak{U}$ of open covers of the set M is said to be a *quasi-uniformity* at M if it possesses the following properties:

(1) if an open cover of M is refined by some cover from $\mathfrak{U}$, then it belongs to $\mathfrak{U}$;
(2) any two covers in $\mathfrak{U}$ have a joint refinement in $\mathfrak{U}$.

For $A \subset M$ and an open cover $\mathcal{U}$ of M, we call the set

$$\operatorname{star}_{\mathcal{U}}(A) = \bigcup\{U \in \mathcal{U}; U \cap A \neq \emptyset\}$$

the *$\mathcal{U}$-neighbourhood of A*.

In the following, the basis of a quasi-uniformity means a basis of the corresponding filter.

Example 2.4. Examples of quasi-uniformities.

(1) If $\{I\}$ is a basis of $\mathfrak{U}$, we call $\mathfrak{U}$ *the coarse quasi-uniformity at M*.
(2) In contrast to the previous example, let $\mathfrak{U}$ be a class of all open covers of M. This quasi-uniformity is said to be *the fine quasi-uniformity at M*.
(3) If a basis of the quasi-uniformity $\mathfrak{U}$ at M is composed of intervals, then we call such quasi-uniformity *the interval quasi-uniformity at M*. If the intervals from the basis are bounded and of the form $\{U_x; x \in M\}$, where the mapping interval $\mapsto$ centre is onto, we get *the symmetric interval quasi-uniformity at M* (the covers $\{U_x; x \in M\}$ are called centered). If the basis of a quasi-uniformity is composed of all such covers, we say that the quasi-uniformity is *fine symmetric*.
(4) Another example can be a quasi-uniformity that we call *the usual metric uniformity on M*: it has for its basis a countable system $\{\mathcal{U}_n\}$ where $\mathcal{U}_n$ consists of all open intervals of length $r_n > 0$ and having its centre in M, where $r_n \to 0$.

A choice of the quasi-uniformity in the definition of generalized absolute continuity affects obtained absolute continuity, but the rules which determine, how to choose the systems $\{(a_k, b_k)\}_K$, have greater importance. Conditions for these rules are included in the following definition:

Definition 2.5. Let $\mathfrak{U}$ be a quasi-uniformity at M. A multivalued mapping Φ of the system $\mathfrak{U}$ to the set of disjoint collections of open subintervals of I is called *acceptable* if it fulfils the following conditions:

(1) every collection from $\Phi(\mathcal{U})$ refines $\mathcal{U}$;
(2) if $A \in \mathcal{A} \in \Phi(\mathcal{U})$, then $\overline{A} \cap M \neq \emptyset$;
(3) $\Phi(\mathcal{U}) \subset \Phi(\mathcal{V})$, provided $\mathcal{U} \prec \mathcal{V}$;
(4) if $\emptyset \neq \mathcal{B} \subset \mathcal{A} \in \Phi(\mathcal{U})$, then $\mathcal{B} \in \Phi(\mathcal{U})$.

We will denote by $D(\Phi)$ the quasi-uniformity $\mathfrak{U}$, where Φ is defined.

Remark. If the set M is not clear from the context, we shall write Φ_M.

Example 2.6. Acceptable mappings.

(1) The largest possible choice is to take for $\Phi(\mathcal{U})$ all systems of disjoint intervals that satisfy the conditions (1) and (2) of acceptable mapping definition (then the remaining conditions are also fulfilled). This acceptable mapping is said to be *full*.

(2) Other possible subsets of a full mapping are the cases when the endpoints of intervals from $\Phi(\mathcal{U})$ belong to some set, which is everywhere dense in some neighbourhood of M (e.g. to the set of rational or irrational numbers). Such a mapping is called *an almost full mapping.*

(3) Suppose $\mathfrak{U}$ to be a symmetric interval quasi-uniformity. We say that Φ defined on $\mathfrak{U}$ is *HK* if Φ has the following property: Whenever $\mathcal{U} = \{U_x; x \in M\} \in \mathfrak{U}$, then for every system $\{(a_k, b_k)\}_K \in \Phi(\mathcal{U})$ it holds that for every $k \in K$ there is an $x \in M$ such that $x \in [a_k, b_k] \subset U_x$.

We say that Φ is *HK-full* if it is HK and any $\Phi(\mathcal{U})$ consists of all systems $\{(a_k, b_k\}_K$ with the above-mentioned property.

Now, we can finally define the main notion of this section, the absolute continuity relative to some acceptable mapping:

Definition 2.7. Let Φ be an acceptable mapping defined on a quasi-uniformity $\mathfrak{U}$ at M. A real function $f\colon I \to \mathbb{R}$ is said to be *absolutely continuous* relative to Φ (briefly $f \in AC(\Phi)$) if for every $\epsilon > 0$ there exists $\mathcal{U} \in D(\Phi)$ and $\delta > 0$ such that for all systems $\{(a_k, b_k)\}_K \in \Phi(\mathcal{U})$ we have $\sum_K |f(b_k) - f(a_k)| < \epsilon$ whenever $\sum_K (b_k - a_k) < \delta$.

Example 2.8. Absolute continuity relative to the mapping Φ.

(1) Every $AC(\Phi)$ is nonempty since it contains constant functions.

(2) Let Φ be a full acceptable mapping on the coarse quasi-uniformity at a bounded set $M = I$. Then the set $AC(\Phi)$ corresponds to classical $AC(I)$.

(3) The foregoing assertion is not true for general acceptable mapping. Suppose $\mathfrak{U}$ to be an arbitrary quasi-uniformity at $M = I = [0, 1]$, C be the Cantor set and f be the Cantor function. If every $V \in \mathcal{V} \in \Phi(\mathcal{U})$ is a subset of $I \setminus C$, then f is absolutely continuous relative to Φ.

(4) For a given set M, let I be the smallest interval containing M and $\mathfrak{U}$ be the coarse quasi-uniformity at M. Suppose that Φ maps $\mathfrak{U}$ to systems of intervals with endpoints in M. Then Φ is acceptable and absolute continuity relative to Φ agrees with absolute continuity in the wide sense (AC) on a set M (cf. [13], p. 223).

(5) In Theorem 2.9 we show that the notion of AC_δ from [1] (see the first part of the proof of this theorem for the definition) coincides with $AC(\Phi)$ for certain Φ.

Theorem 2.9. *Let $f\colon I \to \mathbb{R}$ and $E \subset I$.*

(1) *If the function f is $AC_\delta(E)$ then it is absolutely continuous relative to an HK mapping defined on a fine symmetric interval quasi-uniformity at E.*

(2) *Suppose that $\mathfrak{U}$ is a symmetric interval quasi-uniformity at a set E. Then the function f is $AC_\delta(E)$ provided it is absolutely continuous relative to an HK-full mapping defined on $\mathfrak{U}$.*

PROOF: First, we show that $AC_\delta(E)$ implies the absolute continuity relative to an HK mapping. Let $\epsilon > 0$ be given. By $AC_\delta(E)$ we find a number ν and a gauge

δ such that the implication $\sum_N (d_n - c_n) \le \nu \Rightarrow \sum_N |f(d_n) - f(c_n)| \le \frac{\epsilon}{2}$ holds for every finite system $\{([c_n, d_n], x_n)\}_N$ that is E-subordinate to δ.

For $x \in E$ define $U_x = (x - \delta(x), x + \delta(x))$. Since $\mathfrak{U}$ is a fine symmetric interval quasi-uniformity at M, the system $\mathcal{U} = \{U_x; x \in E\}$ is an element of $\mathfrak{U}$.

