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GLOBAL ADAPTIVE OUTPUT-FEEDBACK CONTROL FOR SWITCHED UNCERTAIN NONLINEAR SYSTEMS

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In this paper, we investigate the problem of global output-feedback regulation for a class of switched nonlinear systems with unknown linear growth condition and uncertain output function. Based on the backstepping method, an adaptive output-feedback controller is designed to guarantee that the state of the switched nonlinear system can be globally regulated to the origin while maintaining global boundedness of the resulting closed-loop switched system under arbitrary switchings. A numerical example is given to demonstrate the effectiveness of the proposed control scheme.

Keywords: switched nonlinear system, output-feedback, adaptive control

Classification: 93D21, 39A13

1. INTRODUCTION

In recent years, the global output-feedback control has become an interesting topic in the field of nonlinear control theory, and therefore has attracted considerable attention [1, 4, 5, 6, 12, 13, 14, 15]. As an important class of hybrid dynamical systems, switched nonlinear systems are usually encountered in practical applications, such as aircraft control systems, robot control systems, and networked control systems [2, 8]. Nevertheless, the global output-feedback control problem of switched nonlinear systems has been limitedly studied in existing literatures [3, 7, 9, 10]. As a consequence, the further investigation of output-feedback stabilization for switched nonlinear systems turns out to be much more important.

In this paper, we consider global output-feedback control problem of switched uncertain nonlinear system:

$$
\begin{aligned}
\dot{\eta}_i &= g_i \eta_{i+1} + \phi_{i,\sigma(t)}(t, \eta, d(t)), \ i = 1, \dots, n-1, \\
\dot{\eta}_n &= g_n u + \phi_{n,\sigma(t)}(t, \eta, d(t)), \\
y &= h_{\sigma(t)}(\eta_1)
\end{aligned}
\tag{1}
$$

where $\eta = (\eta_1, \dots, \eta_n)^T \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are system state, control input and output, respectively. $d : \mathbb{R} \to \mathbb{R}^s$ is a continuous function which denotes uncertain time-varying parameter or disturbance. $\sigma(t)$ is the switching signal taking its values in a finite

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set $M = \{1, \ldots, m\}$ and m is the number of subsystems. The control coefficients $g_i > 0$, $i = 1, \ldots, n$, are unknown constants. The uncertain functions $\phi_{ik} : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$, are continuous and $\phi_{ik}(t, 0, d(t)) = 0$ for $i = 1, \ldots, n-1$ and $k \in M$. The uncertain function $h_k : \mathbb{R} \to \mathbb{R}$, $k \in M$, is $\mathcal{C}^1$ and $h(0) = 0$. Moreover, we assume that the state of system (1) does not jump at the switching instants, i. e., the trajectory $\eta(t)$ is everywhere continuous, and only the system output is measurable.

It is well-known that the backstepping design method is a basic tool to handle global output-feedback stabilization problem of nonlinear systems. Based on such method, many interesting results on stability and stabilization of nonlinear system have been derived, see [5, 6, 10, 12, 14, 15] and the references therein. In the non-switched case, when the growth rate of nonlinearities is unknown, how to design global output-feedback adaptive observer and controller becomes much more important. This has been solvable in [5] by backstepping method and the introduction of one dynamic gain. Then more extensive results have been achieved in [6] by using double dynamic gains. Without additional conditions imposed on the system nonlinearities and control coefficients, global output stabilization problem was further investigated in [12] by introducing a distinct high-gain observer. In the switched case, the delicate construction of common Lyapunov function for all subsystems under arbitrary switchings is of great importance in global output-feedback stabilization. Recently, a great deal of approaches have been proposed to select an appropriate common Lyapunov function. For instance, via bacstepping method, output-feedback stabilization of a class of switched nonlinear systems with unknown control coefficients was studied in [10] by constructing a common Lyapunov function. Subsequently, without precise knowledge of system nonlinearity, global output-feedback stabilization problem for switched uncertain nonlinear systems was solved in [7] by backstepping approach.

However, the precise knowledge of output function is required in the observer design technique of all aforementioned literatures. When output function is uncertain, how to find a proper and general restriction on output function is a main issue. To this end, [14, 15] resolved the problem of global stabilization for non-switched nonlinear systems by restricting the upper and the lower bounds of partial derivative of output function. Up to now, there is no result on control of switched nonlinear systems with uncertain output function. Spontaneously, an interesting problem is raised: *can we find weaker assumptions and an adaptive controller to globally stabilize switched uncertain nonlinear system* (1) *under arbitrary switchings?* In this paper, we aim to solve the adaptive control problem for switched uncertain nonlinear system (1) with unknown linear growth rate and uncertain output function. To deal with this, by a dynamic gain, we first construct a novel observer without information of unmeasurable states. Then based on the backstepping method and the common Lyapunov function idea, a common universal output-feedback controller is designed such that the state of the closed-loop switched uncertain nonlinear system can be globally regulated to the origin while all signals of the closed-loop switched uncertain nonlinear system are bounded. The main contributions of this paper are characterized as follows:

(i) Compared with [14, 15], a novel dynamic high-gain observer is designed owing to unknown growth condition and unknown control coefficients.

(ii) Different from [7], a new adaptive output-feedback controller is designed since output function is uncertain.

(iii) Our results extend existing global stabilization results for non-switched systems to control of switched nonlinear systems.

Notations. Throughout this paper,

$\mathbb{R}^n$ denotes the n-dimensional real space;
$\mathbb{R}_+$ represents the set of all the nonnegative real numbers;
$\mathcal{C}^i$ denotes the set of all functions with continuous ith partial derivatives;
$|X|$ interprets the absolute value of scalar X;
$\|Y\|$ is the Euclidean norm of a vector Y.

For unification of denotation, we take $\prod_{l=j}^{i}(\cdot) = 1$ for $j > i$.

2. PROBLEM FORMULATION

In order to solve the problem of global output-feedback regulation of switched uncertain nonlinear system (1), the following assumptions are required.

Assumption 2.1. For $i = 1, \dots, n$, control coefficients g_i satisfy $\underline{g} \le g_i \le \bar{g}$, where $\underline{g}$ and $\bar{g}$ are known positive constants.

Assumption 2.2. For $i = 1, \dots, n$ and $k \in M$, there exist unknown constants $\tilde{\theta}_k > 0$ such that

$$|\phi_{i,k}(t, \eta, d(t))| \le \tilde{\theta}_k(|\eta_1| + \cdots + |\eta_i|). \tag{2}$$

Assumption 2.3. There exist known positive constants $\underline{\lambda}_k$ and $\bar{\lambda}_k$, $k \in M$ such that

$$\underline{\lambda}_k \le \frac{\partial h_k(\eta_1)}{\partial \eta_1} \le \bar{\lambda}_k, \ \forall \eta_1 \in \mathbb{R}. \tag{3}$$

Remark 2.1. From Assumption 2.2, it is indicated that the upper boundedness of nonlinear function depends on unmeasurable states and unknown switching constant, which is a general linear growth condition. Assumptions 2.1 and 2.3 imply that both the control coefficients and partial derivative of output function are restricted by two positive constants, which plays an important role in later control design.

Now, we introduce the following scaling transformation for system (1):

$$x_i = \frac{\underline{g}^n}{\prod_{j=i}^{n} g_j}\eta_i, \ i = 1, \dots, n. \tag{4}$$

Under transformation (4), system (1) can be rewritten as

$$\begin{aligned} \dot{x}_i &= x_{i+1} + f_{i,\sigma(t)}(t, x, d(t)), \ i = 1, \dots, n-1, \\ \dot{x}_n &= \underline{g}^n u + f_{n,\sigma(t)}(t, x, d(t)), \\ y &= h_{\sigma(t)}(gx_1) \end{aligned} \tag{5}$$

where $x=(x_1,\dots,x_n)^T$, $g=(\prod_{j=1}^{n}g_j)/\underline{g}^n$ and $f_{i,k}=(\underline{g}^n/\prod_{j=i}^{n}g_j)\phi_{i,k}$, $i=1,\dots,n$, $k\in M$. By Assumptions 2.2 and 2.3, it can be not hard to find an unknown positive constant θ_k, and known positive constants $\underline{c}_k$, $\bar{c}_k$, such that

$$|f_{i,k}(t,x,d(t))|\le\theta_k(|x_1|+\cdots+|x_i|), \tag{6}$$

$$\underline{c}_k\le\frac{\partial h_k(gx_1)}{\partial x_1}\le\bar{c}_k,\ \forall x_1\in\mathbb{R} \tag{7}$$

where $\underline{c}_k=\underline{\lambda}_k$ and $\bar{c}_k=(\bar{g}^n/\underline{g}^n)\bar{\lambda}_k$.

