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# On the solvability of systems of linear equations over the ring $\mathbb{Z}$ of integers

Horst Herrlich, Eleftherios Tachtsis

*Abstract.* We investigate the question whether *a system $(E_i)_{i\in I}$ of homogeneous linear equations over $\mathbb{Z}$ is non-trivially solvable in $\mathbb{Z}$ provided that each subsystem $(E_j)_{j\in J}$ with $|J| \leq c$ is non-trivially solvable in $\mathbb{Z}$* where $c$ is a fixed cardinal number such that $c < |I|$. Among other results, we establish the following.

(a) The answer is 'No' in the finite case (i.e., $I$ being finite).

(b) The answer is 'No' in the denumerable case (i.e., $|I| = \aleph_0$ and $c$ a natural number).

(c) The answer in case that $I$ is uncountable and $c \leq \aleph_0$ is 'No relatively consistent with $\mathsf{ZF}$', but is unknown in $\mathsf{ZFC}$. For the above case, we show that "*every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has a non-trivial solution in $\mathbb{Z}$*" **implies** (1) the Axiom of Countable Choice (2) the Axiom of Choice for families of non-empty finite sets (3) the Kinna–Wagner selection principle for families of sets each order isomorphic to $\mathbb{Z}$ with the usual ordering, and is **not implied by** (4) the Boolean Prime Ideal Theorem ($\mathsf{BPI}$) in $\mathsf{ZF}$ (5) the Axiom of Multiple Choice ($\mathsf{MC}$) in $\mathsf{ZFA}$ (6) $\mathsf{DC}_{<\kappa}$ in $\mathsf{ZF}$, for every regular well-ordered cardinal number $\kappa$.

We also show that the related statement "*every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has an uncountable subsystem with a non-trivial solution in $\mathbb{Z}$*" (1) is provable in $\mathsf{ZFC}$ (2) is not provable in $\mathsf{ZF}$ (3) does not imply "every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has a non-trivial solution in $\mathbb{Z}$" in $\mathsf{ZFA}$.

*Keywords:* Axiom of Choice; weak axioms of choice; linear equations with coefficients in $\mathbb{Z}$; infinite systems of linear equations over $\mathbb{Z}$; non-trivial solution of a system in $\mathbb{Z}$; permutation models of $\mathsf{ZFA}$; symmetric models of $\mathsf{ZF}$

*Classification:* Primary 03E25; Secondary 03E35

## 1. Notation, terminology, formulation of the general problem and aim

**Notation 1.** 1. $\omega$ denotes (as usual) the set of natural numbers.

2. $\mathsf{ZF}$ is Zermelo–Fraenkel set theory without the Axiom of Choice ($\mathsf{AC}$).
3. $\mathsf{ZFC}$ is $\mathsf{ZF} + \mathsf{AC}$.

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The second-named author wishes to dedicate this article to the memory of his dear friend and colleague, Horst Herrlich, who passed away on March 13, 2015.

4. $\mathsf{ZFA}$ is $\mathsf{ZF}$ with the Axiom of Extensionality modified in order to allow the existence of atoms.

**Definition 1.** (i) Let $X$ be a set.

1. $X$ is *finite* if there exists an $n \in \omega$ and a bijection (i.e., a one-to-one and onto mapping) $f : X \to n$. Otherwise, $X$ is called *infinite*.
2. $X$ is *denumerable* (or *countably infinite*) if there is a bijection $f : X \to \omega$.
3. $X$ is *countable* if it is finite or denumerable, i.e., if there is an injection $f : X \to \omega$. Otherwise, $X$ is *uncountable*. (Clearly, an uncountable set is infinite.)
4. $X$ is *amorphous* if it is infinite but is not the union of two disjoint infinite sets.
5. $X$ is *Dedekind-finite* if there is no injection $f : \omega \to X$. (Clearly, finite sets and amorphous sets are Dedekind-finite. In $\mathsf{ZFC}$, but not in $\mathsf{ZF}$, Dedekind-finite $\equiv$ finite; see [8].)

(ii) Let $X$ and $Y$ be two sets. '$|X| = |Y|$' means that there is a bijection $f : X \to Y$, '$|X| \leq |Y|$' means that there exists an injection $f : X \to Y$ and '$|X| < |Y|$' means that $|X| \leq |Y|$, but $|X| \neq |Y|$. From the Cantor–Bernstein Theorem (which is provable in $\mathsf{ZF}$, see [9, Theorem 3.2]) it follows that $|X| < |Y|$ if and only if there is an injection $f : X \to Y$, but there is no injection $g : Y \to X$.

(iii) Let $V$ be a model of $\mathsf{ZF}$ and let $\mathrm{On} = \{\alpha \in V : \alpha \text{ is an ordinal}\}$. By transfinite recursion on $\alpha \in \mathrm{On}$, we define $V_\alpha$ as follows: $V_0 = \emptyset$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$ (= the power set of $V_\alpha$), and $V_\alpha = \bigcup\{V_\beta : \beta < \alpha\}$ if $\alpha$ is a limit ordinal. By the axiom of power set and the axiom (scheme) of replacement, we have that for each $\alpha \in \mathrm{On}$, $V_\alpha$ is a set in $V$. Furthermore, by the axiom of foundation we have that $V = \bigcup\{V_\alpha : \alpha \in \mathrm{On}\}$ (see [10, Theorem 4.1, p. 101]).

Now, let $X$ be a set in the model $V$ of $\mathsf{ZF}$. Using the cumulative hierarchy of the sets $V_\alpha$, $\alpha \in \mathrm{On}$, the *cardinality* of $X$, denoted by $|X|$, is defined as the set $\{Y \in V_{\alpha(X)} : \text{there exists a bijection } f : Y \to X\}$, where $\alpha(X)$ is the least ordinal number (also referred to as 'least rank') for which the latter set is non-empty. We note that the above definition of the cardinality of a set also works in $\mathsf{ZFA}$, that is, if the class of all atoms is a set — see also [8, Section 11.2, p. 152, and Section 4.1, pp. 44–45].

A set $c$ is called a *cardinal number* (or simply a *cardinal*) if it is the cardinality of some set. If $c$ and $d$ are cardinals, then '$c \leq d$' (resp. '$c < d$') means that $\forall X \in c$, $\forall Y \in d$, $|X| \leq |Y|$ (resp. $\forall X \in c$, $\forall Y \in d$, $|X| < |Y|$). A cardinal number $c$ is an *aleph* if it is the cardinality of a well-ordered set. $\aleph_0$ denotes the cardinality of $\omega$.

**Definition 2.** 1. Let $X = \{x_i : i \in I\}$ be a set of variables.
A *linear equation over* $\mathbb{Z}$ is an expression of the form $\sum_{j \in J} a_j x_j = b$, where $J$ is a finite subset of $I$, $b \in \mathbb{Z}$, and $a_j \in \mathbb{Z}$ for all $j \in J$. If, in the latter equation, $b = 0$, then the resulting equation $\sum_{j \in J} a_j x_j = 0$ is called a *homogeneous linear equation over* $\mathbb{Z}$.

Note that we consider the sum '$\sum_{j\in J} a_j x_j$' ($J$ a finite subset of $I$) as a *finite formal sum* with indeterminates from $X$ and coefficients in $\mathbb{Z}$, i.e., we consider the set of all functions $f$ from $X$ into $\mathbb{Z}$ with finite support, that is, $|\{x \in X : f(x) \neq 0\}| < \aleph_0$, equipped with pointwise operations. Thus, for $k, m \in \mathbb{Z}$ and $x, y \in X$, we do not distinguish between '$kx+my$' and '$my + kx$'.

2. Let $S$ be a system of linear equations over $\mathbb{Z}$ and let $X = \{x_i : i \in I\}$ ($i \mapsto x_i$, $i \in I$, is a bijection) be the set of all variables appearing in the equations of $S$.
   - A *non-trivial solution* of $S$ in $\mathbb{Z}$ is a family $\{s_i : i \in I\} \subseteq \mathbb{Z} \setminus \{0\}$ such that for every finite set $J \subseteq I$, if $\sum_{j\in J} a_j x_j = b$ is an equation of $S$, then the equation $\sum_{j\in J} a_j s_j = b$ is true in $\mathbb{Z}$. (In other words, a non-trivial solution of $S$ is a function $f : X \to \mathbb{Z} \setminus \{0\}$ such that for every finite $J \subseteq I$, if $\sum_{j\in J} a_j x_j = b$ is an equation of $S$, then the equation $\sum_{j\in J} a_j f(x_j) = b$ is true in $\mathbb{Z}$.)
   - A *non-trivial assignment* of $S$ in $\mathbb{Z}$ is a function $f : Y \to \mathbb{Z} \setminus \{0\}$, where $Y$ is a non-empty subset of the set $X = \{x_i : i \in I\}$ of the variables of the equations of $S$, such that if we replace every $y \in Y$, appearing in the equations of $S$, with its value $f(y)$, then both of the following two conditions are satisfied:
     (a) the equations of $S$ that no longer contain a variable are true in $\mathbb{Z}$,
     (b) the equations of $S$ which still contain variables form a *new* system in which every countable subsystem has a non-trivial solution in $\mathbb{Z}$.