Take a finite system $\{(a_k, b_k)\}_K \in \Phi(\mathcal{U})$ with $\sum_K (b_k - a_k) < \nu$. Since Φ is HK, a system $\{([a_k, b_k], x_k)\}_K$ is E-subordinate to δ (the point x_k is chosen in agreement with the definition of HK mapping). Hence using $AC_\delta(E)$ of f, we get $\sum_K |f(b_k) - f(a_k)| \le \frac{\epsilon}{2} < \epsilon$.

To prove the second part of the theorem, fix $\epsilon > 0$. By HK-absolute continuity of f we find the corresponding ν and a cover $\mathcal{U} = \{(x - \delta(x), x + \delta(x)); x \in E\}$ such that $\sum_K |f(b_k) - f(a_k)| < \epsilon$ for every $\{(a_k, b_k)\}_K \in \Phi(\mathcal{U})$ with $\sum_K (b_k - a_k) < \nu$. Then a map $\delta\colon x \to \delta(x)$ defines a gauge on E.

Let $\{([c_n, d_n], x_n)\}_N$ be a finite system of intervals that is E-subordinate to δ and with $\sum_N (d_n - c_n) \le \frac{\nu}{2}$. Then $\{(c_n, d_n)\}_N \in \Phi(\mathcal{U})$ and using HK-absolute continuity of f, we obtain $\sum_N |f(d_n) - f(c_n)| \le \epsilon$. Hence f is $AC_\delta(E)$. □

Remark. The first part of Theorem 2.9 is not true for usual metric uniformity, since it is possible that $\inf\{\delta(x); x \in E\} = 0$ and hence $\mathcal{U} = \{U_x; x \in E\}$ is not an element of $\mathfrak{U}$.

The following sections are devoted to a study of basic properties of absolutely continuous functions relative to chosen quasi-uniformities and acceptable mappings. Since all the theory presented here is intended for an application in the theory of an integral, we restrict our attention to the interval covers of M.

3. Continuity properties and mapping of null sets

In this section, we look more closely at relationships between our notion of absolute continuity and continuity or uniform continuity. We prove a theorem about boundedness of relative absolutely continuous function on bounded intervals. Finally, we prove assertions about preserving of null and measurable sets.

Classical absolutely continuous functions are continuous. Absolutely continuous functions related to a given acceptable mapping do not necessarily have this property. Let $I = [0, 1]$, $M = I$ and Φ be the almost full mapping, where the endpoints of intervals are taken from rational points of I. Then the Dirichlet function is absolutely continuous relative to Φ, but it is not continuous at M. For the absolute continuity related to some special mapping we can prove the following assertion (we say that f *is continuous at* $a \in M$ *relative to* I if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in I$ with $|x - a| < \delta$ the inequality $|f(x) - f(a)| < \epsilon$ holds):

Theorem 3.1. *Let f be absolutely continuous relative to a mapping Φ defined on a quasi-uniformity $\mathfrak{U}$. Suppose next that Φ has the following property: For every $x \in M$ and $\mathcal{U} \in \mathfrak{U}$ there exists $\delta > 0$ such that for arbitrary $y \in (x - \delta, x]$,*

$z \in [x, x+\delta)$, we have $(y,x) \in \Phi(\mathcal{U})$, $(x,z) \in \Phi(\mathcal{U})$. Then f is continuous on M relative to I.

Proof: Let $a \in M$. We take fixed $\epsilon > 0$ and using absolute continuity of the function f we determine the corresponding δ_ϵ and the cover $\mathcal{U}$. By the condition for the mapping Φ, we find for our a and the cover $\mathcal{U}$ the corresponding $\delta < \delta_\epsilon$. We pick $x \in I$ with $|x-a| < \delta$. Then $(x,a) \in \Phi(\mathcal{U})$ (or $(a,x) \in \Phi(\mathcal{U})$), and therefore $|f(x) - f(a)| < \epsilon$. □

The condition for Φ in Theorem 3.1 is satisfied e.g. by full or HK-full mapping:

Corollary 3.2. *Suppose f to be absolutely continuous relative to a full (HK-full, respectively) mapping Φ. Then f is continuous on M relative to I and hence continuous on M.*

It is well known that classical absolutely continuous functions are uniformly continuous. With some additional assumptions, this remains true also for absolute continuity relative to a mapping. We define uniform continuity relative to a given quasi-uniformity and discuss its basic properties and the relationship to the absolute continuity. With the use of relative uniform continuity, we prove the boundedness of absolutely continuous functions related to some special kinds of acceptable mappings on bounded sets. In what follows,

$$B_r(M) = \{(m-r, m+r); m \in M\}, r > 0.$$

Definition 3.3. A function $f\colon I \to \mathbb{R}$ is *uniformly continuous relative to a quasi-uniformity* $\mathfrak{U}$ at M if for every $\epsilon > 0$ there exists $\mathcal{U} \in \mathfrak{U}$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in U$ for some $U \in \mathcal{U}$.

Example 3.4. Uniform continuity relative to a quasi-uniformity.

(1) Only constant functions are uniformly continuous relative to the coarse quasi-uniformity at a given set M.

(2) Let $M = I$ and the quasi-uniformity $\mathfrak{U}$ consists of covers

$$\mathcal{U}_r = \{B_r(m), m \in M\},$$

where $r > 0$. Then the uniform continuity relative to $\mathfrak{U}$ corresponds to the classical one.

Uniformly continuous functions relative to a quasi-uniformity are of course continuous on M.

Uniform continuity of a function on a bounded interval implies boundedness of this function. As a modification of the theorem about boundedness of uniformly continuous functions on a totally bounded uniform space (see [12], p. 169), we state a theorem with the use of total boundedness of the quasi-uniformity $\mathfrak{U}$ relative to M (we say that *the quasi-uniformity* $\mathfrak{U}$ *is totally bounded* if for every $\mathcal{U} \in \mathfrak{U}$ there exists a finite subsystem $\mathcal{U}' \subset \mathcal{U}$ such that $\mathcal{U}' \in \mathfrak{U}$). In comparison with the classical situation, we obtain the boundedness of f on a neighbourhood of M.

Theorem 3.5. *Let f be uniformly continuous relative to a quasi-uniformity $\mathfrak{U}$ at a set M and $\mathfrak{U}$ be totally bounded. Then f is bounded on some neighbourhood* $\mathrm{star}_{\mathcal{V}}(M)$*, where $\mathcal{V} \in \mathfrak{U}$.*

As a special case, we can take a compact set M because every quasi-uniformity is totally bounded relative to M. The assumptions of Theorem 3.5 are also fulfilled e.g. for a bounded set M and usual metric uniformity.

In the previous theorem, it is not possible to exclude the requirement of total boundedness of the quasi-uniformity $\mathfrak{U}$. Let $I = [0,1], M = \{\frac{1}{n}; n \in \mathbb{N}\}$ and the covers in $\mathfrak{U}$ consist of pairwise disjoint open balls with centres in M. Suppose next that $f(x) = \frac{1}{x}$ for $x \in (0,1]$, $f(0) = 0$.

We show that f is uniformly continuous relative to $\mathfrak{U}$. Fix $\epsilon \in (0,1)$ and take the cover $\mathcal{U} \in \mathfrak{U}$ that is comprised of the balls $B_{r_n}(\frac{1}{n})$ where $r_n = \frac{1}{2}\epsilon(\frac{1}{n} - \frac{1}{n+1})$. Then for the difference $|f(x) - f(y)|$, where $x, y \in B_{r_n}(\frac{1}{n})$, the following inequality holds:

$$\begin{aligned}
|f(x) - f(y)| &< f\left(\frac{1}{n} - \frac{1}{2}\epsilon\Big(\frac{1}{n} - \frac{1}{n+1}\Big)\right) - f\left(\frac{1}{n} + \frac{1}{2}\epsilon\Big(\frac{1}{n} - \frac{1}{n+1}\Big)\right) \\
&= \frac{1}{\frac{1}{n} - \frac{1}{2}\epsilon\Big(\frac{1}{n} - \frac{1}{n+1}\Big)} - \frac{1}{\frac{1}{n} + \frac{1}{2}\epsilon\Big(\frac{1}{n} - \frac{1}{n+1}\Big)} \\
&= \epsilon\left(\frac{4n^2 + 4n}{4n^2 + 8n + 4 - \epsilon^2}\right) < \epsilon,
\end{aligned}$$

since $\frac{4n^2+4n}{4n^2+8n+4-\epsilon^2} \nearrow 1$. Hence f is uniformly continuous relative to $\mathfrak{U}$. But the function f is not bounded on $\mathrm{star}_{\mathcal{U}}(M)$.