In what follows, we will focus on the equivalent system (5). Our objective is to design an adaptive output-feedback controller

$$\dot{\hat{x}}=\psi(\hat{x},L),\ \dot{L}=\varphi(y,\hat{x},L)\text{ and }u=\varrho(y,\hat{x},L),$$

such that the problem of global regulation for switched uncertain nonlinear system (5) is solvable.

3. MAIN RESULTS

In this section, under Assumptions 2.1 − 2.3, it is possible to globally stabilize switched nonlinear system (1) by a universal adaptive output-feedback controller. Thus we are ready to give the main result of present paper.

Theorem 3.1. Under Assumptions 2.1 − 2.3, the problem of global adaptive regulation for switched uncertain nonlinear system (1) under arbitrary switchings is addressed by the following dynamic high-gain observer, and observer-based output-feedback controller:

$$\begin{aligned}\dot{\hat{x}}_i&=\hat{x}_{i+1}-L^ia_i\hat{x}_1,\ i=1,\dots,n-1,\\ \dot{\hat{x}}_n&=\underline{g}^nu-L^na_n\hat{x}_1,\\ \dot{L}&=\frac{y^2}{L^2}+\frac{\hat{x}_1^2}{L^2}+\xi_n^2,\ L(0)=1,\\ u&=-\underline{g}^{-n}L^{n+1}b_n\xi_n\end{aligned} \tag{8}$$

where $(\hat{x}_1,\dots,\hat{x}_n)$ is the observer state, L is a dynamic high gain, $a_i>0$, $i=1,\dots,n$ are coefficients of the Hurwitz polynomial $p(s)=s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n$, and ξ_n is recursively given by

$$\xi_1=\frac{y}{L},\ \xi_i=\frac{\hat{x}_i}{L^i}-\alpha_{i-1},\ \alpha_1=-b_1\xi_1,\ \alpha_{i-1}=-b_{i-1}\xi_{i-1},\ i=3,\dots,n \tag{9}$$

with $b_1,\dots,b_n$ being some appropriate positive constants.

Proof. Considering the equivalent system (5), we first introduce the change of coordinates:

$$\varepsilon_i=\frac{x_i-\hat{x}_i}{L^i},\ z_i=\frac{\hat{x}_i}{L^i},\ i=1,\dots,n. \tag{10}$$

In light of (5), (8), (9) and (10), we obtain

$$\begin{aligned}
\dot{\varepsilon} &= LA\varepsilon + F_k(\cdot) + ax_1 - \frac{\dot{L}}{L}D\varepsilon, \\
\dot{z} &= LAz + \frac{g^n}{L^n}Bu - \frac{\dot{L}}{L}Dz
\end{aligned} \tag{11}$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^T$, $z = (z_1, \ldots, z_n)^T$, $F_k(\cdot) = (f_{1,k}/L, \ldots, f_{n,k}/L^n)^T$, $a = (a_1, \ldots, a_n)^T$ and

$$A = \begin{pmatrix} -a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n-1} & 0 & \cdots & 1 \\ -a_n & 0 & \cdots & 0 \end{pmatrix}_{n\times n}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}_{n\times 1}, \quad D = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & n \end{pmatrix}_{n\times n}.$$

By constructions, A is Hurwitz matrix such that there is a positive definite matrix $P > 0$ satisfying $A^TP + PA \leq -I$ and $DP + PD \geq 0$.

Construct the Lyapunov function $V_0 = \varepsilon^T P\varepsilon$. A simple calculation yields

$$\begin{aligned}
\dot{V}_0 &= L\varepsilon^T(PA + A^TP)\varepsilon + 2\varepsilon^TP(F_k + ax_1) - \frac{\dot{L}}{L}\varepsilon^T(PD + DP)\varepsilon \\
&\leq -L\|\varepsilon\|^2 + 2\varepsilon^TPF_k + 2\varepsilon^TPax_1.
\end{aligned} \tag{12}$$

In what follows, we estimate the last two terms on the right-hand side of (12). Notice that by construction, $\dot{L}(t) \geq 0$, $L(0) = 1$ and therefore $L(t) \geq 1$ for $\forall t \geq 0$, which together with (6) yields

$$\left|\frac{f_{i,k}}{L^i}\right| \leq \frac{\theta_k}{L^i}(|x_1| + \cdots + |x_i|) \leq \theta_k \sum_{j=1}^{i} \frac{|x_j|}{L^j},\ i = 1, \ldots, n,\ k \in M.$$

This leads to

$$\|F_k(\cdot)\| \leq \frac{|f_{1,k}|}{L} + \frac{|f_{2,k}|}{L^2} + \cdots + \frac{|f_{n,k}|}{L^n} \leq n\theta_k \sum_{j=1}^{n} \frac{|x_j|}{L^j},\ k \in M. \tag{13}$$

By the definition of ε_i and z_i, one has

$$\sum_{j=1}^{n} \frac{|x_j|}{L^j} \leq \frac{|x_1|}{L} + \sum_{i=2}^{n} |z_i| + \sqrt{n}\|\varepsilon\|. \tag{14}$$

From (7), it can be deduced that

$$\underline{c}_k|x_1| \leq |y| \leq \bar{c}_k|x_1|,\ k \in M. \tag{15}$$

Combining (13), (14) and (15) gives rise to

$$
\begin{aligned}
2\varepsilon^T P F_k &\leq 2\theta_k \|P\|\|\varepsilon\| \left(n\frac{|\xi_1|}{\underline{c}_k} + n\sum_{i=2}^{n} |z_i| + n\sqrt{n}\|\varepsilon\| \right) \\
&\leq \frac{1}{2}\xi_1^2 + \frac{1}{4}\sum_{i=2}^{n} z_i^2 + \left(\frac{2\theta_k^2 n^2 \|P\|^2}{\underline{c}_k^2} + 2\theta_k n\sqrt{n}\|P\| + 4\theta_k^2 n^2 (n-1)\|P\|^2 \right) \|\varepsilon\|^2, \\
2\varepsilon^T P a x_1 &\leq 2\|\varepsilon\|\|Pa\|| \frac{L\xi_1}{\underline{c}_k}| \leq \frac{L}{2}\|\varepsilon\|^2 + \frac{2L\|Pa\|^2}{\underline{c}_k^2}\xi_1^2.
\end{aligned} \tag{16}
$$

Substituting (16) into (12) yields

$$
\dot{V}_0 \leq -\left(\frac{L}{2} - \Theta_k\right)\|\varepsilon\|^2 + \left(\frac{1}{2} + \frac{2L\|Pa\|^2}{\underline{c}_k^2}\right)\xi_1^2 + \frac{1}{4}\sum_{i=2}^{n} z_i^2 \tag{17}
$$

where $\Theta_k = 2\theta_k^2 n^2 \|P\|^2/\underline{c}_k^2 + 2\theta_k n\sqrt{n}\|P\| + 4\theta_k^2 n^2(n-1)\|P\|^2$ is an unknown constant. □

3.1. Adaptive controller design

In this subsection, we give the design of adaptive output-feedback controller for system (5) by using the backstepping method. The design procedure is summarized as follows.

Step 1: From the definitions of ξ_1 and z_1, one obtains

$$
\begin{aligned}
\dot{\xi}_1 &= \frac{\partial h_k}{\partial x_1}\left(L\varepsilon_2 + Lz_2 + \frac{f_{1,k}}{L}\right) - \frac{\dot{L}}{L}\xi_1, \\
\dot{z}_1 &= Lz_2 - La_1 z_1 - \frac{\dot{L}}{L}z_1.
\end{aligned} \tag{18}
$$

Choose the Lyapunov function $V_1(\varepsilon, z_1, \xi_1) = V_0(\varepsilon) + \frac{z_1^2}{2L} + \frac{\xi_1^2}{2}$, whose derivative is

$$
\begin{aligned}
\dot{V}_1 =& \dot{V}_0 + \xi_1\dot{\xi}_1 + \frac{z_1}{L}\dot{z}_1 - \frac{\dot{L}}{2L^2}z_1^2 \\
\leq& -\left(\frac{L}{2} - \Theta_k\right)\|\varepsilon\|^2 + \left(\frac{1}{2} + \frac{2L\|Pa\|^2}{\underline{c}_k^2}\right)\xi_1^2 + \frac{1}{4}\sum_{i=2}^{n} z_i^2 + \frac{\partial h_k}{\partial x_1}\left(L\varepsilon_2 + Lz_2 + \frac{f_{1,k}}{L}\right)\xi_1 \\
&- \frac{\dot{L}}{L}\xi_1^2 + z_1\left(z_2 - a_1 z_1 - \frac{\dot{L}}{L^2}z_1\right) - \frac{\dot{L}}{2L^2}z_1^2.
\end{aligned} \tag{19}
$$

