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COFINITENESS AND FINITENESS OF LOCAL COHOMOLOGY MODULES OVER REGULAR LOCAL RINGS

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Abstract. Let $(R,\mathfrak{m})$ be a commutative Noetherian regular local ring of dimension d and I be a proper ideal of R such that $\mathrm{mAss}_R(R/I) = \mathrm{Assh}_R(I)$. It is shown that the R-module $H_I^{\mathrm{ht}(I)}(R)$ is I-cofinite if and only if $\mathrm{cd}(I,R) = \mathrm{ht}(I)$. Also we present a sufficient condition under which this condition the R-module $H_I^i(R)$ is finitely generated if and only if it vanishes.

Keywords: cofinite module; Cohen-Macaulay ring; Krull dimension; local cohomology; regular ring

MSC 2010: 13D45, 14B15, 13E05

1. Introduction

Throughout this paper, let R denote a commutative Noetherian local ring (with identity), I a proper ideal of R and M an R-module. The local cohomology modules $H_I^i(M)$ arise as the derived functors of the left exact functor $\Gamma_I(-)$, where for an R-module M, $\Gamma_I(M)$ is the submodule of M consisting of all elements annihilated by some powers of I, i.e. $\bigcup\limits_{n=1}^{\infty}(0:_M I^n)$. There is a natural isomorphism:

$$H_I^i(M) = \varinjlim_{n\geqslant 1} \mathrm{Ext}_R^i(R/I^n, M).$$

It is well-known that if $(R,\mathfrak{m})$ is a regular local ring of dimension $d>0$, then the top local cohomology module $H_{\mathfrak{m}}^d(R)$ is not a finitely generated R-module. But for each $i\geqslant 0$ and each finitely generated module over an arbitrary Noetherian local ring $(R,\mathfrak{m})$ the R-module $H_{\mathfrak{m}}^i(M)$ is Artinian and hence the R-module $\mathrm{Hom}_R(R/\mathfrak{m}, H_{\mathfrak{m}}^i(M))$ is finitely generated. This lead to a conjecture from Grothendieck in [7], that for any ideal I of a Noetherian ring R and any finitely generated

R-module M, the module $\operatorname{Hom}_R(R/I, H^i_I(M))$ is finitely generated. This conjecture is not true in general and several counterexamples are given by several authors (see [8], [4] and [5]). In fact, using [4], Theorem 3.9, it is easy to see that for any Noetherian ring of dimension $d \geqslant 3$ there are an ideal I of R and a finitely generated R-module M, such that the module $\operatorname{Hom}_R(R/I, H^i_I(M))$ is not finitely generated. But for the first time Hartshorne was able to present a counterexample to Grothendieck's conjecture (see [8] for details and the proof). However, he defined an R-module M to be I-cofinite if $\operatorname{Supp} M \subseteq V(I)$ and $\operatorname{Ext}^j_R(R/I, M)$ is finitely generated for all j.

Recall that for an R-module M, the *cohomological dimension of M with respect to I* is defined as

$$\operatorname{cd}(I, M) := \max\{i \in \mathbb{Z} \colon H^i_I(M) \neq 0\}.$$

Let I be a proper ideal of a regular local ring $(R, \mathfrak{m})$. Let $\operatorname{bight}(I)$ denote the biggest height of any minimal prime of I. In [10], Theorem 2.3, it was shown that if $\operatorname{Hom}_R(R/I, H^j_I(R))$ is finitely generated for all $j > r$ for some $r \geqslant \operatorname{bight}(I)$, then $H^j_I(R) = 0$ for all $j > r$. This result implies that if $k = \operatorname{cd}(I, R) > \operatorname{bight}(I)$, then the R-module $\operatorname{Hom}_R(R/I, H^k_I(R))$ is not finitely generated. In particular, the R-module $H^k_I(R))$ is not I-cofinite. The first aim of this paper is to show that if $\operatorname{bight}(I) = \operatorname{ht}(I) = n$, then the R-module $H^n_I(R)$ is I-cofinite if and only if $\operatorname{cd}(I, R) = n$.