In the following theorem, $D(\Phi) \vee \mathfrak{U}$ stands for the quasi-uniformity that has the union of both quasi-uniformities $D(\Phi)$ and $\mathfrak{U}$ as subbasis (the elements of this quasi-uniformity have to intersect the set M).

Theorem 3.6. *Assume that Φ is a full mapping defined on an interval quasi-uniformity $D(\Phi)$ at a set M. Let $\mathfrak{U}$ be a usual metric uniformity at M. Then every $f \in AC(\Phi)$ is uniformly continuous relative to the quasi-uniformity $D(\Phi) \vee \mathfrak{U}$ at M.*

Proof: Fix $\epsilon > 0$. By absolute continuity of f relative to Φ we find the corresponding $\mathcal{V} \in D(\Phi)$ and $\delta > 0$ such that for every collection $\{(a_j, b_j)\}_J \in \Phi(\mathcal{V})$ with $\sum_J (b_j - a_j) < \delta$ the inequality $\sum_J |f(b_j) - f(a_j)| < \frac{\epsilon}{2}$ holds.

Assume that $\mathcal{U} = \{B_r; x \in M\}$ is a cover from usual metric uniformity $\mathfrak{U}$ and $r < \frac{\delta}{2}$ (in other words, the cover $\mathcal{U}$ includes intervals of fixed length smaller than δ). Next, let $\mathcal{W}$ be a joint refinement of covers $\mathcal{U}$ and $\mathcal{V}$ that belongs to the quasi-uniformity $D(\Phi) \vee \mathfrak{U}$ ($\mathcal{W}$ is an interval cover).

Pick $x, y \in W \in \mathcal{W}$ (note that W is an interval). Let $m \in M \cap W$. By fullness of Φ, $(x, m), (y, m) \in \Phi(\mathcal{V})$. Additionally, $|x - m| < \delta$ and $|y - m| < \delta$. Using triangle inequality and absolute continuity of f, we get:

$$\begin{aligned}|f(y) - f(x)| &= |f(y) - f(m) + f(m) - f(x)| \\ &\leq |f(y) - f(m)| + |f(m) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Hence the function $f \in AC(\Phi)$ is uniformly continuous relative to the quasi-uniformity $D(\Phi) \vee \mathfrak{U}$ at M. □

Theorem 3.7. *Suppose Φ to be an HK-full mapping defined on a fine symmetric interval quasi-uniformity $D(\Phi)$ at a set M. Let $\mathfrak{U}$ be a usual metric uniformity at M. Then every $f \in AC(\Phi)$ is uniformly continuous relative to the quasi-uniformity $D(\Phi) \vee \mathfrak{U}$ at M.*

Proof: The procedure is the same as in the case of a full mapping. Besides of that we have to show that the intervals (x, m) and (y, m) are elements of $\Phi(\mathcal{V})$ if Φ is an HK-full mapping.

Let $\mathcal{U}$ and $\mathcal{V}$ be the covers as in the proof of previous theorem. Refine the cover $\mathcal{U}$ by a cover $\{U_x; x \in M\}$. Next, refine the cover $\mathcal{V}$ by a cover $\{V_x; x \in M\}$. Set $\mathcal{W} = \{U_x \cap V_x; x \in M\}$. Then taking $x, y \in W \in \mathcal{W}$ and the point m as a centre of the interval W (note that $m \in M$), we obtain $(x, m), (y, m) \in \Phi(\mathcal{V})$. □

In the previous theorem, it is not possible to take for Φ an almost full mapping (endpoints of intervals in $\mathbb{Q}$). E.g. Dirichlet function is absolutely continuous relative to an almost full mapping Φ on arbitrarily chosen quasi-uniformity (therefore also relative to $D(\Phi) \vee \mathfrak{U}$, where $\mathfrak{U}$ is a usual metric uniformity and $D(\Phi)$ is an arbitrary quasi-uniformity), but it is not uniformly continuous relative to this quasi-uniformity.

As a consequence of Theorems 3.5 and 3.6 (3.7, resp.) we obtain the following assertion about boundedness of absolutely continuous functions:

Corollary 3.8. *Let Φ be a full mapping (HK-full mapping, respectively) on the interval quasi-uniformity at a set M and $\mathfrak{U}$ be a usual metric uniformity at M. Suppose next that the quasi-uniformity $D(\Phi) \vee \mathfrak{U}$ is totally bounded relative to M. Then every $f \in AC(\Phi)$ is bounded on some neighbourhood* $\mathrm{star}_{\mathcal{V}}(M)$, *where $\mathcal{V} \in D(\Phi) \vee \mathfrak{U}$.*

Proof: Since f is absolutely continuous relative to Φ, it is also uniformly continuous relative to $D(\Phi) \vee \mathfrak{U}$ at M (it follows from Theorem 3.6 and 3.7, resp.). The quasi-uniformity $D(\Phi) \vee \mathfrak{U}$ is totally bounded relative to M, hence using Theorem 3.5 we obtain that f is bounded on some $\mathrm{star}_{\mathcal{V}}(M)$, where $\mathcal{V} \in D(\Phi) \vee \mathfrak{U}$. □

Let us turn to another properties of absolutely continuous functions related to an acceptable mapping. Classical absolutely continuous functions fulfil Luzin

(N)-condition or, in other words, map null sets to null sets. Some kinds of absolute continuity, e.g. relative to a full or an HK-full mapping, have the same property. In this section, a null set is a set of zero Lebesgue measure.

Theorem 3.9. *Let Φ_M be a full mapping. If $f \in AC(\Phi_M)$ and $N \subset M$ is a null set, then its image $f(N)$ is also a null set.*

Proof: Fix $\epsilon > 0$. Using absolute continuity of f relative to Φ_M we find $\mathcal{U} \in D(\Phi_M)$ and δ such that every system $\{(a_l, b_l)\}_L \in \Phi_M(\mathcal{U})$ fulfils

$$\sum_L |f(b_l) - f(a_l)| < \epsilon,$$

provided

$$\sum_L (b_l - a_l) < \delta.$$

Since N is a null set, we find an open set $G \supset N$ such that $\mu(G) < \delta$. Let $\mathcal{V} = \{V_x; x \in N\}$ be a joint refinement of $\mathcal{U}$ and $\{G\}$.

Using Besicovitch covering theorem we find a system $\{I_{i,k}\}_k \subset \mathcal{V}$, $i = 1, \dots, n$, such that $I_{i,k} = (x_{i,k} - r_{i,k}, x_{i,k} + r_{i,k})$, the systems $I_{i,k}$ are disjoint for i fixed and the union of these systems covers N. Let $x_{i,k}$ denote the centre of $I_{i,k}$ (the points $x_{i,k}$ are elements of N and hence of M).