By completion of square and (6)–(7), one arrives at

$$
\begin{aligned}
&\frac{\partial h_k}{\partial x_1}L\xi_1\varepsilon_2 \leq \bar{c}_k L|\xi_1\varepsilon_2| \leq \frac{L}{4}\varepsilon_2^2 + \bar{c}_k^2 L\xi_1^2 \leq \frac{L}{4}\|\varepsilon\|^2 + \bar{c}_k^2 L\xi_1^2, \\
&\frac{\partial h_k}{\partial x_1}\xi_1\frac{f_{1,k}}{L} \leq \frac{\bar{c}_k\theta_k}{\underline{c}_k}\xi_1^2, \\
&z_1 z_2 \leq \frac{a_1}{2}z_1^2 + \frac{1}{2a_1}z_2^2.
\end{aligned} \tag{20}
$$

By construction, we have $-\frac{3\dot{L}}{2L^2}z_1^2 \le 0$. Then substituting (20) into (19) yields

$$\begin{aligned}\dot{V}_1 \le &-\left(\frac{L}{4}-\Theta_k\right)\|\varepsilon\|^2+\left(\frac{1}{2}+\frac{\bar{c}_k\theta_k}{\underline{c}_k}+\left(\frac{2\|Pa\|^2}{\underline{c}_k^2}+\bar{c}_k^2\right)L\right)\xi_1^2-\frac{\dot{L}}{L}\xi_1^2+\frac{1}{4}\sum_{i=2}^{n}z_i^2\\&-\frac{a_1}{2}z_1^2+\frac{1}{2a_1}z_2^2+\frac{\partial h_k}{\partial x_1}L\xi_1\alpha_1+\frac{\partial h_k}{\partial x_1}L\xi_1(z_2-\alpha_1).\end{aligned}\tag{21}$$

Choose the virtual controller of the form

$$\alpha_1=-b_1\xi_1,\ b_1\ge\max_{k\in M}\left\{\frac{1}{\underline{c}_k}\left(1+\frac{2\|Pa\|^2}{\underline{c}_k^2}+\bar{c}_k^2\right)\right\},\tag{22}$$

which leads to

$$\begin{aligned}\dot{V}_1 \le &-\left(\frac{L}{4}-\Theta_k\right)\|\varepsilon\|^2-\left(L-\frac{1}{2}-\frac{\bar{c}_k\theta_k}{\underline{c}_k}\right)\xi_1^2-\frac{\dot{L}}{L}\xi_1^2+\frac{1}{4}\sum_{i=2}^{n}z_i^2\\&-\frac{a_1}{2}z_1^2+\frac{1}{2a_1}z_2^2+\frac{\partial h_k}{\partial x_1}L\xi_1(z_2-\alpha_1).\end{aligned}\tag{23}$$

By the fact $\xi_2=z_2-\alpha_1=z_2+b_1\xi_1$, one has $z_2^2\le 2b_1^2\xi_1^2+2\xi_2^2$. Hence

$$\begin{aligned}\dot{V}_1 \le &-\left(\frac{L}{4}-\Theta_k\right)\|\varepsilon\|^2-(L-\omega_{1,k})\,\xi_1^2-\frac{\dot{L}}{L}\xi_1^2+\left(\frac{1}{2}+\frac{1}{a_1}\right)\xi_2^2\\&+\frac{1}{4}\sum_{i=3}^{n}z_i^2-\frac{a_1}{2}z_1^2+\frac{\partial h_k}{\partial x_1}L\xi_1\xi_2\end{aligned}\tag{24}$$

where $\omega_{1,k}=1/2+\bar{c}_k\theta_k/\underline{c}_k+(1/2+1/a_1)b_1^2$ is an unknown positive constant.

Step 2: Choose $V_2(\varepsilon,z_1,\xi_1,\xi_2)=\sigma_1V_1(\varepsilon,z_1,\xi_1)+\frac{1}{2}\xi_2^2$, where $\sigma_1\ge 1$ is a design constant to be determined later. From the definition of ξ_2, it follows that

$$\dot{\xi}_2=b_1\frac{\partial h_k}{\partial x_1}\left(L\varepsilon_2+Lz_2+\frac{f_{1,k}}{L}\right)+Lz_3+a_2(L\varepsilon_1-x_1)-\frac{2\dot{L}}{L}\xi_2+b_1\frac{\dot{L}}{L}\xi_1.\tag{25}$$

Therefore, a direct computation yields

$$\begin{aligned}\dot{V}_2 \le &\sigma_1\Big(-(\frac{L}{4}-\Theta_k)\|\varepsilon\|^2-(L-\omega_{1,k})\xi_1^2-\frac{\dot{L}}{L}\xi_1^2+(\frac{1}{2}+\frac{1}{a_1})\xi_2^2+\frac{1}{4}\sum_{i=3}^{n}z_i^2-\frac{a_1}{2}z_1^2\\&+\frac{\partial h_k}{\partial x_1}L\xi_1\xi_2\Big)+\xi_2\Big(b_1\frac{\partial h_k}{\partial x_1}\left(L\varepsilon_2+Lz_2+\frac{f_{1,k}}{L}\right)+Lz_3+a_2(L\varepsilon_2-x_1)\\&-\frac{2\dot{L}}{L}\xi_2+b_1\frac{\dot{L}}{L}\xi_1\Big).\end{aligned}\tag{26}$$

In the following, we estimate some terms on the right hand of (26). The use of (6)–(7) and the completion of square leads to

$$\begin{aligned}
\sigma_1 \frac{\partial h_k}{\partial x_1} L\xi_1\xi_2 &\leq \sigma_1 \frac{L}{6}\xi_1^2 + \sigma_1 \frac{3\bar{c}_k^2 L}{2}\xi_2^2, \\
a_2\xi_2(L\varepsilon_1 - x_1) &\leq \sigma_1 \frac{L}{6}\xi_1^2 + \sigma_1 \frac{L}{16}\|\varepsilon\|^2 + \frac{1}{\sigma_1}\left(\frac{3a_2^2 L}{2\underline{c}_k^2} + 4a_2^2 L\right)\xi_2^2, \\
b_1 \frac{\partial h_k}{\partial x_1} L\xi_2\varepsilon_2 &\leq \sigma_1 \frac{L}{16}\|\varepsilon\|^2 + \frac{4b_1^2\bar{c}_k^2 L\xi_2^2}{\sigma_1}, \\
b_1 \frac{\partial h_k}{\partial x_1} L\xi_2 z_2 &\leq b_1\bar{c}_k L|\xi_2||\xi_2 - b_1\xi_1| \leq \sigma_1 \frac{L}{6}\xi_1^2 + \left(b_1\bar{c}_k L + \frac{3b_1^4\bar{c}_k^2}{2\sigma_1}L\right)\xi_2^2, \\
b_1 \frac{\partial h_k}{\partial x_1}\frac{f_{1,k}}{L}\xi_2 &\leq \frac{b_1^2\theta_k^2\bar{c}_k^2}{2\sigma_1\underline{c}_k^2}\xi_1^2 + \frac{\sigma_1}{2}\xi_2^2, \\
b_1 \frac{\dot{L}}{L}\xi_1\xi_2 &\leq \frac{b_1^2}{4}\frac{\dot{L}}{L}\xi_1^2 + \frac{\dot{L}}{L}\xi_2^2,
\end{aligned} \tag{27}$$

which together with $\sigma_1 \geq 1$ and (26) yields

$$\begin{aligned}
\dot{V}_2 \leq &- \sigma_1\left(\frac{L}{8} - \Theta_k\right)\|\varepsilon\|^2 - \sigma_1\left(\frac{L}{2} - \omega_{1,k} - \frac{b_1^2\theta_k^2\bar{c}_k^2}{2\underline{c}_k^2}\right)\xi_1^2 - \left(\sigma_1 - \frac{b_1^2}{4}\right)\frac{\dot{L}}{L}\xi_1^2 - \frac{\dot{L}}{L}\xi_2^2 \\
&+ \frac{\sigma_1}{4}\sum_{i=3}^{n} z_i^2 - \frac{\sigma_1 a_1}{2}z_1^2 + \Big(\sigma_1 + \frac{\sigma_1}{a_1} + \Big(\frac{3\sigma_1\bar{c}_k^2}{2} + \frac{3a_2^2}{2\underline{c}_k^2} + 4a_2^2 + 4b_1^2\bar{c}_k^2 + b_1\bar{c}_k \\
&+ \frac{3b_1^4\bar{c}_k^2}{2}\Big)L\Big)\xi_2^2 + L\xi_2\alpha_2 + L\xi_2(z_3 - \alpha_2).
\end{aligned} \tag{28}$$

By designing the virtual controller

$$\alpha_2 = -b_2\xi_2,\ b_2 \geq \max_{k\in M}\left\{1 + \frac{3\sigma_1\bar{c}_k^2}{2} + \frac{3a_2^2}{2\underline{c}_k^2} + 4a_2^2 + 4b_1^2\bar{c}_k^2 + b_1\bar{c}_k + \frac{3b_1^4\bar{c}_k^2}{2}\right\},$$