**Definition 3.** 1. $\mathsf{AC}$ is the Axiom of Choice, i.e., every family of non-empty sets has a choice function.

2. $\mathsf{MC}$ is the Axiom of Multiple Choice, i.e., for every family $\mathcal{A} = \{A_i : i \in I\}$ of non-empty sets there is a function $F$ with domain $\mathcal{A}$ such that $\forall i \in I$, $F(A_i)$ is a non-empty finite subset of $A_i$.
   It is known (see [5], [8, Theorems 9.1 and 9.2]) that $\mathsf{MC}$ is equivalent to $\mathsf{AC}$ in $\mathsf{ZF}$, but not equivalent to $\mathsf{AC}$ in $\mathsf{ZFA}$.
3. Let $\kappa$ be an aleph (i.e., a well-ordered cardinal number).
   (a) $\mathsf{AC}^{\kappa}$ is $\mathsf{AC}$ restricted to $\kappa$-sized families of non-empty sets. In particular, $\mathsf{AC}^{\aleph_0}$ is the Axiom of Countable Choice.
   (b) $\mathsf{DC}_{\kappa}$ is "let $S$ be a non-empty set and let $R$ be a binary relation such that for every $\alpha < \kappa$ and every $\alpha$-sequence $s = (s_\xi)_{\xi<\alpha}$ of elements of $S$ there exists $y \in S$ such that $s\ R\ y$. Then there is a function $f : \kappa \to S$ such that for every $\alpha < \kappa$, $(f \restriction \alpha)\ R\ f(\alpha)$".
   $\mathsf{DC}_{<\kappa}$ is "$\forall \lambda < \kappa$, $\mathsf{DC}_{\lambda}$".
   Note that $\mathsf{DC}_{\aleph_0}$ is a reformulation of the Principle of Dependent Choice $\mathsf{DC}$. Also, for any well-ordered cardinal $\lambda$, $\mathsf{DC}_{\lambda} \to \mathsf{AC}^{\lambda}$ and

"$(\forall\mu)(\mathsf{DC}_\mu)$" (where the parameter $\mu$ represents a well-ordered cardinal) is equivalent to $\mathsf{AC}$; see [8, parts (b) and (c) of Theorem 8.1].

4. The Boolean Prime Ideal Theorem ($\mathsf{BPI}$) is "every non-trivial Boolean algebra has a prime ideal".
5. $\mathsf{AC}_{\mathsf{fin}}$ is $\mathsf{AC}$ restricted to families of non-empty finite sets.
6. $\mathsf{AC}_{\mathsf{fin}}^{\aleph_0}$ is $\mathsf{AC}$ restricted to denumerable families of non-empty finite sets.
7. $\mathsf{AC}_{\mathsf{WO}}$ is $\mathsf{AC}$ restricted to families of non-empty well-orderable sets.
8. van Douwen's Choice Principle ($\mathsf{vDCP}$) is "every family $\mathcal{A} = \{(A_i, \leq_i) : i \in I\}$, where $\forall i \in I$, $(A_i, \leq_i)$ is order isomorphic to $(\mathbb{Z}, \leq)$ ($\leq$ is the usual ordering of the integers), has a choice function".
9. $\mathsf{KW\text{-}vDCP}$ is "every family $\mathcal{A} = \{(A_i, \leq_i) : i \in I\}$, where $\forall i \in I$, $(A_i, \leq_i)$ is order isomorphic to $(\mathbb{Z}, \leq)$, has a Kinna–Wagner selection function, i.e., a function $F$ with domain $\mathcal{A}$ such that $\forall i \in I$, $F(A_i)$ is a non-empty proper subset of $A_i$.
10. $\mathsf{PKW\text{-}vDCP}$ is "every family $\mathcal{A} = \{(A_i, \leq_i) : i \in I\}$, where $\forall i \in I$, $(A_i, \leq_i)$ is order isomorphic to $(\mathbb{Z}, \leq)$, has a partial Kinna–Wagner selection function, i.e., there exists an infinite subfamily $\mathcal{B} \subseteq \mathcal{A}$ with a Kinna–Wagner selection function.

Like every rigorous mathematical discipline, the theory of infinite systems of polynomial equations or of infinite systems of linear equations (over a field) and the existence of solutions of such systems is based on axiomatic set theory. In particular, the Axiom of Choice $\mathsf{AC}$ and weak forms of $\mathsf{AC}$ are indispensable tools for the derivation of results on the existence of solutions. For the reader's convenience and information, we mention a few results in this area. In [7], it is proved that the statement "for every field $F$, for every system $S$ of linear equations over $F$, $S$ has a solution in $F$ if and only if every finite subsystem of $S$ has a solution in $F$", abbreviated as "$\forall F(\mathbf{SLin}(F))$" in [7], is provable in $\mathsf{ZFC}$, and is also relatively consistent with $\mathsf{ZFA} + \neg\mathsf{BPI}$ (see [7, Theorem 4.8]), hence it does not imply $\mathsf{AC}$ in $\mathsf{ZFA}$. It is an open problem whether $\mathsf{BPI}$ implies $\forall F(\mathbf{SLin}(F))$. However, in [7], it has been established that $\mathsf{BPI}$ implies "for every finite field $F$, $\mathbf{SLin}(F)$" (see [7, Theorems 3.13, 3.14]) and therefore, in view of the above result of [7], the latter implication is not reversible in $\mathsf{ZFA}$.

With regard to infinite systems of polynomial equations over a field, it is known (see [1]) that $\mathsf{BPI}$ is equivalent to "a system $S$ of polynomial equations over $\mathbb{Z}_2$ (i.e., the two-element field $\{0, 1\}$) has a solution in $\mathbb{Z}_2$ if and only if every finite subsystem of $S$ has a solution in $\mathbb{Z}_2$", and that $\mathsf{BPI}$ is also equivalent to "for every finite field $F$, a system $S$ of polynomial equations over $F$ has a solution in $F$ if and only if every finite subsystem of $S$ has a solution in $F$" (see [5, Note 30, Theorems 1 and 2, p. 249]).

The current paper also elucidates systems of linear equations, but this time *over the ring $\mathbb{Z}$ of integers*, and studies the problem of the existence of non-trivial solutions in $\mathbb{Z}$ of a system $(E_i)_{i\in I}$ of homogeneous linear equations over $\mathbb{Z}$ such that each subsystem $(E_j)_{j\in J}$ with $|J| \leq c$ is non-trivially solvable in $\mathbb{Z}$ where $c$

is a fixed cardinal such that $c < |I|$. We shall mainly focus on the study of the existence of non-trivial solutions (in $\mathbb{Z}$) of uncountable systems of homogeneous linear Diophantine equations over $\mathbb{Z}$, whose countable subsystems are non-trivially solvable in $\mathbb{Z}$.

At this point, and in view of the forthcoming main results, the reader should carefully see again the definition of 'non-trivial solution in $\mathbb{Z}$' - Definition 2(2). As usual, the term 'non-trivial solution' means 'a non-zero value is assigned to *at least one* variable', however, according to Definition 2(2), the meaning of the aforementioned term in this paper is 'a non-zero value is assigned to *each* one of the variables'. Our motivation for the latter requirement is illuminated by several of the forthcoming main results, such as Lemma 3 and Theorems 3, 4, 5. For example, Lemma 3 witnesses the existence of a model $N$ of $\mathsf{ZF}$, in which there is an uncountable disjoint family $\mathcal{A} = \{A_i : i \in I\}$, where $|A_i| = 2$ for all $i \in I$, which admits no choice function in $N$, though every countable subfamily of $\mathcal{A}$ does have a choice function in $N$. It follows that, in the model $N$, the uncountable system $\sum_{a \in A_i} a = 0$, $i \in I$, which comprises homogeneous linear equations over $\mathbb{Z}$, is such that each of its countable subsystems has a non-trivial solution in $\mathbb{Z}$ (either in the usual meaning or in the meaning required here), hence the above system has a non-trivial solution in $\mathbb{Z}$ in the usual sense, but it has *no* non-trivial solution in $\mathbb{Z}$ in the meaning required in this paper (see Theorem 3, Lemma 3 and Theorem 4). Therefore, Definition 2(2) *naturally emerged* in order to investigate the deductive strength of the existence of non-trivial solutions in $\mathbb{Z}$ of a system of homogeneous linear equations over $\mathbb{Z}$, each of whose subsystems of a fixed smaller cardinality has a non-trivial solution in $\mathbb{Z}$.