We find $y_{i,k}, z_{i,k} \in (x_{i,k} - r_{i,k}, x_{i,k} + r_{i,k})$ such that $2\,|f(y_{i,k}) - f(z_{i,k})| > \sup_{x \in N \cap I_{i,k}} f(x) - \inf_{x \in N \cap I_{i,k}} f(x)$. Suppose $J_{i,k}$ to be an open interval with endpoints $x_{i,k}$ and $y_{i,k}$. Because Φ is a full mapping, $J_{i,k} \in \Phi(\mathcal{U})$. Additionally $\sum_k \mu(J_{i,k}) \le \mu(G) < \delta$. By absolute continuity of f, we obtain

$$\sum_k |f(y_{i,k}) - f(x_{i,k})| < \epsilon.$$

Similarly $\sum_k |f(x_{i,k}) - f(z_{i,k})| < \epsilon$. Hence for a fixed i

$$\begin{aligned}\sum_k \sup_{x \in N \cap I_{i,k}} f(x) - \inf_{x \in N \cap I_{i,k}} &< 2 \sum_k |f(y_{i,k}) - f(z_{i,k})| \\ &\le 2\Big(\sum_k |f(z_{i,k}) - f(x_{i,k})| + \sum_k |f(x_{i,k}) - f(y_{i,k})|\Big) \\ &< 4\epsilon.\end{aligned}$$

Therefore

$$\mu[f(N)] \le 2 \sum_i \sum_k |f(y_{i,k}) - f(z_{i,k})| \le 4n\epsilon.$$

Since ϵ is arbitrary and n is a constant independent of N and $\mathcal{U}$, $\mu[f(N)] = 0$. $\square$

The foregoing theorem is not true for arbitrarily chosen acceptable mapping since the Cantor function maps the null (Cantor) set onto the unit interval but it is absolutely continuous relative to a mapping defined as in Example 2.8(3).

Theorem 3.10. *Assume Φ_M to be an HK-full mapping. If $f \in AC(\Phi_M)$ and $N \subset M$ is a null set, then its image $f(N)$ is also a null set.*

Proof: The proof is the same as in the case of a full mapping, because the intervals $J_{i,k}$ are elements of $\Phi(\mathcal{U})$ where Φ is an HK-full mapping. $\square$

The assertion about mapping of measurable sets is only a consequence of Theorem 3.9 (3.10, resp.):

Theorem 3.11. *Suppose Φ_M to be a full mapping (HK-full mapping, respectively). If $f \in AC(\Phi_M)$ and if $E \subset M$ is a measurable set, then $f(E)$ is a measurable set.*

Proof: The proof is based on the fact that if $E \subset M \subset I$ is a Lebesgue-measurable set, then there exist a null set Z and a sequence $(K_n)_{n=1}^{\infty}$ of compact sets in I such that $E = Z \cup \bigcup_{n=1}^{\infty} K_n$ (see [1], p. 314, for details).

Because $f \in AC(\Phi_M)$ is continuous on M, it is also continuous on $K_n \subset M$ for every index n. Consequently, $f(K_n)$ as an image of the compact set K_n is compact and therefore measurable. According to Theorem 3.9 (3.10, resp.), $f(Z)$ is a null set and hence measurable. Since $f(E) = f(Z) \cup \bigcup_{n=1}^{\infty} f(K_n)$, $f(E)$ is measurable. $\square$

4. Algebraic properties

The class of absolutely continuous functions relative to any acceptable mapping Φ forms a linear space. With some additional requirements, it is also closed under multiplication. In the following, $\mathbb{R}^I$ denotes a usual algebra of all functions $f\colon I \to \mathbb{R}$.

Theorem 4.1. *The class $AC(\Phi)$ is a linear subspace of $\mathbb{R}^I$.*

Proof: For the proof that $AC(\Phi)$ is closed under multiplying by constant fix $\epsilon > 0$. Using definition of absolute continuity of f related to Φ we find for ϵ corresponding $\mathcal{U} \in D(\Phi)$ and $\delta > 0$. Let a system $\{(a_j, b_j)\}_J$ be an element of $\Phi(\mathcal{U})$ and $\sum_J (b_j - a_j) < \delta$. Then

$$\sum_J |cf(b_j) - cf(a_j)| = |c| \sum_J |f(b_j) - f(a_j)| < |c|\,\epsilon.$$

Now let us turn to the addition. Let f and g be absolutely continuous functions relative to Φ. Fix again $\epsilon > 0$. Let $\mathcal{U}_f$ and δ_f be a witness of $f \in AC(\Phi)$ for ϵ. Analogously, absolute continuity of g yields for ϵ corresponding $\mathcal{U}_g$ and δ_g.

Set $\delta = \min\{\delta_f, \delta_g\}$. Suppose next that $\mathcal{U}$ is a joint refinement of the covers $\mathcal{U}_f$ and $\mathcal{U}_g$ that also belongs to $D(\Phi)$ (it exists by condition (2) in quasi-uniformity definition).

Take a system of intervals $\{(a_l, b_l)\}_L \in \Phi(\mathcal{U})$ with the property

$$\sum_L (b_l - a_l) < \delta.$$

Since $\mathcal{U}$ is a joint refinement of covers $\mathcal{U}_f$ and $\mathcal{U}_g$, $\Phi(\mathcal{U}) \subset \Phi(\mathcal{U}_f) \cap \Phi(\mathcal{U}_g)$. Consequently

$$\sum_L |(f+g)(b_l) - (f+g)(a_l)| = \sum_L |f(b_l) - f(a_l)| + \sum_L |g(b_l) - g(a_l)| < 2\epsilon.$$

□

On a compact interval, the class of classical absolutely continuous functions is closed under multiplication. It is not true for unbounded intervals – the function $f(x) = x$ is absolutely continuous on $(0, \infty)$, whereas the function $f^2(x) = x^2$ is not. This leads us to the following theorem:

Theorem 4.2. *Let $r > 0$ and $\mathcal{V} \in D(\Phi)$. If $f, g \in AC(\Phi)$ are bounded on $B_r(M) \cap \mathrm{star}_{\mathcal{V}}(M)$, then $fg \in AC(\Phi)$.*

Proof: Fix $\epsilon > 0$. Find $\mathcal{U}_f$, $\mathcal{U}_g$, δ_f and δ_g as in the proof of closedness for addition in Theorem 4.1. Suppose $\mathcal{U}$ to be a joint refinement of the covers $\mathcal{U}_f$ and $\mathcal{U}_g$. Define $\delta = \min\{\delta_f, \delta_g\}$.

Let f and g be bounded on a neighbourhood $B_r(M) \cap \mathrm{star}_{\mathcal{V}}(M)$ where $r > 0$ is fixed. Hence there exist C and D such that $|f(x)| \leq C$ and $|g(x)| \leq D$ for every $x \in B_r(M) \cap \mathrm{star}_{\mathcal{V}}(M)$. Assume that $\mathcal{W}$ is a joint refinement of $\mathcal{U}$ and $\mathcal{V}$.

Now we take a system of intervals $\{(a_l, b_l)\}_L \in \Phi(\mathcal{W})$ such that $(a_l, b_l) \subset B_r(M)$ for every $l \in L$ and $\sum_L (b_l - a_l) < \delta$. By absolute continuity of f, g and boundedness of both functions, we obtain:

$$\begin{aligned}
\sum_L |fg(b_l) - fg(a_l)| &\leq \sum_L |g(b_l)|\,|f(b_l) - f(a_l)| + \sum_L |f(a_l)|\,|g(b_l) - g(a_l)| \\
&\leq D \sum_L |f(b_l) - f(a_l)| + C \sum_L |g(b_l) - g(a_l)| \\
&< D\epsilon + C\epsilon = (D + C)\epsilon.
\end{aligned}$$

□

The last corollary goes back to the boundedness of absolutely continuous functions relative to a mapping on bounded intervals.