(28) becomes

$$\begin{aligned}
\dot{V}_2 \leq &- \sigma_1\left(\frac{L}{8} - \Theta_k\right)\|\varepsilon\|^2 - \sigma_1\left(\frac{L}{2} - \omega_{2,k}\right)\xi_1^2 - \left(L - \sigma_1 - \frac{\sigma_1}{a_1}\right)\xi_2^2 - \left(\sigma_1 - \frac{b_1^2}{4}\right)\frac{\dot{L}}{L}\xi_1^2 \\
&- \frac{\dot{L}}{L}\xi_2^2 + \frac{\sigma_1}{4}\sum_{i=3}^{n} z_i^2 - \frac{\sigma_1 a_1}{2}z_1^2 + L\xi_2(z_3 - \alpha_2)
\end{aligned} \tag{29}$$

where $\omega_{2,k} = \omega_{1,k} + b_1^2\theta_k^2\bar{c}_k^2/(2\underline{c}_k^2)$ is an unknown positive constant. By the fact $\xi_3 =$

$z_3 - \alpha_2 = z_3 + b_2\xi_2$, one has $z_3^2 \le 2b_2^2\xi_2^2 + 2\xi_3^2$. Hence

$$\begin{aligned}\dot{V}_2 \le &-\sigma_1\left(\frac{L}{8} - \Theta_k\right)\|\varepsilon\|^2 - \sigma_1\left(\frac{L}{2} - \omega_{2,k}\right)\xi_1^2 - (L-\omega_2)\xi_2^2 - \left(\sigma_1 - \frac{b_1^2}{4}\right)\frac{\dot{L}}{L}\xi_1^2\\ &-\frac{\dot{L}}{L}\xi_2^2 + \frac{\sigma_1}{2}\xi_3^2 + \frac{\sigma_1}{4}\sum_{i=4}^{n} z_i^2 - \frac{\sigma_1 a_1}{2} z_1^2 + L\xi_2\xi_3 \end{aligned} \tag{30}$$

where $\omega_2 = \sigma_1 + \sigma_1/a_1 + \sigma_1 b_2^2/2$ is a positive constant.

Inductive step: Suppose at step $i-1$, there are a set of common virtual controllers $\alpha_1, \ldots, \alpha_{i-1}$ defined by

$$\begin{aligned}\alpha_1 &= -b_1\xi_1,\ \xi_2 = z_2 - \alpha_1,\\ \alpha_2 &= -b_2\xi_2,\ \xi_3 = z_3 - \alpha_2,\\ &\vdots\\ \alpha_{i-1} &= -b_{i-1}\xi_{i-1},\ \xi_i = z_i - \alpha_{i-1}\end{aligned} \tag{31}$$

with constants $b_1 > 0, \ldots, b_{i-1} > 0$, and a Lyapunov function $V_{i-1} = \sigma_{i-2}V_{i-2}(\varepsilon, z_1, \xi_1, \ldots, \xi_{i-2}) + \frac{1}{2}\xi_{i-1}^2$ with constants $\sigma_l \ge 1$, $l = 1, \ldots, i-2$ to be determined later, such that

$$\begin{aligned}\dot{V}_{i-1} \le &-\prod_{j=1}^{i-2}\sigma_j\Big(\frac{L}{2^i} - \Theta_k\Big)\|\varepsilon\|^2 - \prod_{j=1}^{i-2}\sigma_j\Big(\frac{L}{2^{i-2}} - \omega_{i-1,k}\Big)\xi_1^2 - \sum_{j=2}^{i-1}\prod_{l=j}^{i-2}\sigma_l\Big(\frac{L}{2^{i-1-j}} - \omega_j\Big)\\ &\times\xi_j^2 - \prod_{j=2}^{i-2}\sigma_j\Big(\sigma_1 - \frac{b_1^2}{4}\Big(1 + \sum_{p=2}^{i-2}\prod_{l=2}^{p} b_l^2\Big)\Big)\frac{\dot{L}}{L}\xi_1^2 - \sum_{j=2}^{i-2}\prod_{l=j+1}^{i-2}\sigma_l\Big(\sigma_j - \frac{1}{4}\sum_{p=j}^{i-2}\prod_{l=j}^{p} b_l^2\Big)\\ &\times\frac{\dot{L}}{L}\xi_j^2 - \frac{\dot{L}}{L}\xi_{i-1}^2 - \frac{\prod_{l=1}^{i-2}\sigma_l a_1}{2}z_1^2 + \frac{\prod_{l=1}^{i-2}\sigma_l}{2}\xi_i^2 + \frac{\prod_{l=1}^{i-2}\sigma_l}{4}\sum_{j=i+1}^{n} z_j^2 + L\xi_{i-1}\xi_i\end{aligned} \tag{32}$$

where $\omega_{i-1,k}$ is an unknown positive constant, and ω_j, $j = 1, \ldots, i-1$, are known positive constants. Next, we will show that (32) still holds at step i. Choose the common Lyapunov function $V_i = \sigma_{i-1}V_{i-1}(\varepsilon, z_1, \xi_1, \ldots, \xi_{i-1}) + \frac{1}{2}\xi_i^2$ with constant $\sigma_{i-1} \ge 1$ to be determined later. From the definition of ξ_i, it is not hard to deduce that

$$\begin{aligned}\dot{\xi}_i =& Lz_{i+1} - \Big(a_i + \sum_{j=2}^{i-1}\prod_{l=j}^{i-1} b_l a_j\Big)(L\varepsilon_1 - x_1) + L\sum_{j=3}^{i}\prod_{l=j-1}^{i-1} b_l z_j + \prod_{l=1}^{i-1} b_l\frac{\partial h_k}{\partial x_1}\Big(L\varepsilon_2 + Lz_2\\ &+\frac{f_{1,k}}{L}\Big) - i\frac{\dot{L}}{L}\xi_i + \sum_{j=1}^{i-1}\prod_{l=j}^{i-1} b_l\frac{\dot{L}}{L}\xi_j.\end{aligned} \tag{33}$$

Hence, it follows that

$$\begin{aligned}
\dot{V}_i \leq &-\prod_{j=1}^{i-1}\sigma_j\Big(\frac{L}{2^i}-\Theta_k\Big)\|\varepsilon\|^2-\prod_{j=1}^{i-1}\sigma_j\Big(\frac{L}{2^{i-2}}-\omega_{i-1,k}\Big)\xi_1^2-\sum_{j=2}^{i-1}\prod_{l=j}^{i-1}\sigma_l\Big(\frac{L}{2^{i-1-j}}-\omega_j\Big)\xi_j^2\\
&-\prod_{j=2}^{i-1}\sigma_j\left(\sigma_1-\frac{b_1^2}{4}\Big(1+\sum_{p=2}^{i-2}\prod_{l=2}^{p}b_l^2\Big)\right)\frac{\dot{L}}{L}\xi_1^2-\sum_{j=2}^{i-2}\prod_{l=j+1}^{i-1}\sigma_l\left(\sigma_j-\frac{1}{4}\sum_{p=j}^{i-2}\prod_{l=j}^{p}b_l^2\right)\frac{\dot{L}}{L}\xi_j^2\\
&-\sigma_{i-1}\frac{\dot{L}}{L}\xi_{i-1}^2-\frac{\prod_{l=1}^{i-1}\sigma_l a_1}{2}z_1^2+\frac{\prod_{l=1}^{i-1}\sigma_l}{2}\xi_i^2+\frac{\prod_{l=1}^{i-1}\sigma_l}{4}\sum_{j=i+1}^{n}z_j^2+\sigma_{i-1}L\xi_{i-1}\xi_i\\
&+\xi_i\Big(Lz_{i+1}-(a_i+\sum_{j=2}^{i-1}\prod_{l=j}^{i-1}b_la_j)\hat{x}_1+L\sum_{j=3}^{i}\prod_{l=j-1}^{i-1}b_lz_j+\prod_{l=1}^{i-1}b_l\frac{\partial h_k}{\partial x_1}\Big(L\varepsilon_2+Lz_2\\
&+\frac{f_{1,k}}{L}\Big)-i\frac{\dot{L}}{L}\xi_i+\sum_{j=1}^{i-1}\prod_{l=j}^{i-1}b_l\frac{\dot{L}}{L}\xi_j\Big).
\end{aligned} \tag{34}$$