As mentioned in the introduction of [9], if R is a regular local ring containing a field, then $H^l_I(R)$ (for $l \geqslant 1$) is finitely generated if and only if it vanishes. This holds because in this family of regular rings we have $\operatorname{injdim}_R(H^i_I(R)) \leqslant \dim \operatorname{Supp}(H^i_I(R))$. In this paper we present a sufficient condition for local Cohen-Macaulay rings under which the same assertion holds.

For each R-module L, we denote by $\operatorname{Assh}_R L$ the set $\{\mathfrak{p} \in \operatorname{Ass}_R L \colon \dim R/\mathfrak{p} = \dim L\}$. Also for any ideal $\mathfrak{a}$ of R we denote $\{\mathfrak{p} \in \operatorname{Spec} R \colon \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. Finally, for any R-module T, $\operatorname{injdim}_R(T)$ denotes the injective dimension of T.

2. Main results

The following theorem is the first main result of this paper.

Theorem 2.1. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension d and I a proper ideal of R such that $\operatorname{bight}(I) = \operatorname{ht}(I) = n$. Then the following statements are equivalent:*

(i) *$H^n_I(R)$ is I-cofinite,*

(ii) $\operatorname{cd}(I, R) = n$.

P r o o f. (ii)$\Rightarrow$(i): It follows from [13], Proposition 3.11.

(i)⇒(ii): Suppose the contrary is true. Let $\mathfrak{p}$ be a minimal element of the set

$$\mathcal{S} := \bigcup_{i=n+1}^{d} \operatorname{Supp}(H_I^i(R)).$$

Then as by hypothesis we have $\operatorname{bight}(I) = \operatorname{ht}(I) = n$ and $\operatorname{cd}(IR_{\mathfrak{p}}, R_{\mathfrak{p}}) > n$, it follows from Grothendieck's vanishing theorem that $\dim(R_{\mathfrak{p}}) > n$ and $\operatorname{bight}(IR_{\mathfrak{p}}) = \operatorname{ht}(IR_{\mathfrak{p}}) = n$. Then replacing $(R, \mathfrak{m})$ with $(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}})$, we may assume that

$$\bigcup_{i=n+1}^{d} \operatorname{Supp}(H_I^i(R)) = \{\mathfrak{m}\} \quad \text{and} \quad \operatorname{bight}(I) = \operatorname{ht}(I) = n.$$

Now as for each $0 \leqslant i \leqslant n$ the R-module $H_I^i(R)$ is I-cofinite, it follows from [11], Corollary 3.5, that the R-module $\operatorname{Hom}_R(R/I, H_I^{n+1}(R))$ is finitely generated with support in $V(\mathfrak{m})$ and so is of finite length. So, by [13], Proposition 4.1, the R-module $H_I^{n+1}(R)$ is I-cofinite. Now since for each $0 \leqslant i \leqslant n+1$ the R-module $H_I^i(R)$ is I-cofinite, again it follows from [11], Corollary 3.5, that the R-module $\operatorname{Hom}_R(R/I, H_I^{n+2}(R))$ is finitely generated with support in $V(\mathfrak{m})$ and so is of finite length. Hence, by [13], Proposition 4.1, the R-module $H_I^{n+2}(R)$ is I-cofinite. Proceeding in the same way we can see that the R-modules $H_I^i(R)$ are I-cofinite for all $i \geqslant 0$. In particular, for all $j > \operatorname{bight}(I) = n$, the R-modules $\operatorname{Hom}_R(R/I, H_I^j(R))$ are finitely generated. Therefore in view of [10], Theorem 2.3 (i), we have $\operatorname{cd}(I, R) = n$, which is a contradiction. □

Corollary 2.2. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension d and $\mathfrak{p}$ a prime ideal of R such that $\operatorname{ht}(\mathfrak{p}) = n$. Then the following statements are equivalent:*
(i) *$H_{\mathfrak{p}}^n(R)$ is $\mathfrak{p}$-cofinite,*
(ii) *$\operatorname{cd}(\mathfrak{p}, R) = n$.*

Proof. The assertion follows from Theorem 2.1. □

For proving the next result we need the following well known lemma and its corollary.