Below, we state the general problem that has been the motivation of the research in this paper.

**The General Problem**: *A system $(E_i)_{i \in I}$ of homogeneous linear equations over $\mathbb{Z}$ is non-trivially solvable in $\mathbb{Z}$ provided that each subsystem $(E_j)_{j \in J}$ with $|J| \leq c$ is non-trivially solvable in $\mathbb{Z}$ where $c$ is a fixed cardinal such that $c < |I|$.*

**The aim of this paper** is to investigate the provability or non-provability of certain cases of the above statement in $\mathsf{ZF}$ and $\mathsf{ZFC}$, as well as their deductive strength and interrelation with certain choice principles. In particular, we will study three cases:

(a) the Finite Case, i.e., $I$ being finite,

(b) the Denumerable Case, i.e., $I = \omega$ and $c$ is any finite set,

(c) the Uncountable Case, where $I$ is uncountable and $c \leq \aleph_0$.

We will show that:

(a') the answer is 'No' in the Finite Case,

(b') the answer is 'No' in the Denumerable Case.

With regard to case (c), the answer is *not known even in the setting of* $\mathsf{ZFC}$, that is, it is unknown whether $\mathsf{AC}$ implies "every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has a non-trivial solution in $\mathbb{Z}$". However, we are able to prove that the latter statement is not a theorem of ZF by establishing that it implies certain weak choice principles, namely the principles $\mathsf{AC}^{\aleph_0}$, $\mathsf{AC}_{\mathsf{fin}}$, and KW-vDCP (see Theorems 1, 3, 5). Furthermore, we will show that the statement is not implied by BPI in ZF, by MC in ZFA, and by $\mathsf{DC}_{<\kappa}$ in ZF, for every regular cardinal number $\kappa$ (see Theorem 2, Corollary 1(b), Theorem 3, and Corollary 4).

We shall also prove the following related result: "*every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has an uncountable subsystem with a non-trivial solution in $\mathbb{Z}$*" is provable in ZFC, but not provable in ZF (see Theorems 6 and 8). In addition, and among other results, we will establish that the latter statement does not imply "every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has a non-trivial solution in $\mathbb{Z}$" in ZFA set theory (see Theorem 7).

## 2. Main results

**2.1 The Finite Case.** (ZF) For each positive integer $n$, there exists a system of $n+1$ linear homogeneous equations over $\mathbb{Z}$, which has no non-trivial solution in $\mathbb{Z}$, though each subsystem of $n$ equations does.

Indeed, let $n \in \omega \setminus \{0\}$ and $X = \{x_0, x_1, \ldots, x_n\}$ be a set of $n+1$ variables. Then, the following system is as required.

$$\begin{aligned} x_{i+1} - 2x_i &= 0, \quad i = 0, 1, \ldots, n-1, \\ x_0 - 2x_n &= 0. \end{aligned}$$

**2.2 The Denumerable Case.** (ZF) There exists a denumerable system of linear homogeneous equations over $\mathbb{Z}$, which has no non-trivial solution in $\mathbb{Z}$, though each finite subsystem does.

Indeed, let $X = \{x_n : n \in \omega\}$ be a denumerable set of variables (the map $n \mapsto x_n$, $n \in \omega$, is a bijection). Below, we present several counterexamples, whose ideas shall be used in the uncountable case in order to derive results on the deductive strength of the corresponding statement.

**Example 1.** Consider the following system over $\mathbb{Z}$:

$$x_n - 2x_{n+1} = 0, \quad n \in \omega \setminus \{0\}.$$

**Example 2.** Consider the following system over $\mathbb{Z}$:

$$nx_n - mx_m = 0, \quad n, m \in \omega,\ n \neq m.$$

**Example 3.** Consider the following system over $\mathbb{Z}$:

$$nx_n + (n+1)x_{n+1} = 0, \quad n \in \omega.$$

**Example 4.** Consider the following system over $\mathbb{Z}$:

$$2x_n + 3x_{n+1} = 0, \quad n \in \omega.$$

**2.3 The Uncountable Case: $I$ is uncountable and $c \leq \aleph_0$.** Here, we consider the statement: *Every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has a non-trivial solution in $\mathbb{Z}$.*

As mentioned in Section 1, we do not know whether the above assertion is provable in $\mathsf{ZFC}$. However, we have a clear picture of the situation with uncountable system of inequalities. In particular, we have the following example in $\mathsf{ZF}$.

**Example 5.** The following uncountable system of homogeneous inequalities with coefficients in $\mathbb{Z}$ has no solution in $\mathbb{Z}$, though each countable subsystem does: For each pair of elements $a$ and $b$ in $\aleph_1$ with $a < b$, consider the inequality $a \neq b$.

We now present our results on the deductive strength of "every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has a non-trivial solution in $\mathbb{Z}$".

**Lemma 1** (see [3], [5])**.** $\mathsf{AC}^{\aleph_0}$ *if and only if every denumerable family $\mathcal{A}$ of non-empty sets has a partial choice function, i.e., $\mathcal{A}$ has an infinite subfamily with a choice function.*

**Theorem 1.** *The statement "every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has a non-trivial solution in $\mathbb{Z}$" implies $\mathsf{AC}^{\aleph_0}$.*

Proof: Assume the hypothesis and let $\mathcal{A} = \{A_i : i \in \omega\}$ be a denumerable family of non-empty sets. Without loss of generality, assume that $\mathcal{A}$ is disjoint and, towards a proof by contradiction, assume that $\mathcal{A}$ has no partial choice function (see Lemma 1). We consider the following equations over $\mathbb{Z}$:

$$na + (n+1)b = 0,\ n \in \omega \setminus \{0\},\ a \in A_n,\ b \in A_{n+1}, \tag{1}$$

and we let $S$ be the linear homogeneous system of all equations of the form (1).

Since $\mathcal{A}$ has no partial choice function, we have that $S$ is uncountable. To see this, assume the contrary. Then (since $S$ is infinite) $S$ is denumerable (see Definition 1), so let $(E_i = 0)_{i\in\omega}$ be an enumeration of the equations of $S$, and also let $X_i$ be the set of variables of $E_i$, $i \in \omega$. Then $\forall i \in \omega$, $\forall j \in \omega$, we have that $|X_i \cap A_j| = 1$ or $|X_i \cap A_j| = 0$. From the latter observation, as well as, from equation (1), and the fact that $\bigcup\{X_i : i \in \omega\} = \bigcup \mathcal{A}$, it is fairly easy to construct via mathematical induction a partial choice function of $\mathcal{A}$. This contradicts our assumption on $\mathcal{A}$. Thus $S$ is uncountable.

Using similar reasoning, we may prove that for every countable subsystem $L$ of $S$, the set of all variables of the equations of $L$ must necessarily be contained in some finite union of the $A_i$'s (since $\mathcal{A}$ has no partial choice function). Based on the latter fact, we may easily show that $L$ has a non-trivial solution in $\mathbb{Z}$.

Therefore, by our hypothesis, $S$ has a non-trivial solution. However, this is easily seen to be false (without invoking any choice form). Thus, $\mathcal{A}$ has a partial choice function. The conclusion now follows from Lemma 1. □

**Theorem 2.** $\mathsf{BPI}$ *does not imply "every uncountable system of linear homogeneous equations over* $\mathbb{Z}$*, each of its countable subsystems having a non-trivial solution in* $\mathbb{Z}$*, has a non-trivial solution in* $\mathbb{Z}$*" in* $\mathsf{ZF}$.

Proof: The result follows from Theorem 1 and the fact that $\mathsf{BPI}$ does not imply $\mathsf{AC}^{\aleph_0}$ in $\mathsf{ZF}$, e.g. the basic Cohen model (Model $\mathcal{M}1$ in [5]) satisfies $\mathsf{BPI} + \neg\mathsf{AC}^{\aleph_0}$, see [5] or [8]. □

**Lemma 2.** *The following statements are equivalent.*

1. $\mathsf{AC}_{\mathsf{fin}}$.
2. *For every set* $\mathcal{A}$ *of non-empty finite sets there is a function* $F$ *with domain* $\mathcal{A}$ *such that for all* $A \in \mathcal{A}$*, if* $|A| \geq 2$ *then* $F(A)$ *is a non-empty proper subset of* $A$.

Proof: The proof is given in [5, Note 70]. □

**Theorem 3.** *The statement "every uncountable system of linear homogeneous equations over* $\mathbb{Z}$*, each of its countable subsystems having a non-trivial solution in* $\mathbb{Z}$*, has a non-trivial solution in* $\mathbb{Z}$*" implies* $\mathsf{AC}_{\mathsf{fin}}$.