Similar to Step 2, based on the fact $\sigma_l \geq 1$, $l=1,\ldots,i-1$ and (6) − (10), using the completion of square, we obtain

$$\begin{aligned}
&\sigma_{i-1}L\xi_{i-1}\xi_i \leq \frac{\sigma_{i-1}}{6}L\xi_{i-1}^2+\frac{3\sigma_{i-1}}{2}L\xi_i^2,\\
&-\left(a_i+\sum_{j=2}^{i-1}\prod_{l=j}^{i-1}b_la_j\right)\xi_i(L\varepsilon_1-x_1)\leq\left(a_i+\sum_{j=2}^{i-1}\prod_{l=j}^{i-1}b_la_j\right)|\xi_i|\left(L|\varepsilon_1|+\frac{L|\xi_1|}{\underline{c}_k}\right)\\
&\leq\frac{\prod_{l=1}^{i-1}\sigma_lL}{2^{i+2}}\|\varepsilon\|^2+\frac{\prod_{l=1}^{i-1}\sigma_lL}{2^i}\xi_1^2+\left(2^i+\frac{2^{i-2}}{\underline{c}_k^2}\right)\left(a_i+\sum_{j=2}^{i-1}\prod_{l=j}^{i-1}b_la_j\right)^2L\xi_i^2,\\
&\sum_{j=1}^{i-1}\prod_{l=j}^{i-1}b_l\frac{\dot{L}}{L}\xi_j\xi_i\leq(i-1)\frac{\dot{L}}{L}\xi_i^2+\frac{b_1^2\prod_{l=2}^{i-1}b_l^2}{4}\frac{\dot{L}}{L}\xi_1^2+\sum_{j=2}^{i-1}\frac{\prod_{l=j}^{i-1}b_l^2}{4}\frac{\dot{L}}{L}\xi_j^2,\\
&\prod_{l=1}^{i-1}b_l\frac{\partial h_k}{\partial x_1}L\varepsilon_2\xi_i\leq\frac{\prod_{l=1}^{i-1}\sigma_l}{2^{i+2}}L\|\varepsilon\|^2+2^i\prod_{l=1}^{i-1}b_l^2\bar{c}_k^2L\xi_i^2,\\
&\prod_{l=1}^{i-1}b_l\frac{\partial h_k}{\partial x_1}Lz_2\xi_i\leq\frac{\prod_{l=1}^{i-1}\sigma_l}{2^i}L\xi_1^2+\frac{\prod_{l=2}^{i-1}\sigma_l}{2^{i-1}}L\xi_2^2+\big(2^{i-2}b_1^4\bar{c}_k^2+2^{i-3}b_1^2\bar{c}_k^2\big)\prod_{l=2}^{i-1}b_l^2L\xi_i^2,\\
&\prod_{l=1}^{i-1}b_l\frac{\partial h_k}{\partial x_1}\frac{f_{1,k}}{L}\xi_i\leq\prod_{l=1}^{i-1}b_l\bar{c}_k\theta_k\frac{1}{\underline{c}_k}|\xi_1||\xi_i|\leq\frac{\bar{c}_k^2\theta_k^2\prod_{l=1}^{i-1}b_l^2}{2\underline{c}_k^2}\xi_1^2+\frac{\prod_{l=1}^{i-1}\sigma_l}{2}\xi_i^2,\\
&Lb_{i-1}z_i\xi_i\leq\left(b_{i-1}+\frac{3}{2}b_{i-1}^4\right)L\xi_i^2+\frac{\sigma_{i-1}}{6}L\xi_{i-1}^2,
\end{aligned}$$

$$L\sum_{j=3}^{i-1}\prod_{l=j-1}^{i-1} b_l z_j \xi_i \leq \frac{1}{6}\sigma_{i-1}L\xi_{i-1}^2 + \frac{\prod_{l=2}^{i-1}\sigma_l}{2^{i-1}}L\xi_2^2 + \sum_{j=3}^{i-2}\frac{\prod_{l=j}^{i-1}\sigma_l}{2^{i-j}}L\xi_j^2 + 3\sum_{j=3}^{i-1}2^{i-j-1}(1+b_{j-1}^2)\prod_{l=j-1}^{i-1}b_l^2 L\xi_i^2. \tag{35}$$

Substituting (35) into (34) yields

$$\begin{aligned}\dot{V}_i \leq & -\prod_{j=1}^{i-1}\sigma_j\Big(\frac{L}{2^{i+1}}-\Theta_k\Big)\|\varepsilon\|^2 - \prod_{j=1}^{i-1}\sigma_j\Big(\frac{L}{2^{i-1}}-\omega_{i,k}\Big)\xi_1^2 - \sum_{j=2}^{i-1}\prod_{l=j}^{i-1}\sigma_l\Big(\frac{L}{2^{i-j}}-\omega_j\Big)\xi_j^2\\ & -\prod_{j=2}^{i-1}\sigma_j\Big(\sigma_1-\frac{b_1^2}{4}\Big(1+\sum_{p=2}^{i-1}\prod_{l=2}^{p}b_l^2\Big)\Big)\frac{\dot{L}}{L}\xi_1^2 - \sum_{j=2}^{i-1}\prod_{l=j+1}^{i-1}\sigma_l\Big(\sigma_j-\frac{1}{4}\sum_{p=j}^{i-1}\prod_{l=j}^{p}b_l^2\Big)\frac{\dot{L}}{L}\xi_j^2\\ & -\frac{\dot{L}}{L}\xi_i^2 - \frac{\prod_{l=1}^{i-1}\sigma_l a_1}{2}z_1^2 + \frac{\prod_{l=1}^{i-1}\sigma_l}{4}\sum_{j=i+1}^{n}z_j^2 + \xi_i^2\Big(\prod_{l=1}^{i-1}\sigma_l + L\Big(\frac{3\sigma_{i-1}}{2}+\big(2^i+\frac{2^{i-2}}{\underline{c}_k^2}\big)\\ & \times\big(a_i+\sum_{j=2}^{i-1}\prod_{l=j}^{i-1}b_l a_j\big)^2 + b_{i-1} + \frac{3}{2}b_{i-1}^4 + \big(2^{i-2}b_1^4\bar{c}_k^2 + 2^{i-3}b_1^2\bar{c}_k^2 + 2^i b_1^2\bar{c}_k^2\big)\prod_{l=2}^{i-1}b_l^2\\ & + 3\sum_{j=3}^{i-1}2^{i-j-1}(1+b_{j-1}^2)\prod_{l=j-1}^{i-1}b_l^2\Big)\Big) + L\xi_i\alpha_i + L\xi_i\xi_{i+1}\end{aligned} \tag{36}$$

where $\omega_{i,k} = \omega_{i-1,k} + \bar{c}_k^2\theta_k^2\prod_{l=1}^{i-1}b_l^2/(2\underline{c}_k^2)$ is an unknown constant.

Choosing the virtual controller $\alpha_i = -b_i\xi_i$ with

$$\begin{aligned}b_i \geq \max_{k\in M}\Big\{&1+\frac{3\sigma_{i-1}}{2}+\big(2^i+\frac{2^{i-2}}{\underline{c}_k^2}\big)\times\big(a_i+\sum_{j=2}^{i-1}\prod_{l=j}^{i-1}b_l a_j\big)^2 + b_{i-1}+\frac{3}{2}b_{i-1}^4\\ &+\big(2^{i-2}b_1^4\bar{c}_k^2+2^{i-3}b_1^2\bar{c}_k^2+2^i b_1^2\bar{c}_k^2\big)\prod_{l=2}^{i-1}b_l^2 + 3\sum_{j=3}^{i-1}2^{i-j-1}(1+b_{j-1}^2)\prod_{l=j-1}^{i-1}b_l^2\Big)\Big\}\end{aligned}$$

and $z_{i+1} \leq 2\xi_{i+1}^2 + 2b_i^2\xi_i^2$, one arrives at

$$\begin{aligned}\dot{V}_i \leq & -\prod_{j=1}^{i-1}\sigma_j\left(\frac{L}{2^{i+1}}-\Theta_k\right)\|\varepsilon\|^2 - \prod_{j=1}^{i-1}\sigma_j\left(\frac{L}{2^{i-1}}-\omega_{i,k}\right)\xi_1^2 - \sum_{j=2}^{i}\prod_{l=j}^{i-1}\sigma_l\left(\frac{L}{2^{i-j}}-\omega_j\right)\xi_j^2\\ & -\prod_{j=2}^{i-1}\sigma_j\left(\sigma_1-\frac{b_1^2}{4}\Big(1+\sum_{p=2}^{i-1}\prod_{l=2}^{p}b_l^2\Big)\right)\frac{\dot{L}}{L}\xi_1^2 - \sum_{j=2}^{i-1}\prod_{l=j+1}^{i-1}\sigma_l\left(\sigma_j-\frac{1}{4}\sum_{p=j}^{i-1}\prod_{l=j}^{p}b_l^2\right)\frac{\dot{L}}{L}\xi_j^2\\ & -\frac{\dot{L}}{L}\xi_i^2 - \frac{\prod_{l=1}^{i-1}\sigma_l a_1}{2}z_1^2 + \frac{\prod_{l=1}^{i-1}\sigma_l}{2}\xi_{i+1}^2 + \frac{\prod_{l=1}^{i-1}\sigma_l}{4}\sum_{j=i+2}^{n}z_j^2 + L\xi_i\xi_{i+1}\end{aligned} \tag{37}$$

where $\omega_i = (1 + b_i^2/2)\prod_{l=1}^{i-1}\sigma_l$.