Lemma 2.3. *Let $(R, \mathfrak{m})$ be a Noetherian local Cohen-Macaulay ring of dimension d and I a nonzero proper ideal of R such that $\operatorname{grade}(I, R) = t$. If $0 = Q_1 \cap \ldots \cap Q_r$ with $\operatorname{Ass}_R(R/Q_i) = \mathfrak{q}_i$ is a minimal primary decomposition of the zero ideal of R and*

$$T = \{\mathfrak{q} \in \operatorname{Ass}_R(R) \colon \dim R/(I + \mathfrak{q}) = \dim R/I\},$$

then $0 :_R H_I^t(R) = \bigcap_{\mathfrak{q}_i \in T} Q_i$.

Proof. See [2], Theorem 2.2. □

Corollary 2.4. *Let $(R,\mathfrak{m})$ be a Noetherian local Cohen-Macaulay ring of dimension d and $\mathfrak{p}$ a prime ideal of R such that* $\operatorname{grade}(\mathfrak{p},R)=t\geqslant 0$. *Then $0:_R H^t_{\mathfrak{p}}(R)=0$ if and only if $Z_R(R)\subseteq\mathfrak{p}$, where $Z_R(R)$ is the set of all zero divisors of R.*

Proof. The assertion follows immediately from Lemma 2.3. $\square$

Now we are ready to state and prove our next main result.

Theorem 2.5. *Let $(R,\mathfrak{m})$ be a Noetherian local Cohen-Macaulay ring of dimension d and $\mathfrak{p}$ a prime ideal of R such that* $\operatorname{ht}(\mathfrak{p})=n=\operatorname{cd}(\mathfrak{p},R)$. *Then $Z_R(R)\subseteq\mathfrak{p}$.*

Proof. Since $\operatorname{ht}(\mathfrak{p})=n=\operatorname{cd}(\mathfrak{p},R)$, it follows that $\operatorname{grade}(\mathfrak{p},R)=n$ and hence we have $H^i_{\mathfrak{p}}(R)\neq 0$ if and only if $i=n$. Now we show that $H^{d-n}_{\mathfrak{m}}(H^n_{\mathfrak{p}}(R))\cong H^d_{\mathfrak{m}}(R)$. Since $\operatorname{grade}(\mathfrak{p},R)=n$, it follows that $\mathfrak{p}$ contains an R-regular sequence such as $x_1,\ldots,x_n$. In particular, this sequence is a $\mathfrak{p}$-filter regular sequence for R. Set $H:=H^n_{(x_1,\ldots,x_n)}(R)$. By [12], Proposition 1.2, $\Gamma_{\mathfrak{p}}(H)=H^0_{\mathfrak{p}}(H)=H^n_{\mathfrak{p}}(R)$ and for each $i\geqslant 1$, $H^i_{\mathfrak{p}}(H)=H^{n+i}_{\mathfrak{p}}(R)=0$. Therefore for each $i\geqslant 1$, $H^i_{\mathfrak{p}}(H)=0$ and $\Gamma_{\mathfrak{p}}(H)=H^n_{\mathfrak{p}}(R)$. On the other hand, $\Gamma_{\mathfrak{p}}(H/\Gamma_{\mathfrak{p}}(H))=0$ and for all $i\geqslant 1$, $H^i_{\mathfrak{p}}(H/\Gamma_{\mathfrak{p}}(H))\cong H^i_{\mathfrak{p}}(H)=0$. Then for all $i\geqslant 1$,

$$\emptyset=\operatorname{Supp}H^i_{\mathfrak{p}}(H/\Gamma_{\mathfrak{p}}(H))\subseteq V(\mathfrak{m})=\mathfrak{m}.$$