Proof: Assume that every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has a non-trivial solution in $\mathbb{Z}$, and let $\mathcal{A} = \{A_i : i \in I\}$ be a family of non-empty finite sets. By Theorem 1, we may assume, without loss of generality, that $\mathcal{A}$ is uncountable. Moreover, we may assume that $\mathcal{A}$ is disjoint and that $\forall i \in I$, $|A_i| \geq 2$. Consider the following uncountable system of linear homogeneous equations over $\mathbb{Z}$:

$$\sum_{a \in A_i} a = 0, \; i \in I. \tag{2}$$

**Claim 1.** Every countable subsystem of (2) has a non-trivial solution in $\mathbb{Z}$.

Proof: Let $J$ be a countable subset of $I$ and consider the subsystem

$$\sum_{a \in A_j} a = 0, \; j \in J, \tag{3}$$

of (2). We consider the following two cases.

(a) $J$ is finite. In this case, it is straightforward to verify that (3) has a non-trivial solution in $\mathbb{Z}$.

(b) $J$ is denumerable. Then $\mathcal{A} = \{A_j : j \in J\}$ is a denumerable family of non-empty finite sets. Thus, by Theorem 1, $\mathcal{A}$ has a choice function, say $f$. For each $j \in J$, let

$$w_j = |A_j| - 1.$$

(Note that $\forall j \in J$, $w_j \neq 0$.)

We may define now a non-trivial solution $(g(x))_{x \in \bigcup \mathcal{A}}$ of the system (3) as follows: For $j \in J$ and $x \in A_j$, let

$$g(x) = \begin{cases} w_j & \text{if } x = f(A_j) \\ -1 & \text{if } x \in A_j \setminus \{f(A_j)\}. \end{cases}$$

It is clear that $(g(x))_{x \in \bigcup \mathcal{A}}$ is a non-trivial solution of the system (3).

The above cases complete the proof of the claim. □

By Claim 1 and our assumption, the system (2) has a non-trivial solution in $\mathbb{Z}$, say $(s(x))_{x \in \bigcup \mathcal{A}}$. Due to equations (2), we have that for all $i \in I$, the restriction $s \upharpoonright A_i$ of $s$ on $A_i$ must take on both positive and negative values in $\mathbb{Z}$. Therefore, $\forall i \in I$, the set $B_i = \{b \in A_i : s(b) < 0\}$ is a non-empty proper subset of $A_i$. Thus, we have proved that given a family $\mathcal{A}$ of non-empty finite sets, there is a function $F$ with domain $\mathcal{A}$ such that for all $A \in \mathcal{A}$, if $|A| \geq 2$, then $F(A)$ is a non-empty proper subset of $A$. By Lemma 2 we conclude that $\mathsf{AC}_{\mathsf{fin}}$ holds, finishing the proof of the theorem. □

**Corollary 1.** (a) *In every permutation model of* $\mathsf{ZFA}$, *"every uncountable system of linear homogeneous equations over* $\mathbb{Z}$*, each of its countable subsystems having a non-trivial solution in* $\mathbb{Z}$*, has a non-trivial solution in* $\mathbb{Z}$*" implies* $\mathsf{AC}_{\mathsf{WO}}$.

(b) $\mathsf{MC}$ *does not imply "every uncountable system of linear homogeneous equations over* $\mathbb{Z}$*, each of its countable subsystems having a non-trivial solution in* $\mathbb{Z}$*, has a non-trivial solution in* $\mathbb{Z}$*" in* $\mathsf{ZFA}$.

Proof: (a) It is known (see [5], [8]) that in every Fraenkel–Mostowski (FM) model, $\mathsf{AC}_{\mathsf{fin}} \leftrightarrow \mathsf{AC}_{\mathsf{WO}}$. The conclusion now follows from Theorem 3.

(b) This follows from Theorem 3 and the fact that $\mathsf{MC}$ does not imply $\mathsf{AC}_{\mathsf{fin}}$ in $\mathsf{ZFA}$ (see the Second Fraenkel Model, Model $\mathcal{N}2$ in [5]). □

We prove next that for every regular cardinal $\kappa$, $\mathsf{DC}_{<\kappa}$ does not imply "every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has a non-trivial solution in $\mathbb{Z}$" in $\mathsf{ZF}$ set theory. Although we could derive the result using Theorem 3 above and Theorem 8.3 in [8], in the proof of which a permutation model $\mathcal{V}$ of $\mathsf{ZFA}$ is built satisfying $\mathsf{DC}_\lambda$ for every $\lambda < \aleph_\alpha$ ($\aleph_\alpha$ being a regular cardinal) and "there is a family of $\aleph_\alpha$ pairs without a choice function", which is then embedded in a symmetric model of $\mathsf{ZF}$ — via the Second Embedding Theorem (see [8, Theorem 6.8, p. 94]) — with the required properties, we prefer to give our own proof using a direct forcing construction. Indeed, we have the following.

**Lemma 3.** *Assume* $\aleph_\alpha$ *is a regular cardinal. There exists a model* $N$ *of* $\mathsf{ZF}$*, in which for every cardinal* $\lambda < \aleph_\alpha$*,* $\mathsf{DC}_\lambda$ *is true (hence, for every* $\lambda < \aleph_\alpha$*,* $\mathsf{AC}^\lambda$ *is true in* $N$*), but there is an* $\aleph_\alpha$*-sized family* $\mathcal{A} = \{A_i : i < \aleph_\alpha\}$ *of pairs having no choice function.*

Proof: Assume the hypothesis and let $M$ be a transitive model of $\mathsf{ZFC}$. We shall construct a symmetric extension model $N$ of $M$ with the required properties.

Let $\mathbb{P} = \mathrm{Fn}(\aleph_\alpha \times 2 \times \aleph_\alpha \times \aleph_\alpha, 2, \aleph_\alpha)$ be the set of all partial functions $p$ with $|p| < \aleph_\alpha$, $\mathrm{dom}(p) \subseteq \aleph_\alpha \times 2 \times \aleph_\alpha \times \aleph_\alpha$ and $\mathrm{ran}(p) \subseteq 2 = \{0,1\}$, partially ordered by reverse inclusion, i.e., $p \leq q$ if and only if $p \supseteq q$. $\mathbb{P}$ has the empty function as its maximum element, which we denote by $\mathbf{1}$. Further, since $\aleph_\alpha$ is a regular cardinal, it follows from [10, Lemma 6.13, p. 214] that $(\mathbb{P}, \leq)$ is a $\aleph_\alpha$-closed poset. Hence, forcing with $\mathbb{P}$ adds only new subsets of $\aleph_\alpha$ and no new subsets of cardinals $\lambda < \aleph_\alpha$, see [10, Theorem 6.14, p. 214].

Let $G$ be a $\mathbb{P}$-generic set over $M$ and $M[G]$ the corresponding generic extension model of $M$. In $M[G]$, we define the following sets along with their canonical names.

1. $a_{n,t,i} = \{j \in \aleph_\alpha : \exists p \in G, p(n,t,i,j) = 1\}$, $n \in \aleph_\alpha$, $t \in 2$, $i \in \aleph_\alpha$,
   $\overline{a_{n,t,i}} = \{(\check{j}, p) : j \in \aleph_\alpha, p \in \mathbb{P}, p(n,t,i,j) = 1\}$.
2. $A_{n,t} = \{a_{n,t,i} : i \in \aleph_\alpha\}$, $n \in \aleph_\alpha$, $t \in 2$,
   $\overline{A_{n,t}} = \{(\overline{a_{n,t,i}}, \mathbf{1}) : i \in \aleph_\alpha\}$.
3. $A_n = \{A_{n,0}, A_{n,1}\}$, $n \in \aleph_\alpha$,
   $\overline{A_n} = \{(\overline{A_{n,0}}, \mathbf{1}), (\overline{A_{n,1}}, \mathbf{1})\}$.
4. $\mathcal{A} = \{A_n : n \in \aleph_\alpha\}$,
   $\overline{\mathcal{A}} = \{(\overline{A_n}, \mathbf{1}) : n \in \aleph_\alpha\}$.