Corollary 4.3. *If the conditions of Theorem 3.8 are fulfilled, then the class $AC(\Phi)$ is a subalgebra of $\mathbb{R}^I$.*

5. Dependence on Φ

Main result of this section is a generalisation of the fact that the absolute continuity relative to a full mapping on the coarse quasi-uniformity at a bounded set $M = I$ coincides with the definition of classical absolute continuity.

We begin with two simple but useful observations.

(1) If a quasi-uniformity $\mathfrak{V}$ at M is contained in a quasi-uniformity $\mathfrak{U}$ at M ($\mathfrak{V} \subset \mathfrak{U}$) and Φ is an acceptable mapping on $\mathfrak{U}$, then its restriction Ψ on $\mathfrak{V}$ is also acceptable and $AC(\Psi) \subset AC(\Phi)$.

(2) Let Φ, Ψ be acceptable mappings with the same domain and $\Phi \subset \Psi$ (for arbitrary $\mathcal{U} \in D(\Phi)$, $\Phi(\mathcal{U}) \subset \Psi(\mathcal{U})$). Then $AC(\Psi) \subset AC(\Phi)$.

Theorem 5.1. *Let M be dense in I and Φ be a full acceptable mapping. If the functions from $AC(\Phi)$ are continuous on I, then the classes $AC(\Phi)$ and $AC(\Psi)$ coincide, where Ψ is a restriction of Φ to the quasi-uniformity of all $D(\Phi)$-neighbourhoods of M.*

Proof: The relation $AC(\Psi) \subset AC(\Phi)$ follows from the fact that the quasi-uniformity of all $D(\Phi)$-neighbourhoods is smaller than $\mathfrak{U}$ and from the first observation above.

To prove the converse relation, take $f \in AC(\Phi)$. Fix $\epsilon > 0$ and using the absolute continuity of f relative to Φ find corresponding δ and $\mathcal{U} \in D(\Phi)$. We have $\Phi(\mathcal{U}) \subset \Phi(\mathrm{star}_{\mathcal{U}}(M)) = \Psi(\mathrm{star}_{\mathcal{U}}(M))$, since $\mathcal{U} \prec \mathrm{star}_{\mathcal{U}}(M)$.

Take a system of intervals $\{(a_j,b_j)\}_J \in \Psi(\mathrm{star}_{\mathcal{U}}(M)) \setminus \Phi(\mathcal{U})$ with $\sum_J (b_j - a_j) < \delta$. Using continuity of f, we find for every interval (a_j,b_j) a closed subinterval $[a'_j,b'_j]$ such that $\sum_J \left|f(a'_j) - f(a_j)\right| < \epsilon$ and $\sum_J \left|f(b_j) - f(b'_j)\right| < \epsilon$.

Since $[a'_j,b'_j] \subset (a_j,b_j)$, the cover $\mathcal{U}$ from the definition of absolute continuity covers $[a'_j,b'_j]$. The interval $[a'_j,b'_j]$ is compact, hence there exists a finite subcover $\mathcal{U}'$ of $\mathcal{U}$ that also covers $[a'_j,b'_j]$. The cover $\mathcal{U}'$ is finite, hence we may pick a minimal subcover of $\mathcal{U}'$. Therefore we will assume that $\mathcal{U}'$ has this property and that only the neighbouring intervals have a nonempty intersection. Let $\mathcal{U}' = \{U_p\}_{p=1}^{q}$. Set $r_{j,0} = a'_j$, $r_{j,q} = b'_j$ and $r_{j,p} \in U_p \cap U_{p+1}$. Using this procedure, we obtain for a fixed j a system of intervals $\{[r_{j,p}, r_{j,p+1}]\}_{p=0}^{q-1}$ such that $\bigcup_{p=0}^{q-1}[r_{j,p}, r_{j,p+1}] = [a'_j,b'_j]$ and $\{(r_{j,p}, r_{j,p+1})\}_{p=0}^{q-1} \in \Phi(\mathcal{U})$.

Construction of the intervals $[r_{j,p}, r_{j,p+1}]$ then gives

$$\sum_J \sum_{p=0}^{q-1} (r_{j,p+1} - r_{j,p}) = \sum_J (b'_j - a'_j) < \sum_J (b_j - a_j) < \delta,$$

and hence (using continuity of f)

$$\begin{aligned}\epsilon > \sum_J \sum_{p=0}^{q-1} |f(r_{j,p+1}) - f(r_{j,p})| &\geq \sum_J \left|\sum_{p=0}^{q-1} \big(f(r_{j,p+1}) - f(r_{j,p})\big)\right| \\ &\geq \sum_J \left|f(b'_j) - f(a'_j)\right|.\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\sum_J |f(b_j) - f(a_j)| \leq \sum_J \left|f(a'_j) - f(a_j)\right| &+ \sum_J \left|f(b'_j) - f(a'_j)\right| \\ &+ \sum_J \left|f(b_j) - f(b'_j)\right| < 3\epsilon.\end{aligned}$$

The function f is therefore absolutely continuous relative to the acceptable mapping Ψ. $\square$

Let us mention an important consequence of the theorem. If we define the absolute continuity for full mappings on a given quasi-uniformity at dense subsets of I, it is not necessary to specify the mapping Φ and the cover from its domain (under assumption of continuity of functions from $AC(\Phi)$).

The procedure of a division of intervals $[a'_j, b'_j]$ into smaller intervals from Theorem 5.1 cannot be used without the assumption of density of M in I. Let us show an example. Set $M = \{2, 6\}$, $\mathfrak{U}$ to be the fine quasi-uniformity and $\mathcal{U} = \{U_1, U_2\}$ where $U_1 = (0, 4)$, $U_2 = (3, 7)$. Hence $\mathrm{star}_{\mathcal{U}}(M) = (0, 7)$ and for the interval $(1, 5) \in \Psi(\mathrm{star}_{\mathcal{U}}(M))$ it is not possible to find its division into smaller intervals such that every interval is an element of $\Phi(\mathcal{U})$, since every part of $(1, 5)$ that refines U_2 does not intersect M.

Since the intervals (r_{jp}, r_{jp+1}) from Theorem 5.1 are parts of open sets from $\mathcal{U}$, we can always move their endpoints a bit (with the exception of points a_j, b_j), e.g. in the way that they belong to some set which is everywhere dense in a neighbourhood of M. This possibility can be used when proving the variant of the previous theorem for almost full mappings.

Theorem 5.2. *Suppose Φ to be an almost full mapping such that the endpoints of intervals from $\Phi(\mathcal{U})$ belong to $I \cap \mathbb{Q}$. Then the conclusion of Theorem 5.1 remains true for Φ.*

6. Dependence on M

In this section we show the relationships between $AC(\Phi_M)$ and $AC(\Phi_N)$ where Φ_M and Φ_N are two unrelated acceptable mappings defined at the sets M and N, respectively. It is the analogy to theorems about classical absolute continuity on a union of intervals or classical absolute continuity on subintervals of given interval.

Theorem 6.1. *If $N \subset M$ and for every $\mathcal{U} \in D(\Phi_M)$ there exists $\mathcal{V} \in D(\Phi_N)$ such that $\mathcal{V}$ refines $\mathcal{U}$ and $\Phi_N(\mathcal{V}) \subset \Phi_M(\mathcal{U})$, then $AC(\Phi_M) \subset AC(\Phi_N)$.*

Proof: Let $N \subset M$ and the assumptions of the theorem hold. Suppose $f \in AC(\Phi_M)$. The aim is to show that $f \in AC(\Phi_N)$.

Fix $\epsilon > 0$. By absolute continuity of the function f relative to the mapping Φ_M there exist a cover $\mathcal{U} \in D(\Phi_M)$ and $\delta > 0$ such that whenever the system $\{(a_j, b_j)\}_J \in \Phi(\mathcal{U}_M)$ satisfies $\sum_J (b_j - a_j) < \delta$, then $\sum_J |f(b_j) - f(a_j)| < \epsilon$.