Hence by [1], Theorem 3.1, $H^i_{\mathfrak{m}}(H/\Gamma_{\mathfrak{p}}(H))\cong H^i_{\mathfrak{p}}(H/\Gamma_{\mathfrak{p}}(H))=0$. Also by [1], Theorem 4.5, $H^{d-n}_{(x_{n+1},\ldots,x_d)}(H)\cong H^d_{(x_1,\ldots,x_d)}(R)\cong H^d_{\mathfrak{m}}(R)$. Since H is $(x_1,\ldots,x_n)$-torsion, it follows that $H^i_{(x_{n+1},\ldots,x_d)}(H)\cong H^i_{(x_1,\ldots,x_d)}(H)\cong H^i_{\mathfrak{m}}(H)$. Consequently, for all $i\geqslant 1$, $H^{d-n}_{\mathfrak{m}}(H)\cong H^{d-n}_{(x_{n+1},\ldots,x_d)}(H)\cong H^d_{\mathfrak{m}}(R)$. There is a short exact sequence

$$0\to\Gamma_{\mathfrak{p}}(H)=H^n_{\mathfrak{p}}(H)\to H\to H/\Gamma_{\mathfrak{p}}(H))\to 0.$$

Now from this short exact sequence and the fact that for all $i\geqslant 0$, $H^i_{\mathfrak{p}}(H/\Gamma_{\mathfrak{p}}(H))=0$, we conclude that

$$H^{d-n}_{\mathfrak{m}}(H^n_{\mathfrak{p}}(R))\cong H^{d-n}_{\mathfrak{m}}(H)\cong H^d_{\mathfrak{m}}(R).$$

Therefore, using the fact that $\operatorname{Ass}_R(R)=\operatorname{Assh}_R(R)$, we used from reference [3], Corollary 2.9, that

$$0:_R H^n_{\mathfrak{p}}(R)\subseteq 0:_R H^{d-n}_{\mathfrak{m}}(H^n_{\mathfrak{p}}(R))\subseteq 0:_R H^d_{\mathfrak{m}}(R)=0.$$

Hence, we have $0:_R H^n_{\mathfrak{p}}(R)=0$ and so the assertion follows from Corollary 2.4. $\square$

The following proposition is needed in the proof of Theorem 2.7.

Proposition 2.6. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geqslant 2$ and let X and Y be nonempty subsets of $\mathrm{Ass}_R(R)$ such that $\mathrm{Ass}_R(R) = X \cup Y$ and $X \cap Y = \emptyset$. Let n be a positive integer such that $1 \leqslant n \leqslant d-1$. Then there exists a prime ideal Q of R such that $\mathrm{ht}(Q) = n$ and $\bigcap_{\mathfrak{p}\in X} \mathfrak{p} \subseteq Q$ and $\bigcap_{\mathfrak{q}\in Y} \mathfrak{q} \nsubseteq Q$. In particular, $\mathrm{cd}(Q, R) > \mathrm{ht}(Q) = n$.*

Proof. Since $\bigcap_{\mathfrak{q}\in Y} \mathfrak{q} \nsubseteq \bigcup_{\mathfrak{p}\in X} \mathfrak{p}$, it follows that there exists an element $y \in \bigcap_{\mathfrak{q}\in Y} \mathfrak{q}$ such that $y \notin \bigcup_{\mathfrak{p}\in X} \mathfrak{p}$. Then y is a part of a system of parameters for the R-module R/J, where $J := \bigcap_{\mathfrak{p}\in X} \mathfrak{p}$. So there are elements $x_1, \ldots, x_n \in \mathfrak{m}$, where the elements $y, x_1, \ldots, x_n$ are a part of a system of parameters for R/J. So $\dim(R/(J + (x_1, \ldots, x_n))) = d - n$, which implies that there exists a prime ideal Q in $\mathrm{Assh}_R(R/(J + (x_1, \ldots, x_n)))$ such that $\mathrm{ht}(Q) = n$ and $y \notin Q$ and so $\bigcap_{\mathfrak{q}\in Y} \mathfrak{q} \nsubseteq Q$ and $\bigcap_{\mathfrak{p}\in X} \mathfrak{p} \subseteq Q$. In particular, $Z_R(R) \nsubseteq Q$ and so by Theorem 2.5 we have $\mathrm{ht}(Q) = n < \mathrm{cd}(Q, R)$. □