Every permutation $\phi$ of $\aleph_\alpha \times 2 \times \aleph_\alpha$ induces an order-automorphism of $(\mathbb{P}, \leq)$ by requiring for every $p \in \mathbb{P}$,

$$\mathrm{dom}\,\phi(p) = \{(\phi(n,t,i), j) : (n,t,i,j) \in \mathrm{dom}(p)\}, \tag{4}$$

$$\phi(p)(\phi(n,t,i), j) = p(n,t,i,j). \tag{5}$$

Let $\mathcal{G}$ be the group of all order-automorphisms of $(\mathbb{P}, \leq)$ induced (as in equations (4), (5)) by all those permutations $\phi$ of $\aleph_\alpha \times 2 \times \aleph_\alpha$ such that $\forall (n,t,i) \in \aleph_\alpha \times 2 \times \aleph_\alpha$, $\phi(n,t,i) = (n,t',i')$, and

$$\forall n \in \aleph_\alpha, \text{ either } (\forall i \in \aleph_\alpha, t' = t), \text{ or } (\forall i \in \aleph_\alpha, t' = 1-t). \tag{6}$$

It follows that $\forall \phi \in \mathcal{G}$, $\forall n \in \aleph_\alpha$, and $\forall t \in 2$,

$$\phi(\overline{A_{n,t}}) = \overline{A_{n,t}} \text{ or } \overline{A_{n,(1-t)}},\ \phi(\overline{A_n}) = \overline{A_n},\ \phi(\overline{\mathcal{A}}) = \overline{\mathcal{A}}. \tag{7}$$

For every subset $E \subseteq \aleph_\alpha \times 2 \times \aleph_\alpha$ with $|E| < \aleph_\alpha$, let $\mathrm{fix}_{\mathcal{G}}(E) = \{\phi \in \mathcal{G} : \forall e \in E, \phi(e) = e\}$ and let $\Gamma$ be the normal filter of subgroups of $\mathcal{G}$ generated by $\{\mathrm{fix}_{\mathcal{G}}(E) : E \subseteq \aleph_\alpha \times 2 \times \aleph_\alpha, |E| < \aleph_\alpha\}$. An element $x \in M$ is called *symmetric* if there exists a subset $E \subseteq \aleph_\alpha \times 2 \times \aleph_\alpha$ with $|E| < \aleph_\alpha$ such that $\forall \phi \in \mathrm{fix}_{\mathcal{G}}(E)$, $\phi(x) = x$. Under these circumstances, we call $E$ a *support* of $x$. An element $x \in M$ is called *hereditarily symmetric* if $x$ and every element of the transitive closure of $x$ is symmetric. Let HS be the set of all hereditarily symmetric names in $M$ and let $N = \{\tau_G : \tau \in \mathrm{HS}\} \subset M[G]$, where $\tau_G$ is the value of the name $\tau$ as given in [10, Definition 2.7, p. 189], be the symmetric extension model of $M$.

**Claim 2.** The sets $a_{n,t,i}$, $A_{n,t}$, $A_n$, and $\mathcal{A}$, where $n, i \in \aleph_\alpha$ and $t \in 2$, are elements of $N$. Moreover, $\mathcal{A}$ is $\aleph_\alpha$-sized in $N$.

PROOF: Fix $n, i \in \aleph_\alpha$ and $t \in 2$. It is fairly straightforward to see that $E = \{(n,t,i)\}$ is a support of $\overline{a_{n,t,i}}$ and $\overline{A_{n,t}}$. Now, by equation (7) we have that $\forall \phi \in \mathcal{G}$, $\phi(\overline{A_n}) = \overline{A_n}$ and $\phi(\overline{\mathcal{A}}) = \overline{\mathcal{A}}$. Thus, $a_{n,t,i}$, $A_{n,t}$, $A_n$, and $\mathcal{A}$ all belong to $N$. Furthermore, $\dot{f} = \{(\mathrm{op}(\check{n}, \overline{A_n}), \mathbf{1}) : n \in \aleph_\alpha\}$, where $\mathrm{op}(\sigma, \tau)$ is the name for the ordered pair $(\sigma_G, \tau_G)$ given in [10, Definition 2.16, p. 191], is an HS-name for the mapping $f = \{(n, A_n) : n \in \aleph_\alpha\}$ (in $M[G]$), since $\forall \phi \in \mathcal{G}$, $\phi(\dot{f}) = \dot{f}$. Thus, $|\mathcal{A}| = \aleph_\alpha$ in $N$, finishing the proof of the claim. □

**Claim 3.** For every cardinal $\lambda < \aleph_\alpha$, $\mathsf{DC}_\lambda$ is true in the model $N$. Hence, for every $\lambda < \aleph_\alpha$, $\mathsf{AC}^\lambda$ is true in $N$.

PROOF: Fix a cardinal $\lambda < \aleph_\alpha$. Since $\mathbb{P}$ is $\aleph_\alpha$-closed, it can be shown as in [8, Lemma 8.5, p. 124] that if $\lambda < \aleph_\alpha$ and $f \in M[G]$ is a function on $\lambda$ with values in $N$, then $f \in N$. It follows that if $X \in N$ and $R \in N$ is a relation satisfying the assumptions of $\mathsf{DC}_\lambda$ in $N$, then by $\mathsf{AC}$ in $M[G]$, there is a function $f : \lambda \to X$ in $M[G]$ such that $\forall \mu < \lambda$, $(f \restriction \mu)\ R\ f(\mu)$. By the above observation we have that $f \in N$. Thus, $\mathsf{DC}_\lambda$ is true in $N$. □

**Claim 4.** In $N$, $\mathsf{AC}^{\aleph_\alpha}$ fails for the family of pairs, $\mathcal{A} = \{A_n : n \in \aleph_\alpha\}$. (However, note that by Claim 3, $\forall \lambda < \aleph_\alpha$, $\forall \mathcal{B} \in [\mathcal{A}]^\lambda = \{\mathcal{C} \subseteq \mathcal{A} : |\mathcal{C}| = \lambda\}$, $\mathcal{B}$ has a choice function in $N$.)

PROOF: Towards a proof by contradiction, assume that $f$ is a choice function of $\mathcal{A}$ in $N$. Let $\dot{f}$ be a HS-name for $f$ and let $p \in G$ be such that

$$p \Vdash \text{“}\dot{f} \text{ is a choice function of } \overline{\mathcal{A}}\text{”}. \tag{8}$$

Let $E \subseteq \aleph_\alpha \times 2 \times \aleph_\alpha$, $|E| < \aleph_\alpha$, be a support for $\dot{f}$. Since $|E| < \aleph_\alpha$, there exist ordinals $n \in \aleph_\alpha \setminus \mathrm{dom}(\mathrm{dom}(E))$ and $t \in 2$ such that $f(A_n) = A_{n,t}$. (Note that $(n,t) \notin \mathrm{dom}(E)$.) Let $q \in G$ be such that $q \leq p$ and

$$q \Vdash \dot{f}(\overline{A_n}) = \overline{A_{n,t}}. \tag{9}$$

Since $|q| < \aleph_\alpha$, there exists an ordinal $k \in \aleph_\alpha$ such that $\forall i \in \aleph_\alpha$ with $i \geq k$ and $\forall u \in 2$, $(n,u,i) \notin \mathrm{dom}(q)$. Let $\phi_n : [0,k] \to [k,2k]$ be an order isomorphism. We define an element $\psi \in \mathcal{G}$ as follows:

$$\psi(m,u,i) = \begin{cases} (n, 1-u, \phi_n(i)) & \text{if } m = n \text{ and } i \in [0,k], \\ (n, 1-u, (\phi_n)^{-1}(i)) & \text{if } m = n \text{ and } i \in [k,2k], \\ (n, 1-u, i) & \text{if } m = n \text{ and } 2k < i, \\ (m,u,i) & \text{if } m \neq n. \end{cases}$$

It can be easily verified that $\psi \in \mathrm{fix}_{\mathcal{G}}(E)$, hence $\psi(\dot{f}) = \dot{f}$, $\psi(\overline{A_{n,t}}) = \overline{A_{n,(1-t)}}$, and that $q$ and $\psi(q)$ are compatible conditions. It follows that $q \cup \psi(q)$ is a well-defined extension of $q$, $\psi(q)$, and $p$. Furthermore, by equation (9), we obtain that

$$(10) \qquad \psi(q) \Vdash \dot{f}(\overline{A_n}) = \overline{A_{n,(1-t)}},$$

and by equations (9) and (10), we conclude that

$$(11) \qquad q \cup \psi(q) \Vdash (\dot{f}(\overline{A_n}) = \overline{A_{n,t}}) \wedge (\dot{f}(\overline{A_n}) = \overline{A_{n,(1-t)}}).$$

But then, the equations (8) and (11) yield a contradiction as it can be easily checked via standard forcing arguments (note that $q \cup \psi(q) \Vdash$ "$\dot{f}$ is a choice function of $\overline{\mathcal{A}}$", since $q \cup \psi(q) \leq p$). Thus, $\mathcal{A}$ has no choice function in the model $N$, finishing the proof of Claim 4. □

The above complete the proof of the lemma. □

**Theorem 4.** *Assume $\aleph_\alpha$ is a regular cardinal. Then there exists a model $N$ of* ZF *which satisfies* $\mathsf{DC}_\lambda$ *for every $\lambda < \aleph_\alpha$ and "there is an uncountable linear homogeneous system over $\mathbb{Z}$ which has no non-trivial solution in $\mathbb{Z}$, although each of its countable subsystems has a non-trivial solution in $\mathbb{Z}$".*

Proof: The result follows immediately from Theorem 3 and Lemma 3. □

**Theorem 5.** *The statement "every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has a non-trivial solution in $\mathbb{Z}$" implies* KW-vDCP.