Using the assumptions, we find for $\mathcal{U}$ a refinement $\mathcal{V} \in D(\Phi_N)$. We take a system $\{(a_k, b_k)\}_K \in \Phi_N(\mathcal{V})$ with the property $\sum_K (b_k - a_k) < \delta$.

Since $\Phi_N(\mathcal{V}) \subset \Phi_M(\mathcal{U})$, the system $\{(a_k, b_k)\}_K$ is an element of $\Phi_M(\mathcal{U})$. Then $\sum_K |f(b_k) - f(a_k)| < \epsilon$ and the function f is absolutely continuous relative to the acceptable mapping Φ_N. $\square$

Theorem 6.2. *Suppose that the mappings Φ_M, Φ_N, $\Phi_{M\cup N}$ satisfy the following condition: For every $\mathcal{U}_M \in D(\Phi_M)$, $\mathcal{U}_N \in D(\Phi_N)$ there exists $\mathcal{U} \in D(\Phi_{M\cup N})$ such that*

(1) $\mathcal{U} \prec \mathcal{U}_M \cup \mathcal{U}_N$;
(2) *if* $\mathcal{A} \in \Phi_{M\cup N}(\mathcal{U})$, *then* $\mathcal{A}_M = \{J \in \mathcal{A}; J \cap M \neq \emptyset\} \in \Phi_M(\mathcal{U}_M)$ *and* $\mathcal{A}_N = \{J \in \mathcal{A}; J \cap N \neq \emptyset\} \in \Phi_N(\mathcal{U}_N)$.

Then $AC(\Phi_M) \cap AC(\Phi_N) \subset AC(\Phi_{M\cup N})$.

Proof: Let $f \in AC(\Phi_M)$, $f \in AC(\Phi_N)$ and the assumptions of the theorem be true. We show that $f \in AC(\Phi_{M\cup N})$.

Fix $\epsilon > 0$. Using definitions of $f \in AC(\Phi_M)$ and $f \in AC(\Phi_N)$ we find the corresponding δ_M, δ_N and $\mathcal{U}_M \in D(\Phi_M), \mathcal{U}_N \in D(\Phi_N)$ of desired properties. Set $\delta = \min\{\delta_M, \delta_N\}$. Under above assumptions for covers $\mathcal{U}_M$ and $\mathcal{U}_N$, there exists a cover $\mathcal{U} \in D(\Phi_{M\cup N})$ such that $\mathcal{U} \prec \mathcal{U}_M \cup \mathcal{U}_N$.

We take a system $\mathcal{A} = \{(a_k, b_k)\}_K \in \Phi_{M\cup N}(\mathcal{U})$ with $\sum_K (b_k - a_k) < \delta$. Let $\mathcal{A}_M$ be a system of intervals $(a_k, b_k) \in \mathcal{A}$ such that $(a_k, b_k) \cap M \neq \emptyset$. Analogously, $\mathcal{A}_N$ is a system of intervals $(a_k, b_k) \in \mathcal{A}$ with $(a_k, b_k) \cap N \neq \emptyset$. Thus

$$\sum_{(a_k,b_k)\in\mathcal{A}_M} (b_k - a_k) \leq \sum_{(a_k,b_k)\in\mathcal{A}} (b_k - a_k) < \delta.$$

The same inequality holds for every system of intervals from $\mathcal{A}_N$.

Absolute continuity of f related to Φ_M and Φ_N yields

$$\sum_{(a_k,b_k)\in\mathcal{A}} |f(b_k) - f(a_k)| \leq \sum_{(a_k,b_k)\in\mathcal{A}_M} |f(b_k) - f(a_k)| + \sum_{(a_k,b_k)\in\mathcal{A}_N} |f(b_k) - f(a_k)| < 2\epsilon.$$

□

Corollary 6.3. *Let M, N satisfy the assumptions of Theorem 6.2 and the pairs $(M, M\cup N)$, $(N, M\cup N)$ fulfil the conditions of Theorem 6.1. Then $AC(\Phi_M) \cap AC(\Phi_N) = AC(\Phi_{M\cup N})$.*

Proof: Since the sets M, N fulfil the assumptions of Theorem 6.2, $AC(\Phi_M) \cap AC(\Phi_N) \subset AC(\Phi_{M\cup N})$. Suppose next $f \in AC(\Phi_{M\cup N})$. We show that $f \in AC(\Phi_M)$. Let the tupple $(M, M \cup N)$ satisfy the conditions of Theorem 6.1. Hence $AC(\Phi_{M\cup N}) \subset AC(\Phi_M)$ and $f \in AC(\Phi_M)$. By the similar argument, $f \in AC(\Phi_N)$. Thus $f \in AC(\Phi_M) \cap AC(\Phi_N)$, and consequently $AC(\Phi_{M\cup N}) \subset AC(\Phi_M) \cap AC(\Phi_N)$. □

In the next corollary, the notion of *the restriction* of a mapping Φ_X to Φ_Y on a smaller set $Y \subset X$ is used. Let $\mathcal{U}_X$ be a cover of X. Then the cover $\mathcal{U}_Y$ of Y is a set $\{U \in \mathcal{U}_X; U \cap Y \neq \emptyset\}$. The elements of the restriction of Φ_X to the set Y are defined as follows: $\Phi_Y(\mathcal{U}_Y) = \{\mathcal{A} \in \Phi_X(\mathcal{U}_X); A \in \mathcal{A} \Rightarrow \overline{A} \cap Y \neq \emptyset\}$. We denote the restriction of Φ_X to Y by $\Phi_X|Y$.

Corollary 6.4. *For all acceptable mappings Φ related to the set $M \cup N$ and their restrictions Φ_M, Φ_N to M, N, respectively, the equality $AC(\Phi_M) \cap AC(\Phi_N) = AC(\Phi)$ holds.*

PROOF: To prove this theorem, it is sufficient to verify that defined restrictions satisfy the assumptions of Theorems 6.1 and 6.2 and to use Corollary 6.3. Set $\Phi_M = \Phi_{M\cup N}|M$ and $\Phi_N = \Phi_{M\cup N}|N$.

We show first that the tupple $(M, M \cup N)$ fulfils the conditions of Theorem 6.1 (then the same holds for $(N, M \cup N)$). Let $\mathcal{U}$ be a cover of $M \cup N$ and $\mathcal{V}$ be a restriction of $\mathcal{U}$ to M. Clearly $M \subset M \cup N$, from the construction of the cover $\mathcal{V}$ it follows that $\mathcal{V} \prec \mathcal{U}$. Next, if a system $\{(a_k, b_k)\}_K$ is contained in $\Phi_M(\mathcal{V})$, construction of Φ_M yields $\{(a_k, b_k)\}_K \in \Phi_{M\cup N}(\mathcal{U})$ (we only omit some elements in the cover).

The mappings Φ_M and Φ_N have also the properties essential for using of Theorem 6.2. For arbitrary $\mathcal{U}_M \in D(\Phi_M)$ and $\mathcal{U}_N \in D(\Phi_N)$, $\mathcal{U} = \mathcal{U}_M \cup \mathcal{U}_N$, hence $\mathcal{U} \prec \mathcal{U}_M \cup \mathcal{U}_N$. The second property follows from the construction of restrictions $\Phi_{M\cup N}|M$ and $\Phi_{M\cup N}|N$. □

7. Absolute continuity and derivative

The relationship between absolute continuity of a function relative to a mapping and existence of its derivative is not so lucid as in the classical case. For that reason, in this section only assertions for absolute continuities relative to one concrete mapping (mainly full or HK) are stated, rather than general theorems. For these special absolute continuities we obtain similar assertions as for the classical absolute continuity.