Theorem 2.7. *Let $(R, \mathfrak{m})$ be a Noetherian local Cohen-Macaulay ring of dimension $d \geqslant 2$ and n an integer such that $1 \leqslant n \leqslant d-1$. If for any prime ideal $\mathfrak{p}$ of R with $\mathrm{ht}(\mathfrak{p}) = n$ we have $\mathrm{cd}(\mathfrak{p}, R) = n$, then $\mathrm{Ass}_R(R)$ has exactly one element.*

Proof. The assertion follows from Proposition 2.6. □

As mentioned in the introduction of [9], if R is a regular local ring containing a field, then $H_I^l(R)$ (for $l \geqslant 1$) is finitely generated if and only if it vanishes. In this section we present a condition under which the same assertion holds for a given Cohen-Macaulay local ring.

Theorem 2.8. *Let $(R, \mathfrak{m})$ be a Noetherian Cohen-Macaulay local ring of dimension $d \geqslant 1$ such that*

$$\mathfrak{m} H_J^{\mathrm{ht}(J)}(R) = H_J^{\mathrm{ht}(J)}(R)$$

for every proper ideal J of R with $\mathrm{ht}(J) \geqslant 1$. Let I be an ideal of R such that $H_I^l(R)$ (for $l \geqslant 1$) is finitely generated. Then $H_I^l(R) = 0$.

Proof. Suppose that the contrary is true and $H_I^l(R)$ is nonzero and finitely generated. Since $H_I^l(R) \neq 0$, it follows that I is a proper and non-nilpotent ideal of R. If $l = \mathrm{ht}(I)$, then as $l \geqslant 1$, it follows from the hypothesis that $H_I^l(R) = \mathfrak{m} H_I^l(R)$ and so by NAK lemma (Nakayama's lemma) it follows that $H_I^l(R) = 0$, which is a contradiction. Therefore, using [6], Theorem 6.2.7, we have $l > \mathrm{ht}(I) = \mathrm{grade}(I, R)$.

So, if $l = 1$, then $\operatorname{ht}(I) = 0$. Moreover, as the R-module $H^1_I(R)$ is finitely generated, it follows that the set $\operatorname{Ass}_R(H^1_I(R))$ is finite. Let

$$\operatorname{Ass}_R(H^1_I(R)) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}.$$

Then it follows from the Grothendieck's vanishing theorem that $\operatorname{ht}(\mathfrak{p}_i) \geqslant 1$ for all $i = 1, \ldots, n$. Therefore there exists an element $x \in \bigcap_{i=1}^{n} \mathfrak{p}_i$ such that x is an R-sequence. Now it is easy to see that x is a system of parameters for the R-module R/I. In particular, $\dim(R/(I + Rx)) = d - 1$, which implies that $\operatorname{ht}(I + Rx) = 1$. As $x \in \bigcap_{i=1}^{n} \mathfrak{p}_i$, it follows that $H^0_{Rx}(H^1_I(R)) = H^1_I(R)$. Consequently, from the exact sequence

$$0 \to H^1_{Rx}(H^0_I(R)) \to H^1_{I+Rx}(R) \to H^0_{Rx}(H^1_I(R)) \to 0$$

(see [14], Corollary 3.5) we get the exact sequence

$$(*) \qquad 0 \to H^1_{Rx}(H^0_I(R)) \to H^1_{I+Rx}(R) \to H^1_I(R) \to 0.$$

Now as by hypothesis we have

$$\mathfrak{m}H^1_{I+Rx}(R) = H^1_{I+Rx}(R),$$

from the exact sequence $(*)$, $H^1_I(R) = \mathfrak{m}H^1_I(R)$ and so by NAK lemma it follows that $H^1_I(R) = 0$, which is a contradiction. Thus, we have $l > \operatorname{grade}(I, R) \geqslant 1$. Now in view of [9], Theorem 3, there exists an ideal $J \supseteq I$ of $\operatorname{grade}(J, R) = l - 1$ such that