Proof: Assume the hypothesis and let $\mathcal{A} = \{(A_i, \leq_i) : i \in I\}$ be a family as in KW-vDCP, for which — without loss of generality — we assume that it is disjoint, and further we may assume that $I$ is uncountable (due to Theorem 1). Towards a proof by contradiction, assume that $\mathcal{A}$ has no Kinna–Wagner selection function. Consider the following linear homogeneous system over $\mathbb{Z}$:

$$(12) \qquad \forall i \in I, \forall x \in A_i, \forall y \in A_i \text{ such that } \nexists z \in A_i \text{ with } x < z < y,\ x + y = 0.$$

The following hold:

1. The system (12) is uncountable. Assume the contrary, then letting $X$ be the set of variables of the equations of (12), we have that $X$ is a countable union of pairs, thus by Theorem 1, $X$ is countable. Since $X = \bigcup \mathcal{A}$, we conclude that $\bigcup \mathcal{A}$ is countable, thus $\mathcal{A}$ has a choice function, which contradicts our assumption on $\mathcal{A}$.
2. Every countable subsystem of (12) has a non-trivial solution. Let $L$ be a countable subsystem of (12) and let $X_L$ be the set of variables of the equations in $L$. Then (by Theorem 1) $X_L$ is countable, thus $|Y_L| \leq \aleph_0$, where $Y_L = \{i \in I : X_L \cap A_i \neq \emptyset\}$. For each $y \in Y_L$, let $\{a_{y,z} : z \in \mathbb{Z}\}$ be an enumeration of $A_y$ by $\mathbb{Z}$ (note that we have used here Theorem 1 again, in case $|Y_L| = \aleph_0$). It is straightforward to define now a non-trivial solution of $L$. We leave the details to the reader.

From the above observations and our hypothesis, we have that the system (12) has a non-trivial solution, say $\mathbf{s}$. Then $\forall i \in I$, $\mathbf{s} \restriction A_i$ takes on both positive and negative values in $\mathbb{Z}$. It follows that

$$F = \{(A_i, \{x \in A_i : \mathbf{s}(x) < 0\}) : i \in I\}$$

is a Kinna–Wagner selection function for $\mathcal{A}$, finishing the proof of the theorem. $\square$

**2.3.1 A weaker statement which is a theorem of $\mathsf{ZFC}$, but not a theorem of $\mathsf{ZF}$.** In this part of the paper, we will study the set-theoretic strength of the statement "*every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has an uncountable subsystem with a non-trivial solution in $\mathbb{Z}$*", which is clearly derivable from "every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has a non-trivial solution in $\mathbb{Z}$".

We will prove that the above statement is a theorem of $\mathsf{ZFC}$, but not a theorem of $\mathsf{ZF}$. Furthermore, the ideas of the proof that the statement is derivable from the $\mathsf{ZFC}$ axioms shall be crucial in showing that it is strictly weaker than "every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has a non-trivial solution in $\mathbb{Z}$" in $\mathsf{ZFA}$ set theory.

We start with the establishment of the following auxiliary result.

**Proposition 1.** $\mathsf{AC}^{\aleph_0}$ *implies "every system of linear equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has a non-trivial assignment in $\mathbb{Z}$".*

Proof: Assume $\mathsf{AC}^{\aleph_0}$ and let $S$ be a system of linear equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$. Let $X$ be the set of all variables appearing in the equations of $S$. We want to show that $S$ has a non-trivial assignment in $\mathbb{Z}$. To this end, let $x$ be an arbitrary element of $X$, which we fix for the rest of the proof. In order to achieve our goal, it suffices to show that there exists a non-zero integer $z$ such that $\{(x,z)\}$ is an assignment of $S$. Towards a proof by contradiction, assume that $\forall z \in \mathbb{Z} \setminus \{0\}$, $f_z = \{(x,z)\}$ is not an assignment of $S$. Then $S$ cannot have an equation of the form '$a \cdot x = b$' ($a \neq 0$); otherwise, since every countable subsystem of $S$ has a non-trivial solution in $\mathbb{Z}$, hence '$a \cdot x = b$' has a unique non-trivial solution in $\mathbb{Z}$, say $s$, it can be easily verified that $\{(x,s)\}$ is a non-trivial assignment of $S$, which is a contradiction. It follows that for each $z \in \mathbb{Z} \setminus \{0\}$, if we replace any monomial $a \cdot x$ in (the equations of) $S$ by $a \cdot f_z(x)$ $(= a \cdot z)$, the new system that is formed, say $S_z$, has a countable subsystem with no non-trivial solutions in $\mathbb{Z}$.

By $\mathsf{AC}^{\aleph_0}$, pick for each $z \in \mathbb{Z} \setminus \{0\}$, a countable subsystem $T_z$ of $S_z$ which has no non-trivial solution in $\mathbb{Z}$. By $\mathsf{AC}^{\aleph_0}$ again, $T := \bigcup_{z \in \mathbb{Z}} T_z$ is a countable system. Replacing for every $z \in \mathbb{Z} \setminus \{0\}$, any expression $a \cdot f_z(x)$ appearing in $T$ by $a \cdot x$, we obtain a countable subsystem of $S$ which clearly does not have any non-trivial solutions in $\mathbb{Z}$ (since for no $z \in \mathbb{Z} \setminus \{0\}$ does $T_z$ have a non-trivial solution in $\mathbb{Z}$, and we have used all $z \in \mathbb{Z} \setminus \{0\}$!). This contradicts our assumption on the system $S$ that each of its countable subsystems has a non-trivial solution in $\mathbb{Z}$ and completes the proof of the proposition. □

The combinatorial result of the subsequent Lemma 4 is known as the "$\Delta$-*system Lemma*" and is provable in ZFC (see [9, Theorem 9.18, p. 118] or [10, Theorem 1.5, p. 49]). In [6], it has been shown that the latter result is not a theorem of ZF; in particular, in Corollary 2.5 of [6] it has been shown that the $\Delta$-system Lemma is equivalent to the conjunction of CUT (the Countable Union Theorem, i.e., "a countable union of countable sets is countable") and PC ("every uncountable collection of countable sets has an uncountable subcollection with a choice function"). Let us recall here the notion of the $\Delta$-system and the statement of the $\Delta$-system Lemma.

**Definition 4.** A family $\mathcal{A}$ of sets is called a $\Delta$-*system* if there is a fixed set $r$, called the *root* of the $\Delta$-system, such that $a \cap b = r$ whenever $a$ and $b$ are distinct members of $\mathcal{A}$.

**Lemma 4** ($\Delta$-system Lemma)**.** *If $\mathcal{A}$ is an uncountable family of finite sets, then there is an uncountable family $\mathcal{B} \subseteq \mathcal{A}$ which forms a $\Delta$-system.*

**Theorem 6** (ZFC)**.** *Every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has an uncountable subsystem with a non-trivial solution in $\mathbb{Z}$.*

Proof: Fix $(E_i)_{i \in I}$ (the mapping $i \mapsto E_i$, $i \in I$, is a bijection) an uncountable homogeneous system over $\mathbb{Z}$ such that each of its countable subsystems has a non-trivial solution in $\mathbb{Z}$. For each $i \in I$, let $X_i$ be the finite set of variables of the equation $E_i$. Let $\mathcal{A} = \{X_i : i \in I\}$. Then $\mathcal{A}$ is an uncountable family of finite sets (otherwise, and since $|\mathbb{Z}| = \aleph_0$ and linear equations are built using finite formal sums, we would have that $(E_i)_{i \in I}$ is countable), thus by Lemma 4, there is uncountable subset $J \subseteq I$, such that $\mathcal{B} = \{X_j : j \in J\}$ forms a $\Delta$-system with root $r = \{x_1, \ldots, x_k\}$. Then $(E_j)_{j \in J}$ is an uncountable subsystem of $(E_i)_{i \in I}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$. As in the proof of Proposition 1, there is a non-zero integer $z_1$ such that $f_1 = \{(x_1, z_1)\}$ is a non-trivial assignment of the system $(E_j)_{j \in J}$. Let $S_1$ be the system resulting from $(E_j)_{j \in J}$ by substituting $x_1$ by $z_1$. Then every countable subsystem of $S_1$ has a non-trivial solution in $\mathbb{Z}$, thus there is a non-zero integer $z_2$ such that $f_2 = \{(x_2, z_2)\}$ is a non-trivial assignment of $S_1$. Let $S_2$ be the system resulting from $S_1$ by substituting $x_2$ by $z_2$. Continuing by induction we may conclude with an uncountable system $S_k$ such that all the variables $x_m$, $1 \leq m \leq k$, in the root $r$ have been substituted by non-zero values $z_m$, and every countable subsystem of $S_k$ has a non-trivial solution in $\mathbb{Z}$.