This completes the inductive proof.

Step n: Employing the inductive argument, there is a positive definite and proper Lyapunov function $V_n = \sigma_{n-1}V_{n-1}(\varepsilon, z_1, \xi_1, \dots, \xi_{n-1}) + \frac{1}{2}\xi_n^2$ with constant $\sigma_{n-1} \geq 1$ to be determined later, such that

$$\begin{aligned}
\dot{V}_n \leq & -\prod_{j=1}^{n-1}\sigma_j\Big(\frac{L}{2^{n+1}} - \Theta_k\Big)\|\varepsilon\|^2 - \prod_{j=1}^{n-1}\sigma_j\Big(\frac{L}{2^{n-1}} - \omega_{n,k}\Big)\xi_1^2 - \sum_{j=2}^{n-1}\prod_{l=j}^{n-1}\sigma_l\Big(\frac{L}{2^{n-j}} - \omega_j\Big)\xi_j^2 \\
& - \prod_{j=2}^{n-1}\sigma_j\Big(\sigma_1 - \frac{b_1^2}{4}\Big(1 + \sum_{p=2}^{n-1}\prod_{l=2}^{p}b_l^2\Big)\Big)\frac{\dot{L}}{L}\xi_1^2 - \sum_{j=2}^{n-1}\prod_{l=j+1}^{n-1}\sigma_l\Big(\sigma_j - \frac{1}{4}\sum_{p=j}^{n-1}\prod_{l=j}^{p}b_l^2\Big)\frac{\dot{L}}{L}\xi_j^2 \\
& - \frac{\dot{L}}{L}\xi_n^2 - \frac{\prod_{l=1}^{i-1}\sigma_l a_1}{2}z_1^2 + \xi_n^2\Big(\prod_{l=1}^{n-1}\sigma_l + L\Big(\frac{3\sigma_{n-1}}{2} + \big(2^n + \frac{2^{n-2}}{\underline{c}_k^2}\big) \times \big(a_n \\
& + \sum_{j=2}^{n-1}\prod_{l=j}^{i-1}b_l a_j\big)^2 + b_{n-1} + \frac{3}{2}b_{n-1}^4 + \big(2^{n-2}b_1^4\bar{c}_k^2 + 2^{n-3}b_1^2\bar{c}_k^2 + 2^n b_1^2\bar{c}_k^2\big)\prod_{l=2}^{n-1}b_l^2 \\
& + 3\sum_{j=3}^{n-1}2^{n-j-1}(1 + b_{j-1}^2)\prod_{l=j-1}^{n-1}b_l^2\Big)\Big) + \frac{g^n}{L^n}\xi_n u
\end{aligned} \tag{38}$$

where $\omega_{n,k} = \omega_{n-1,k} + \bar{c}_k^2\theta_k^2\prod_{l=1}^{n-1}b_l^2/(2\underline{c}_k^2)$ is an unknown constant.

Designing the controller

$$u = -\underline{g}^{-n}L^{n+1}b_n\xi_n = -\underline{g}^{-n}\sum_{j=2}^{n}L^{n+1-j}\prod_{l=j}^{n}b_l\hat{x}_j - \underline{g}^{-n}L^n\prod_{l=1}^{n}b_l y \tag{39}$$

where

$$\begin{aligned}
b_n \geq \max_{k\in M}\Big\{ & 1 + \frac{3\sigma_{n-1}}{2} + \big(2^n + \frac{2^{n-2}}{\underline{c}_k^2}\big) \times \big(a_n + \sum_{j=2}^{n-1}\prod_{l=j}^{n-1}b_l a_j\big)^2 + b_{n-1} + \frac{3}{2}b_{n-1}^4 \\
& + \big(2^{n-2}b_1^4\bar{c}_k^{-2} + 2^{n-3}b_1^2\bar{c}_k^{-2} + 2^n b_1^2\bar{c}_k^{-2}\big)\prod_{l=2}^{n-1}b_l^2 + 3\sum_{j=3}^{n-1}2^{n-j-1}(1 + b_{j-1}^2)\prod_{l=j-1}^{n-1}b_l^2\Big\}
\end{aligned}$$

leads to

$$\begin{aligned}
\dot{V}_n \leq & -\prod_{j=1}^{n-1}\sigma_j\Big(\frac{L}{2^{n+1}} - \Theta_k\Big)\|\varepsilon\|^2 - \prod_{j=1}^{n-1}\sigma_j\Big(\frac{L}{2^{n-1}} - \omega_{n,k}\Big)\xi_1^2 - \sum_{j=2}^{n}\prod_{l=j}^{n-1}\sigma_l\Big(\frac{L}{2^{n-j}} - \omega_j\Big)\xi_j^2 \\
& - \prod_{j=2}^{n-1}\sigma_j\Big(\sigma_1 - \frac{b_1^2}{4}\Big(1 + \sum_{p=2}^{n-1}\prod_{l=2}^{p}b_l^2\Big)\Big)\frac{\dot{L}}{L}\xi_1^2 - \sum_{j=2}^{n-1}\prod_{l=j+1}^{n-1}\sigma_l\Big(\sigma_j - \frac{1}{4}\sum_{p=j}^{n-1}\prod_{l=j}^{p}b_l^2\Big)\frac{\dot{L}}{L}\xi_j^2
\end{aligned}$$

$$-\frac{\dot{L}}{L}\xi_n^2 - \frac{\prod_{l=1}^{n-1}\sigma_l a_1}{2}z_1^2 \tag{40}$$

where $\omega_n = \prod_{l=1}^{n-1}\sigma_l$.

Finally, we select σ_j, $j = 1, \dots, n-1$ satisfying

$$\begin{aligned}\sigma_1 &\geq \max\left\{1, \frac{b_1^2}{4}\Big(1 + \sum_{p=2}^{n-1}\prod_{l=2}^{p} b_l^2\Big)\right\},\\ \sigma_j &\geq \max\left\{1, \frac{1}{4}\sum_{p=j}^{n-1}\prod_{l=j}^{p} b_l^2\right\},\ j = 2, \dots, n-1,\end{aligned} \tag{41}$$

and take

$$\begin{aligned}\bar{\Theta} &= \max_{k\in M}\Big\{\prod_{l=1}^{n-1}\sigma_l\Theta_k,\ \prod_{l=1}^{n-1}\sigma_l\omega_{n,k},\ \prod_{l=j}^{n-1}\sigma_l\omega_j, j = 2, \dots, n\Big\},\\ \rho &= \min\left\{\frac{\prod_{l=1}^{n-1}\sigma_l}{2^{n+1}}, \frac{\prod_{l=1}^{n-1}\sigma_l a_1}{2}, \frac{\prod_{l=j}^{n-1}\sigma_l}{2^{n-j}}, j = 2, \dots, n\right\},\end{aligned}$$

such that (40) becomes

$$\dot{V}_n \leq -(\rho L - \bar{\Theta})(\|\varepsilon\|^2 + \|\xi\|^2) - \rho z_1^2 \tag{42}$$

where $\xi = (\xi_1, \dots, \xi_n)^T$.

Remark 3.1. Dynamic gain L increases a freedom degree of switched uncertain nonlinear system. Merged with common Lyapunov idea and observer construction technique, it can effectively dominate all the possible uncertainties. In addition, it will play a key role in the proof that all signals of closed-loop switched system are bounded. However, the introduction of L gives rise to complicated control design and stability analysis.

3.2. Stability analysis

In this subsection, Let $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)^T$. Our objective is that starting from any initial condition $(\eta(0), \hat{x}(0)) \in \mathbb{R}^n \times \mathbb{R}^n$ and $L(0) = 1$, there is an adaptive output-feedback controller such that

(i) *the solution* $(\eta(t), \hat{x}(t), L(t))$ *of closed-loop system well-defined on* $[0, \infty)$ *is unique and globally bounded;*

(ii) $\lim_{t\to+\infty}(\eta(t), \hat{x}(t)) = 0$ *and* $\lim_{t\to+\infty} L(t) = \bar{L} \in \mathbb{R}_+$.