If a function $f\colon I \to \mathbb{R}$ has a finite derivative $f'(x)$ at the point $x \in I$, then for every $\epsilon > 0$ there exists $\delta_x > 0$ such that

$$(f'(x) - \epsilon)(z - y) < f(z) - f(y) < (f'(x) + \epsilon)(z - y) \tag{1}$$

for $x - \delta_x < y \le x \le z < x + \delta_x$.

If $f'(x)$ is finite on a set M, $\{(x - \delta_x, x + \delta_x); x \in M\}$ is a symmetric interval cover of M and we call it $\mathrm{der}(f, \epsilon)$-*cover*.

It is well known that a classically absolutely continuous function on I has a derivative everywhere on I with the exception of a null set. But a function need not to be absolutely continuous on I even if it possesses a finite derivative everywhere on this interval. An easy example of such a function can be $f(x) = \frac{1}{x}$ on the interval $(0, 1)$. This or more complicated examples show that it is necessary to give limiting requirements to f.

Theorem 7.1. *Let $\mathfrak{U}$ be the fine symmetric quasi-uniformity at $M \subset I$ and Φ possesses the HK-property. Let $f\colon I \to \mathbb{R}$ have a finite derivative on $M \subset I$. Then $f \in AC(\Phi)$ if either f' is bounded on M or M is a null set.*

PROOF: Fix $\epsilon > 0$. The function f has a finite derivative at any point of the set M, therefore there exists a der(f,ϵ)-cover of M. Let a system $\mathcal{D} = \{(x-\delta_x, x+\delta_x); x \in M\}$ be this cover. Since $\mathfrak{U}$ is fine symmetric, $\mathcal{D} \in \mathfrak{U}$.

The proof will be divided into two parts. Let us first prove the theorem for the case when f' is bounded on M. Since f' is bounded on M, there exists a real number L such that $|f'(x)| \leq L$ on M. Take $\delta < \epsilon$ and a disjoint system $\{(a_k,b_k)\}_K \in \Phi(\mathcal{D})$ with $\sum_K (b_k - a_k) < \delta$. Then using (1), HK-property of Φ and boundedness of $f'(x)$ on M, we obtain for arbitrarily chosen interval (a_k, b_k):

$$(-L-\epsilon)(b_k - a_k) < f(b_k) - f(a_k) < (L+\epsilon)(b_k - a_k).$$

Hence

$$\sum_K |f(b_k) - f(a_k)| < \sum_K (L+\epsilon)(b_k - a_k) < (L+\epsilon)\delta < \epsilon(L+\epsilon),$$

and the function f is absolutely continuous relative to the acceptable mapping Φ.

We now turn us to the case when the set M is a null set. For this purpose define $M_n = \{x \in M; n \leq |f'(x)| < n+1\}$. Then M_n are disjoint sets and $M = \bigcup_n M_n$. For arbitrary n, M_n is a null set and it may be covered by the system $\mathcal{S} = \{S_l\}_L$ of disjoint open intervals with the sum of lenghts $s_n < \frac{\epsilon}{(n+1)2^n}$. Let $\mathcal{V}$ be a centered cover of M such that $\mathcal{V} \prec \mathcal{S} = \{S_n\}$.

We find a joint refinement of covers $\mathcal{V}$ and $\mathcal{D}$, which is contained in $\mathfrak{U}$ (it exists since the cover $\mathcal{V}$ is an element of $\mathfrak{U}$). Let $\mathcal{P}$ be this cover. We take $\delta < \epsilon$ and a system of intervals $\{(a_k,b_k)\}_K \in \Phi(\mathcal{P})$ with $\sum_K (b_k - a_k) < \delta$. Hence ($x_k$ is an element of M included in the interval $[a_k, b_k]$):

$$\begin{aligned}
\sum_K |f(b_k) - f(a_k)| &< \sum_K (|f'(x_k)| + \epsilon)(b_k - a_k) \\
&= \sum_{n=0}^{\infty} \left(\sum_{x_k \in M_n} |f'(x_k)| \,(b_k - a_k) \right) + \epsilon \sum_K (b_k - a_k) \\
&< \sum_{n=0}^{\infty} (n+1) \left(\sum_{x_k \in M_n} (b_k - a_k) \right) + \epsilon\delta \\
&< \left(\sum_{n=0}^{\infty} (n+1) \frac{\epsilon}{(n+1)2^n} \right) + \epsilon^2 = 2\epsilon + \epsilon^2 = \epsilon(2+\epsilon),
\end{aligned}$$

and the function f is absolutely continuous relative to the mapping Φ. □

Remark. The assumption of HK mapping can be omitted for finite sets. A cover by intervals $(x-\delta_x, x+\delta_x)$ can be constructed in such way that intervals $(x-\delta_x, x+\delta_x)$ are disjoint. Hence these intervals contain only one point of a set M – this will be the centre of interval, the point x. Corresponding "refinements" have then desired properties, the point from this finite set belongs to the closed interval $[a_k, b_k]$.

In the following, assume M, N to be arbitrary subsets of I.

Corollary 7.2. *Let $f \in AC(\Phi_N)$ have finite derivative f' on a set M. Then $f \in AC(\Phi_{M\cup N})$, provided assumptions of Theorem 7.1 are satisfied and Theorem 6.2 holds for $\Phi_{M\cup N}$ related to $M \cup N$.*

PROOF: Suppose that the function f has a finite derivative on M and the assumptions of Theorem 7.1 hold. Then $f \in AC(\Phi_M)$. If Theorem 6.2 is true for $\Phi_{M\cup N}$ relative to $M \cup N$, then $AC(\Phi_M) \cap AC(\Phi_N) \subset AC(\Phi_{M\cup N})$. Hence $f \in AC(\Phi_{M\cup N})$. □

Theorem 7.3 is only a consequence of the foregoing one, but for its importance we formulate it as a theorem. Let us denote by $M \div N$ the symmetric difference of sets M and N.

Theorem 7.3. *Assume that $f \in AC(\Phi_M)$, $g \in AC(\Phi_N)$ and there exist derivatives f', g' on $N \setminus M$, $M \setminus N$, respectively, and one of these assumptions is satisfied:*

(1) *$M \div N$ is a null set;*
(2) *the derivative f' is bounded on $N \setminus M$ and $M \setminus N$ is a null set;*
(3) *the derivative g' is bounded on $M \setminus N$ and $N \setminus M$ is a null set;*
(4) *the derivatives f' on $N \setminus M$ and g' on $M \setminus N$ are bounded.*

If the restriction of acceptable mapping Φ related to $M \cup N$ has the HK-property also on $N \setminus M$ and on $M \setminus N$, then $f + g \in AC(\Phi)$.

PROOF: Using the assumptions of the theorem, we have $f \in AC(\Phi_{N\setminus M})$, $g \in AC(\Phi_{M\setminus N})$, respectively. By Theorem 7.1, $f \in AC(\Phi_N)$ and $g \in AC(\Phi_M)$. Corollary 6.3 yields $f, g \in AC(\Phi_{M\cup N}) = AC(\Phi)$. Linearity gives then $f + g \in AC(\Phi)$. □

Now, let us come to the most important theorem of this section. There are many possibilities how to prove this theorem (see [7, p. 104], [11, p. 30],l and [13, p. 225], for instance). We show a longer but elementary proof that is motivated by [1] (proof of Theorem 14.11, p. 236).