Since for $j, j' \in J$ with $j \neq j'$ we have that $(X_j \cap X_{j'}) \setminus r = \emptyset$, and since every equation in $S_k$ has a non-trivial solution in $\mathbb{Z}$, we may pick, via AC, a non-trivial solution $s_j$ of the $j$-equation of $S_k$, $j \in J$. Note that $\forall j \in J$, $\forall m \leq k$, we have that $s_j(x_m) = z_m$. Then

$$\mathbf{s} = \bigcup_{j\in J} s_j$$

is a non-trivial solution of the uncountable subsystem $(E_j)_{j\in J}$ of $(E_i)_{i\in I}$, finishing the proof of the theorem. $\square$

**Remark 1.** Note that the root $r$ of the $\Delta$-system $\mathcal{B}$ in the proof of Theorem 6 can be the empty set. In this case, the uncountable subsystem $(E_j)_{j\in J}$ of $(E_i)_{i\in I}$ has again (via $\mathsf{AC}$) a non-trivial solution in $\mathbb{Z}$; simply, for each $j \in J$, choose (via $\mathsf{AC}$) a non-trivial solution $s_j$ of the $j$-equation $E_j$. Then $\mathbf{s} = \bigcup_{j\in J} s_j$ is a non-trivial solution of the system $(E_j)_{j\in J}$.

**Lemma 5.** *The statement "every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has an uncountable subsystem with a non-trivial solution in $\mathbb{Z}$" is true in the Basic Fraenkel Model of* $\mathsf{ZFA} + \neg\mathsf{AC}$.

PROOF: We recall first the description of the Basic Fraenkel Model, which is labeled as 'Model $\mathcal{N}1$' in [5]: We start with a ground model $\mathcal{M}$ of $\mathsf{ZFA} + \mathsf{AC}$ with a denumerable set $A$ of atoms. The group $G$ of permutations of $A$ used to define the model is the group of all permutations of $A$. For any element $x$ of $\mathcal{M}$, $\mathrm{fix}_G(x)$ denotes the subgroup $\{\phi \in G : \forall t \in x, \phi(t) = t\}$ of $G$ and $\mathrm{Sym}_G(x)$ denotes the subgroup $\{\phi \in G : \phi(x) = x\}$ of $G$. Let $\Gamma$ be the normal filter of subgroups of $G$ generated by the filter base $\{\mathrm{fix}_G(E) : E \in [A]^{<\omega}\}$, where $[A]^{<\omega}$ is the set of finite subsets of $A$. An element $x$ of $\mathcal{M}$ is called *symmetric* if $\mathrm{Sym}_G(x) \in \Gamma$, hence $x$ is symmetric if there is some finite set $E \subset A$ such that $\mathrm{fix}_G(E) \subseteq \mathrm{Sym}_G(x)$. Under these circumstances, $E$ is called *a support* of $x$. The element $x$ of $\mathcal{M}$ is called *hereditarily symmetric* if $x$ and every element in the transitive closure of $x$ is symmetric. $\mathcal{N}1$ is the FM model determined by $\mathcal{M}$, $G$ and $\Gamma$, that is, $\mathcal{N}1$ is the model which consists exactly of the hereditarily symmetric elements of $\mathcal{M}$.

The following facts are known to be true in the model $\mathcal{N}1$ (see [2], [5], [8]) and they will be useful to our proof.

1. The set $A$ of the atoms is amorphous (i.e., the power set $\mathcal{P}(A)$ of $A$ in $\mathcal{N}1$ consists solely of the finite and the cofinite subsets of $A$). Thus, $A$ is a Dedekind-finite set in $\mathcal{N}1$ (i.e., $\aleph_0 \not\leq |A|$ in $\mathcal{N}1$).
2. The power set of a well-orderable set is well-orderable (this is true in every FM model of $\mathsf{ZFA}$; see [5], [8]).
3. If a set $x \in \mathcal{N}1$ is not well-orderable, then there exists an infinite subset $B \subseteq A$ (thus $B$ is cofinite) such that $|B| \leq |x|$.
4. A well-orderable union of well-orderable sets is well-orderable; in particular, a countable union of countable sets is countable.

We turn now to the proof of our result. Let $\mathbf{S} = (E_i)_{i\in I}$ be an uncountable system of linear homogeneous equations over $\mathbb{Z}$ in $\mathcal{N}1$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$. For each $i \in I$ let $X_i$ be the (finite) set of variables appearing in equation $E_i$ and let $F \subset A$ be a finite support for $\mathbf{S}$

(hence, $F$ is also a support for $\{X_i : i \in I\}$). There are two cases for the index set $I$.

**Case 1.** $I$ is well-orderable in $\mathcal{N}1$. Then both $\mathbf{S}$ and $\{X_i : i \in I\}$ are well-orderable in $\mathcal{N}1$, hence by item (2.) we have that the power set $\mathcal{P}(\mathbf{S})$ of $\mathbf{S}$ is also well-orderable in $\mathcal{N}1$. We may follow now the proof of Proposition 1, using the fact that $\mathcal{P}(\mathbf{S})$ is well-orderable (instead of $\mathsf{AC}^{\aleph_0}$ in that proof) and item (4.) in order to verify that $\mathbf{S}$ has a non-trivial assignment in $\mathbb{Z}$. Furthermore, since $\{X_i : i \in I\}$ is a well-orderable uncountable set, we have that $\aleph_1 \leq |\{X_i : i \in I\}|$, and since, $\forall i \in I$, $X_i$ is finite, the statement of item (4.) is true in $\mathcal{N}1$ and $\aleph_1$ is a regular cardinal in $\mathcal{N}1$, we have that the proof of the $\Delta$-system Lemma as given in [9, Theorem 9.18, p. 118] applies in order to obtain an uncountable subfamily of $\{X_i : i \in I\}$ which is a $\Delta$-system. Applying now the proof of Theorem 6, we may conclude that $\mathbf{S}$ has an uncountable subsystem in the model $\mathcal{N}1$ with a non-trivial solution in $\mathbb{Z}$. (The reader should note here that since $|\mathbb{Z}| = \aleph_0$, $\{X_i : i \in I\}$ is well-orderable and $\forall i \in I$, $X_i$ is finite (hence the solution set of each equation of $\mathbf{S}$ is well-orderable), all we need in order to apply the argument in the last paragraph of the proof of Theorem 6 is the Axiom of Choice for well-orderable families of non-empty well-orderable sets, which is true in $\mathcal{N}1$ due to item (4.).)

**Case 2.** $I$ is not well-orderable in $\mathcal{N}1$. Then by item (3.) we have that there exists a cofinite set $B \subseteq A$ such that $|B| \leq |I|$ in $\mathcal{N}1$. Without loss of generality we assume that $B \subseteq I$. Let $F' \supseteq F$ be a support for the uncountable subsystem $\mathbf{T} = (E_b)_{b\in B}$ of $\mathbf{S}$. Since $B$ is Dedekind-finite, it is easy to verify that the set $V = \bigcup\{X_b : b \in B\}$ is not well-orderable in $\mathcal{N}1$, hence by item (3.) again, $V$ contains a cofinite copy of the atoms. For simplicity, and without loss of generality, assume that $V \cap A = B$. Since $F'$ is finite and $B$ is infinite, it follows that there is an element $b \in B$ such that $W_b \neq \emptyset$, where $W_b = (A \cap X_b) \setminus F'$. Let $a \in W_b$, let $F'' = (X_b \cup F') \setminus \{a\}$, and also let