Now, we will show that the solution $(\eta(t), \hat{x}(t), L(t))$ exists and is unique on the maximal interval $[0, t_f)$ for $0 < t_f \leq +\infty$. This can be done by a contradiction argument. Firstly, we claim $L(t)$ is bounded on $[0, t_f)$. Suppose $\lim_{t\to t_f} L(t) = +\infty$. Since $\dot{L}(t) \geq$

$0,\ \forall t \geq 0$, $L(t)$ is a monotone nondecreasing function. Hence, there is a finite time $T \in [0, t_f)$ such that $L(t) \geq (\rho + \bar{\Theta})/\rho$, $\forall t \in [T, t_f)$. This together with (42) yields

$$\dot{V} \leq -\rho(\|\varepsilon\|^2 + \|\xi\|^2) - \rho z_1^2,\ \forall t \in [T, t_f). \tag{43}$$

As a result,

$$+\infty = L(t_f) - L(T) = \int_T^{t_f} \dot{L}(t)dt \leq -\int_T^{t_f} \frac{\dot{V}(t)}{\rho}\,\mathrm{d}t \leq \frac{V(T)}{\rho} < +\infty,$$

which is a contradiction. This implies that $L(t)$ is bounded on $[0, t_f)$ and $\lim_{t\to t_f} L(t) = \bar{L}$.

In the following, we will show the boundedness of z on $[0, t_f)$. Consider the Lyapunov function $V(z) = z^T P z$ for the z-dynamic system of (11). A simple calculation leads to

$$\begin{aligned}\dot{V}(z) &= Lz^T(PA + A^T P)z + \frac{2}{L^n} z^T PBu - \frac{\dot{L}}{L} z^T(PD + DP)z \\ &\leq -\frac{L}{2}\|z\|^2 + 2Lb_n^2\|P\|^2\xi_n^2 \leq -\frac{1}{2}\|z\|^2 + 2b_n^2\|P\|^2 L\dot{L}.\end{aligned} \tag{44}$$

Thus, for $\forall t \in [0, t_f)$, one gives

$$z^T(t)Pz(t) \leq z^T(0)Pz(0) + b_n^2\|P\|^2\bar{L}^2,$$

from which, it follows that for $\forall t \in [0, t_f)$

$$\begin{aligned}\|z(t)\|^2 &\leq \frac{1}{\lambda_{\min}(P)}\big(z^T(0)Pz(0) + b_n^2\|P\|^2\bar{L}^2\big), \\ \int_0^t \|z(s)\|^2\,\mathrm{d}s &\leq 2(z^T(0)Pz(0) + b_n^2\|P\|^2\bar{L}^2).\end{aligned} \tag{45}$$

This implies the boundedness of $z(t)$ and $\int_0^t \|z(s)\|^2\,\mathrm{d}s$ on $[0, t_f)$.

Then, we will claim that ε is bounded on $[0, t_f)$. To this end, we introduce the change of coordinates

$$\bar{\varepsilon}_i = \frac{x_i - \hat{x}_i}{L^{*i}},\ i = 1, \ldots, n$$

where L^* is a positive constant satisfying

$$L^* = \max_{k\in M}\Big\{\bar{L}, \Theta_k + 3\Big\}. \tag{46}$$

As a consequence, the error dynamic system (11) is transformed into

$$\bar{\varepsilon} = L^* A\bar{\varepsilon} + L^* a\bar{\varepsilon}_1 - L\Lambda_1 a\bar{\varepsilon}_1 + \Lambda_2 a x_1 + F_k^* \tag{47}$$

where $\bar{\varepsilon} = (\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_n)$, $\Lambda_1 = \mathrm{diag}\{1, L/L^*, \ldots, (L/L^*)^{n-1}\}$,
$\Lambda_2 = \mathrm{diag}\{L/L^*, \ldots, (L/L^*)^n\}$ and $F_k^* = (f_{1,k}/L^*, \cdots, f_{n,k}/L^{*n})^T$.

Choosing the Lyapunov function of (47) on $[0, t_f)$ of the form $V(\bar{\varepsilon}) = \bar{\varepsilon}^T P \bar{\varepsilon}$, one obtains

$$\dot{V}(\bar{\varepsilon}) = -L^* \|\bar{\varepsilon}\|^2 + 2L^* \bar{\varepsilon}^T P a \bar{\varepsilon}_1 - 2L \bar{\varepsilon}^T P \Lambda_1 a \bar{\varepsilon}_1 + 2\bar{\varepsilon}^T P \Lambda_2 a x_1 + 2\bar{\varepsilon}^T P F_k^*. \tag{48}$$

By completion of square and $L/L^* \le 1$, it follows that

$$\begin{aligned}
&2L^* \bar{\varepsilon}^T P a \bar{\varepsilon}_1 \le L^{*2} \|Pa\|^2 \bar{\varepsilon}_1^2 + \|\bar{\varepsilon}\|^2, \\
&2L \bar{\varepsilon}^T P \Lambda_1 a \bar{\varepsilon}_1 \le L^2 \|P\Lambda_1 a\|^2 \bar{\varepsilon}_1^2 + \|\bar{\varepsilon}\|^2, \\
&2\bar{\varepsilon}^T P \Lambda_2 a x_1 \le \frac{L^2 \|P\Lambda_2 a\|^2}{\underline{c}_k^2} \xi_1^2 + \|\bar{\varepsilon}\|^2, \\
&2\bar{\varepsilon}^T P F_k^* \le 2\theta_k \|P\| \|\bar{\varepsilon}\| \left(n \frac{|\xi_1| L}{\underline{c}_k L^*} + n \sum_{i=2}^{n} \Big(\frac{L}{L^*}\Big)^i |z_i| + n\sqrt{n} \|\bar{\varepsilon}\| \right) \le \frac{1}{2}\xi_1^2 + \frac{1}{4}\sum_{i=2}^{n} z_i^2 \\
&\qquad + \left(\frac{2\theta_k^2 n^2 \|P\|^2 L^2}{\underline{c}_k^2 L^{*2}} + 2\theta_k n\sqrt{n} \|P\| + 4\theta_k^2 n^2 (n-1) \Big(\frac{L}{L^*}\Big)^{2i} \|P\|^2 \right) \|\bar{\varepsilon}\|^2 \\
&\qquad \le \frac{1}{2}\xi_1^2 + \frac{1}{4}\sum_{i=2}^{n} z_i^2 + \Theta_k \|\bar{\varepsilon}\|^2,
\end{aligned} \tag{49}$$

Substituting (49) into (48) on $[0, t_f)$ yields

$$\begin{aligned}
\dot{V}(\bar{\varepsilon}) \le& (L^* - \Theta_k - 3)\|\bar{\varepsilon}\|^2 + \frac{1}{4}\sum_{i=2}^{n} z_i^2 + \Big(\frac{1}{2} + \frac{L^2 \|P\Lambda_2 a\|^2}{\underline{c}_k^2}\Big)\xi_1^2 + \Big(L^{*2}\|Pa\|^2 \\
&+ L^2 \|P\Lambda_1 a\|^2\Big)\bar{\varepsilon}_1^2 \le -\|\bar{\varepsilon}\|^2 + \frac{1}{4}\|z(t)\|^2 + \Big(\frac{1}{2} + \frac{L^2 \|P\Lambda_2 a\|^2}{\underline{c}_k^2}\Big)\xi_1^2 \\
&+ \Big(L^{*2}\|Pa\|^2 + L^2 \|P\Lambda_1 a\|^2\Big)\Big(\frac{2L^2 \xi_1^2}{L^{*2}\underline{c}_k^2} + \frac{2L^2 z_1^2}{L^{*2}}\Big) \\
\le& -\|\bar{\varepsilon}\|^2 + \frac{1}{4}\|z(t)\|^2 + \tilde{\Theta}\xi_1^2 + \tilde{\Theta} z_1^2 \le -\|\bar{\varepsilon}\|^2 + \frac{1}{4}\|z(t)\|^2 + \tilde{\Theta}\dot{L}
\end{aligned} \tag{50}$$

where

$$\tilde{\Theta} = \max_{k \in M} \Big\{ \frac{1}{2} + \frac{\bar{L}^2 \|P\Lambda_2 a\|^2}{\underline{c}_k^2} + 2\bar{L}^2 \frac{\|Pa\|^2}{\underline{c}_k^2} + 2\bar{L}^2 \frac{\|P\Lambda_1 a\|^2}{\underline{c}_k^2},\ 2\bar{L}^2 \|Pa\|^2 + 2\bar{L}^2 \|P\Lambda_1 a\|^2 \Big\}.$$

From (50), it can be concluded that for $\forall t \in [0, t_f)$

$$\bar{\varepsilon}^T(t) P \bar{\varepsilon}(t) \le \bar{\varepsilon}^T(0) P \bar{\varepsilon}(0) - \int_0^t \|\bar{\varepsilon}(s)\|^2 \,\mathrm{d}s + \tilde{\Theta}(L(t) - L(0)) + \frac{1}{4}\int_0^t \|z(s)\|^2 \,\mathrm{d}s,$$

which implies

$$\begin{aligned}
&\|\bar{\varepsilon}(t)\|^2 \le \frac{1}{\lambda_{\min}(P)} \Big(\bar{\varepsilon}^T(0) P \bar{\varepsilon}(0) + \tilde{\Theta}\bar{L} + \frac{1}{4}\int_0^t \|z(s)\|^2 \,\mathrm{d}s \Big), \\
&\int_0^t \|\bar{\varepsilon}(s)\|^2 \,\mathrm{d}s \le \bar{\varepsilon}^T(0) P \bar{\varepsilon}(0) + \tilde{\Theta}\bar{L} + \frac{1}{4}\int_0^t \|z(s)\|^2 \,\mathrm{d}s.
\end{aligned} \tag{51}$$

Since $z(t)$ and $\int_0^t \|z(s)\|^2\,\mathrm{d}s$ are bounded on $[0,t_f)$, it can be seen from (51) that $\bar{\varepsilon}(t)$ and $\int_0^t \|\bar{\varepsilon}(s)\|^2\,\mathrm{d}s$ are bounded on $[0,t_f)$. By the boundedness of $L(t)$ and the definition of $\bar{\varepsilon}_i$, ε_i, $i=1,\ldots,n$, it can be easily concluded that $\varepsilon(t)$ and $\int_0^t \|\varepsilon(s)\|^2\,\mathrm{d}s$ are bounded on $[0,t_f)$.