Theorem 7.4. *Suppose that M is a null set, $f \in AC(\Phi_M)$ related to the full mapping Φ on the fine quasi-uniformity at M and a finite derivative $f'(x)$ exists and is nonnegative for all $x \in I \setminus M$. Then f is non-decreasing on I.*

PROOF: Fix $\epsilon > 0$. By definition of $AC(\Phi_M)$, we find for ϵ corresponding δ and $\mathcal{U}_M \in D(\Phi_M)$. As the set M is a null set, we find $G \supset M$ open such that $\mu(G) < \delta$. Let $\mathcal{V}$ be a centered cover of M by intervals contained in G. Let $\mathcal{W}_M$ be a joint refinement of the covers $\mathcal{U}_M$ and $\mathcal{V}_M$ belonging to $D(\Phi_M)$.

Since the derivative of the function f exists on $I \setminus M$, we find for ϵ a $\mathrm{der}(f, \epsilon)$-cover of $I \setminus M$. Let $\mathcal{D}_{I\setminus M} = \{(x - \delta_x, x + \delta_x); x \in I \setminus M\}$ be this cover. Then the system of sets $\mathcal{W}_M \cup \mathcal{D}_{I\setminus M}$ covers the interval I.

Let $r, s \in I$ and $r < s$. We construct a cover of the bounded interval $[r, s]$.

(1) For $x \in I \setminus M$, pick an interval $(x - \delta_x, x + \delta_x)$ from the cover $\mathcal{D}_{I\setminus M}$.
(2) For $x \in M$, yet uncovered by previous intervals, we choose an interval from $\mathcal{W}_M$ that contains x.

We choose from this cover a finite subcover of the interval $[r, s]$. Let the system $\mathcal{P} = \{P_p\}_{p=0}^q = \{(c_p, d_p)\}_{p=0}^q$ be this cover. As the system $\mathcal{P}$ is finite, we may assume that this cover is minimal. Hence only neighbouring intervals have a nonempty intersection.

Now set $y_0 = r$, $y_q = s$ and $y_p \in P_p \cap P_{p+1}$. If the interval (c_{p+1}, d_p) contains a point x_p as a centre of the interval $(c_p, d_p) \in \mathcal{D}_{I\setminus M}$, we take $y_p \in [x_p, d_p)$. Analogously, if the interval (c_{p+1}, d_p) contains a point x_{p+1} as a centre of the interval $(c_{p+1}, d_{p+1}) \in \mathcal{D}_{I\setminus M}$, we pick $y_p \in (c_{p+1}, x_{p+1}]$. In the case that the interval (c_{p+1}, d_p) includes the points x_p, x_{p+1} resp., as the centres of intervals (c_p, d_p), $(c_{p+1}, d_{p+1}) \in \mathcal{D}_{I\setminus M}$ resp., we take $y_p \in [x_p, x_{p+1}]$. Let S be the system of these intervals. We estimate

$$f(r) - f(s) = \sum_{(y_p, y_{p+1}) \in S} (f(y_p) - f(y_{p+1})).$$

Set D a system of intervals (y_p, y_{p+1}) that are included in some interval of the cover $\mathcal{D}_{I\setminus M}$, let E be a system of remaining intervals; these intervals refine the cover $\mathcal{W}_M$. Remember that Φ_M is full, hence an arbitrary subinterval of the cover $\mathcal{W}_M$ is included in $\Phi_M(\mathcal{U}_M)$.

Since $\sum_{(y_p, y_{p+1}) \in E} (y_{p+1} - y_p) < \delta$, absolute continuity of the function f yields

$$\sum_{(y_p, y_{p+1}) \in E} (f(y_p) - f(y_{p+1})) \le \sum_{(y_p, y_{p+1}) \in E} |f(y_p) - f(y_{p+1})| < \epsilon. \tag{2}$$

For an arbitrary interval $(y_p, y_{p+1}) \in D$ the following inequality holds:

$$(f'(x_p) - \epsilon)(y_{p+1} - y_p) < f(y_{p+1}) - f(y_p) < (f'(x_p) + \epsilon)(y_{p+1} - y_p), \tag{3}$$

because $y_p \le x_p \le y_{p+1}$ and the interval (y_p, y_{p+1}) refines the interval $(x_p - \delta_{x_p}, x_p + \delta_{x_p})$. Then (using inequality (3) and the assumption that $f'(x)$ is nonnegative on $I \setminus M$):

$$\begin{aligned} \sum_{(y_p, y_{p+1}) \in D} (f(y_p) - f(y_{p+1})) &< \sum_{(y_p, y_{p+1}) \in D} (-f'(x_p) + \epsilon)(y_{p+1} - y_p) \\ &< \sum_{(y_p, y_{p+1}) \in D} \epsilon (y_{p+1} - y_p) \le \epsilon(s - r). \end{aligned} \tag{4}$$

Combining inequalities (2) and (4) for intervals from sets D and E, we get:

$$\begin{aligned} f(r)-f(s) &= \sum_{(y_p,y_{p+1})\in S} (f(y_p)-f(y_{p+1})) \\ &\leq \sum_{(y_p,y_{p+1})\in D} (f(y_p)-f(y_{p+1})) + \sum_{(y_p,y_{p+1})\in E} (f(y_p)-f(y_{p+1})) \\ &< \epsilon(s-r+1), \end{aligned}$$

therefore $f(r)-f(s)\leq 0$ (ϵ is an arbitrary positive number) and the function f is non-decreasing on I. □

Theorem 7.5. *Assume that M is a null set, $f\in AC(\Phi_M)$ related to HK-full mapping on a fine symmetric interval quasi-uniformity at M and a finite derivative $f'(x)$ exists and is nonnegative for all $x\in I\setminus M$. Then f is non-decreasing on I.*

Proof: The proof of Theorem 7.5 almost copies the previous one. The only difference is that the intervals (y_p,y_{p+1}) have to meet more requirements, they have to refine the cover $\mathcal{D}_{I\setminus M}$ or have to be images of some neighbourhoods of points from M in HK-full mapping (with no loss of generality, we may assume that these neighbourhoods are symmetric).

The idea of construction of intervals (y_p,y_{p+1}) is based on the fact that every interval (y_p,y_{p+1}) has to contain the centre of the interval from $\mathcal{D}_{I\setminus M}$ or $\mathcal{W}_M$ that this given interval covers. Let us show how to choose the point y_p in the intersection of two intervals such that the "left" interval is an element from $\mathcal{W}_M$ and the "right" interval belongs to $\mathcal{D}_{I\setminus M}$ (in other cases, we only combine these techniques). If the interval (c_{p+1},d_p) contains a point x_p as a centre of the interval $(c_p,d_p)\in\mathcal{W}_M$, we take $y_p\in[x_p,d_p)$. Analogously, if the interval (c_{p+1},d_p) contains a point x_{p+1} as a centre of the interval $(c_{p+1},d_{p+1})\in\mathcal{D}_{I\setminus M}$, we pick $y_p\in(c_{p+1},x_{p+1}]$. In the case that the interval (c_{p+1},d_p) includes the points x_p, x_{p+1}, resp., as centres of intervals $(c_p,d_p)\in\mathcal{W}_M$, $(c_{p+1},d_{p+1})\in\mathcal{D}_{I\setminus M}$, resp., we take $y_p\in[x_p,x_{p+1}]$. Finally, if the interval (c_{p+1},d_p) contains neither x_p nor x_{p+1}, we can choose the point y_p arbitrarily in (c_{p+1},d_p). Using this procedure, we obtain a division of $[r,s]$ of desired properties. □

Corollary 7.6. *Let f be an absolutely continuous function relative to the full mapping Φ_M on a fine quasi-uniformity at a null set M (or relative to the HK-full mapping on a fine symmetric quasi-uniformity at M) and $f'(x)=0$ for all $x\in I\setminus M$. Then f is constant on I.*

Proof: Under the above assumptions, the function f is by Theorem 7.4 (or 7.5) both non-decreasing and non-increasing on interval I, and therefore constant on I. □

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