$$H^l_I(R) \cong H^l_J(R).$$

So replacing I with J we may assume that $\operatorname{grade}(I, R) = l - 1 \geqslant 1$ and the R-module $H^l_I(R)$ is nonzero and finitely generated. Now as the R-module $H^l_I(R)$ is finitely generated, it follows that the set $\operatorname{Ass}_R(H^l_I(R))$ is finite. Let

$$\operatorname{Ass}_R(H^l_I(R)) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_t\}.$$

Then it follows from the Grothendieck's vanishing theorem that $\operatorname{ht}(\mathfrak{q}_i) \geqslant l$ for all $i = 1, \ldots, t$. Therefore there exists an element $z \in \bigcap_{i=1}^{t} \mathfrak{q}_i$ such that $\operatorname{ht}(I + Rz) = l$. As $z \in \bigcap_{i=1}^{t} \mathfrak{q}_i$, it follows that $H^0_{Rz}(H^l_I(R)) = H^l_I(R)$. Consequently, from the exact sequence

$$0 \to H^1_{Rz}(H^{l-1}_I(R)) \to H^l_{I+Rz}(R) \to H^0_{Rz}(H^l_I(R)) \to 0$$

(see [14], Corollary 3.5) we get the exact sequence

$$(**)\qquad 0 \to H^1_{Rz}(H^{l-1}_I(R)) \to H^l_{I+Rz}(R) \to H^l_I(R) \to 0.$$

Now as by hypothesis we have

$$\mathfrak{m}H^l_{I+Rz}(R) = H^l_{I+Rz}(R),$$

from the exact sequence $(**)$, $H^l_I(R) = \mathfrak{m}H^l_I(R)$ and so by NAK lemma it follows that $H^l_I(R) = 0$, which is a contradiction. $\square$

The following result is an application of Theorem 2.8.

Theorem 2.9. *Let $(R, \mathfrak{m})$ be a Noetherian regular local ring of dimension $d \geqslant 1$ such that*

$$\operatorname{injdim}_R H^{\operatorname{ht}(J)}_J(R) < d$$

for every proper nonzero ideal J of R. Let I be an ideal of R such that $H^l_I(R)$ (for $l \geqslant 1$) is finitely generated. Then $H^l_I(R) = 0$.

P r o o f. In view of Theorem 2.8 it is enough to prove that

$$\mathfrak{m}H^{\operatorname{ht}(J)}_J(R) = H^{\operatorname{ht}(J)}_J(R)$$

for every proper ideal J of R with $\operatorname{ht}(J) \geqslant 1$. To do this, suppose that J is a proper and nonzero ideal of R such that $\mathfrak{m}H^{\operatorname{ht}(J)}_J(R) \neq H^{\operatorname{ht}(J)}_J(R)$. Then there is an exact sequence

$$0 \to K \to H^{\operatorname{ht}(J)}_J(R) \to R/\mathfrak{m} \to 0$$

for some submodule K of $H^{\operatorname{ht}(J)}_J(R)$, which induces the exact sequence

$$\operatorname{Ext}^d_R(R/\mathfrak{m}, H^{\operatorname{ht}(J)}_J(R)) \to \operatorname{Ext}^d_R(R/\mathfrak{m}, R/\mathfrak{m}) \to 0.$$

(Note that since R is a regular local ring of dimension d, it follows $\operatorname{injdim}_R(K) \leqslant d$.) Now as $\operatorname{Ext}^d_R(R/\mathfrak{m}, R/\mathfrak{m}) \neq 0$, it follows that $\operatorname{Ext}^d_R(R/\mathfrak{m}, H^{\operatorname{ht}(J)}_J(R)) \neq 0$ and hence $\operatorname{injdim}_R H^{\operatorname{ht}(J)}_J(R) = d$, which is a contradiction. $\square$

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