Up to now, the bondedness of $(z(t),\varepsilon(t),L(t))$ have been proved on maximal interval $[0,t_f)$. With the definition of z_i and ε_i, $i=1,\ldots,n$ in mind, one obtains that $(\eta(t),\hat{x}(t),L(t))$ is bounded on maximal interval $[0,t_f)$.

Moreover, it can be shown that $t_f=+\infty$. This can be also done by a contradiction argument. Suppose $t_f<+\infty$. Then t_f would be a finite-escape time, which means that at least one component of the solution $(x(t),\hat{x}(t),L(t))$ would tend to infinity when $t\to t_f$. However, the continuity of the solution guarantees $(x(t),\hat{x}(t),L(t))$ is bounded at $t=t_f$ owing to the boundedness of $(x(t),\hat{x}(t),L(t))$ on $[0,t_f)$. This is an apparent contradiction. As a consequence, the solution of the closed-loop system is bounded over $[0,+\infty)$.

On the other hand, by the boundedness of $(z(t),\varepsilon(t),L(t))$ on $[0,+\infty)$, it can be deduced that $\dot{z}(t)$ and $\dot{\varepsilon}(t)$ are bounded on $[0,+\infty)$. Noting that $\int_0^{+\infty}\|z(t)\|^2\,\mathrm{d}t<+\infty$ and $\int_0^{+\infty}\|\varepsilon(t)\|^2\,\mathrm{d}t<+\infty$, by Barbalat's lemma, one has $\lim_{t\to+\infty} z(t)=0$ and $\lim_{t\to+\infty}\varepsilon(t)=0$. From the definition of $L(t)$, z_i and ε_i, $i=1,\ldots,n$, it follows that $\lim_{t\to+\infty}\eta(t)=\lim_{t\to+\infty}x(t)=0$, $\lim_{t\to+\infty}\hat{x}(t)=0$ and $\lim_{t\to+\infty}L(t)=\bar{L}\in\mathbb{R}_+$.

4. AN ILLUSTRATIVE EXAMPLE

Consider the following switched uncertain nonlinear system:

$$\begin{aligned}
\dot{\eta}_1 &= g_1\eta_2+\phi_{1,\sigma(t)}(t,\eta,d(t)),\\
\dot{\eta}_2 &= g_2u+\phi_{2,\sigma(t)}(t,\eta,d(t)),\\
y &= c_{1,\sigma(t)}\eta_1+c_{2,\sigma(t)}\sin\eta_1
\end{aligned}\tag{52}$$

where $\sigma(t):[0,+\infty)\to M=\{1,2\}$, $\phi_{11}(t,\eta,d(t))=\theta_{11}\eta_1+d_{11}(t)\eta_1\sin^2\eta_2$, $\phi_{1,2}(t,\eta,d(t))=\theta_{21}d_{21}(t)\eta_1\sin\eta_1$, $\phi_{21}(t,\eta,d(t))=\theta_{12}\eta_2\sin\eta_1+d_{12}(t)\theta_{13}\ln(1+\theta_{14}\eta_2^2)$, $\phi_{22}(t,\eta,d(t))=\frac{\theta_{22}\eta_2}{1+\eta_1^4}+d_{22}(t)\eta_1$ with $\theta_{11},\theta_{12},\theta_{13},\theta_{14},\theta_{21},\theta_{22}$ being unknown constants and $d_{11},d_{12},d_{21},d_{22}$ being uncertain bounded parameters. $0.2\le\lambda_{11},\lambda_{21}\le 1$, $1.1\le\lambda_{12},\lambda_{22}\le 2$ and $0.9\le g_1,g_2\le 1.8$. It can be shown that the switching nonlinear system (52) satisfies Assumptions 2.1-2.3. Hence, by Theorem 3.1, a dynamic high-gain observer and output-feedback controller can be designed as follows:

$$\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2-2L\hat{x}_1,\\
\dot{\hat{x}}_2 &= 0.9^2u-L^2\hat{x}_1,\\
u &= 0.9^{-2}(-12.436L\hat{x}_2-426.828L^2y),\\
\dot{L} &= \frac{\hat{x}_1^2}{L^2}+\frac{y^2}{L^2}+\left(\frac{\hat{x}_2}{L^2}+34.322\frac{y}{L}\right)^2.
\end{aligned}\tag{53}$$

In the simulation, we choose $g_1=g_2=1$, $\theta_{11}=1.5$, $\theta_{12}=0.8$, $\theta_{13}=0.5$, $\theta_{14}=0.7$, $\theta_{21}=0.4$, $\theta_{22}=0.2$ $d_{11}=\sin t$, $d_{12}=0.6$, $d_{21}=0.5$, $d_{22}=0.3$, and $c_{11}=0.2$,

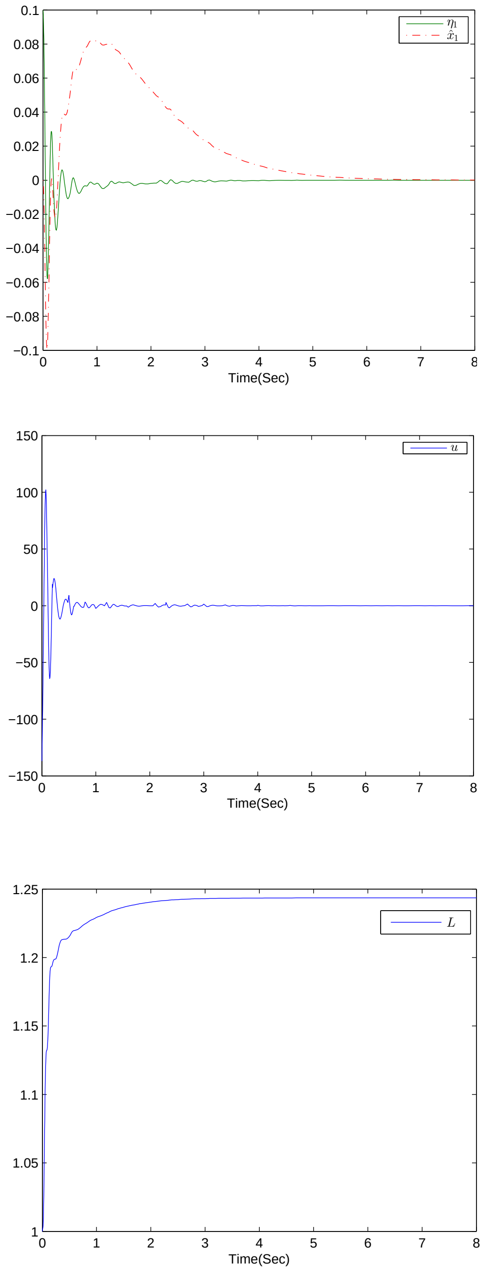

Fig. 1. The responses of the closed-loop system (52) – (53).

$c_{21} = 1$, $c_{12} = 1.1$, $c_{22} = 0.5$. With the initial condition $(\eta_1(0), \eta_2(0)) = (0.1, -0.3)$ and $(\hat{x}_1(0), \hat{x}_2(0), L(0)) = (0, 0, 1)$, Fig. 1 demonstrates the effectiveness of the proposed control scheme.

5. CONCLUSION

This paper has discussed the problem of global output-feedback regulation for a class of switched uncertain nonlinear systems under arbitrary switchings. This problem has been solvable using the suitable observer and controller, which can be explicitly constructed. It can be indicated that an appropriate choice of dynamic high gain will enable us to achieve global asymptotic regulation of the closed-loop switched system. A remaining problem to be investigated is how to design an adaptive output-feedback controller for a class of high-order switched uncertain nonlinear systems with weaker assumptions.

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