$$\mathbf{U} = \{\phi(E_b) : \phi \in \mathrm{fix}_G(F'')\}.$$

Note that $\mathrm{fix}_G(F'') \in \Gamma$, since $F''$ is a finite set in $\mathcal{N}1$, hence it is well-orderable in $\mathcal{N}1$, and consequently there is a finite set $Q \subset A$ such that $\mathrm{fix}_G(Q) \subseteq \mathrm{fix}_G(F'')$ (see [8, Equation (4.2), p. 47]). It follows that $\mathbf{U} \in \mathcal{N}1$, since $\mathrm{fix}_G(F'') \subseteq \mathrm{Sym}_G(\mathbf{U})$. Now, $F' \subseteq F''$ implies that $\mathrm{fix}_G(F'') \subseteq \mathrm{fix}_G(F')$, and since $\mathrm{fix}_G(F') \subseteq \mathrm{Sym}_G(\mathbf{T})$ (for, $F'$ is a support of $\mathbf{T}$), we have that $\mathrm{fix}_G(F'') \subseteq \mathrm{Sym}_G(\mathbf{T})$, hence $\mathbf{U}$ is a subsystem of $\mathbf{T}$ and therefore it is a subsystem of $\mathbf{S}$ (which clearly contains equation $E_b$). Furthermore, if $\phi_1, \phi_2 \in \mathrm{fix}_G(F'')$ are such that $\phi_1(a) \neq \phi_2(a)$, and if $\lambda a$ is the term of equation $E_b$ that contains $a$, then the left-hand sides of the equations $\phi_1(E_b)$ and $\phi_2(E_b)$ differ only in the terms $\lambda\phi_1(a)$ and $\lambda\phi_2(a)$. It follows that $\mathbf{U}$ has the same cardinality with the $\mathrm{fix}_G(F'')$-orbit of $a$, i.e., with the set

$$\mathrm{Orb}_{\mathrm{fix}_G(F'')}(a) = \{\phi(a) : \phi \in \mathrm{fix}_G(F'')\},$$

which is uncountable, since it is a cofinite subset of $A$.

From our hypothesis that every countable subsystem of $\mathbf{S}$ has a non-trivial solution in $\mathbb{Z}$, it follows that equation $E_b$ also has a non-trivial solution in $\mathbb{Z}$, say $\mathbf{s}$. In view of the observations in the previous paragraph, it follows that the system $\mathbf{U}$ has a non-trivial solution in $\mathbb{Z}$; indeed, define a mapping $\mathbf{t} : X_b \cup \mathrm{Orb}_{\mathrm{fix}_G(F'')}(a) \to \mathbb{Z}$ (note that $X_b \cup \mathrm{Orb}_{\mathrm{fix}_G(F'')}(a)$ is the set of variables of the equations of the system $\mathbf{U}$) as follows:

$$\mathbf{t}(x) = \begin{cases} \mathbf{s}(x) & \text{if } x \in X_b \setminus \{a\} \\ \mathbf{s}(a) & \text{if } x \in \mathrm{Orb}_{\mathrm{fix}_G(F'')}(a). \end{cases}$$

It is clear that $\mathbf{t}$ is a non-trivial solution of $\mathbf{U}$.

The above two cases complete the proof of the lemma. □

**Theorem 7.** *"Every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has an uncountable subsystem with a non-trivial solution in $\mathbb{Z}$" does not imply "every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has a non-trivial solution in $\mathbb{Z}$" in* $\mathsf{ZFA}$.

PROOF: The independence result follows from Lemma 5, the fact that the Axiom of Countable Choice $\mathsf{AC}^{\aleph_0}$ is false in the Basic Fraenkel Model (see [5], [8]), and Theorem 1. □

**Theorem 8.** *The statement "every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has an uncountable subsystem with a non-trivial solution in $\mathbb{Z}$" implies $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$, thus it is not provable in* $\mathsf{ZF}$.

PROOF: Assume the hypothesis. Since $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ is equivalent to its partial version $\mathsf{PAC}^{\aleph_0}_{\mathsf{fin}}$, i.e., "every denumerable family of non-empty finite sets has a partial choice function" (see [3], [5]), it suffices to show that our hypothesis implies $\mathsf{PAC}^{\aleph_0}_{\mathsf{fin}}$. By way of a contradiction, assume that there exists a denumerable disjoint family $\mathcal{A} = \{A_n : n \in \omega\}$ of non-empty finite sets having no partial choice function. We consider the following system of linear equations over $\mathbb{Z}$:

$$x + ny = 0,\ x \in A_0,\ y \in A_n,\ n \in \omega \setminus \{0\}. \tag{13}$$

Similarly to the proof of Theorem 1, one shows that (13) is an uncountable system such that each of its countable subsystems is necessarily finite. Moreover, it is easy to see that every finite subsystem of (13) has a non-trivial solution in $\mathbb{Z}$. Thus, by our hypothesis, (13) has an uncountable subsystem with a non-trivial solution in $\mathbb{Z}$. However, no infinite subsystem of (13) has a non-trivial solution in $\mathbb{Z}$ and we have reached a contradiction. Thus, $\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ holds, finishing the proof of the theorem. □

**Corollary 2.** MC *does not imply "every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has an uncountable subsystem with a non-trivial solution in $\mathbb{Z}$" in* ZFA.

Proof: The Second Fraenkel Model (Model $\mathcal{N}2$ in [5]) satisfies $\mathsf{MC} + \neg\mathsf{AC}^{\aleph_0}_{\mathsf{fin}}$ (see [5], [8]). Hence, by Theorem 8, "every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has an uncountable subsystem with a non-trivial solution in $\mathbb{Z}$" is false in the Second Fraenkel Model. □

**Theorem 9.** *The statement "every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has an uncountable subsystem with a non-trivial solution in $\mathbb{Z}$" implies* PKW-vDCP.

Proof: Assume the hypothesis and let $\mathcal{A} = \{(A_i, \leq_i) : i \in I\}$ be a family as in PKW-vDCP, for which we assume — without loss of generality — that it is disjoint, and further we assume that $I$ is uncountable (due to Theorem 1). Towards a proof by contradiction, assume that $\mathcal{A}$ has no partial Kinna–Wagner selection function. Consider the following linear homogeneous system over $\mathbb{Z}$:

(14) $\quad \forall i \in I, \forall x \in A_i, \forall y \in A_i$ such that $\nexists z \in A_i$ with $x < z < y,\ x + y = 0.$

As in the proof of Theorem 5, the system (14) is uncountable and each of its countable subsystems has a non-trivial solution in $\mathbb{Z}$. By our hypothesis, there is an uncountable subsystem $S$ of (14) with a non-trivial solution in $\mathbb{Z}$, say $\mathbf{s}$. Let $X_S$ be the set of the variables of the equations of $S$. Since $S$ is uncountable and $\forall i \in I$, $|A_i| = \aleph_0$, we may conclude that the set

$$I' = \{i \in I : X_S \cap A_i \neq \emptyset\}$$

is infinite. Then

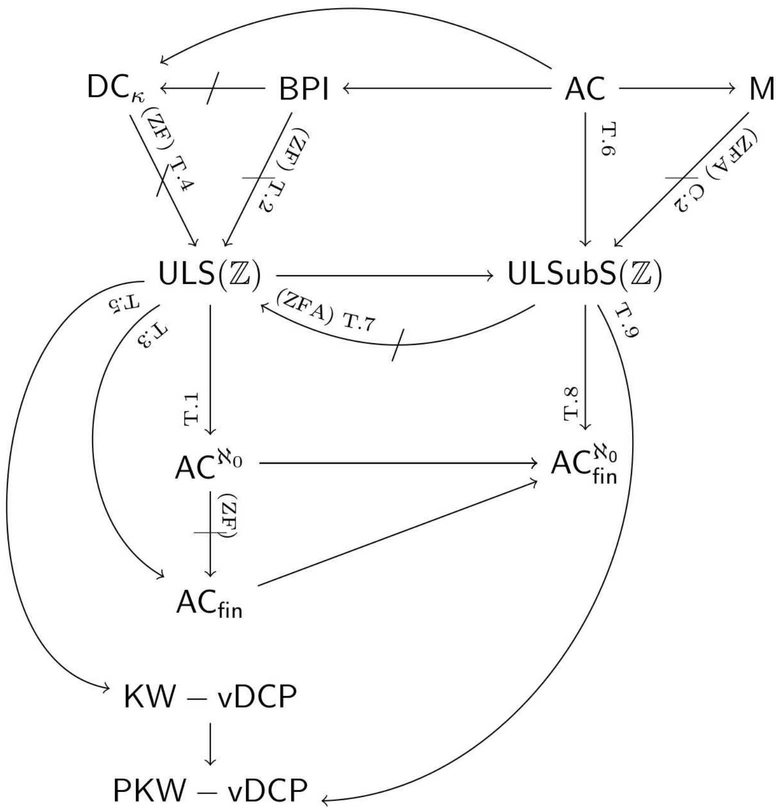
$$g = \{(A_i, \{x \in A_i : \mathbf{s}(x) < 0\}) : i \in I'\}$$

is a partial Kinna–Wagner selection function for $\mathcal{A}$, finishing the proof of the theorem. □

## 3. Diagram of results

In the following diagram, we summarize main results of our paper. Unlabeled arrows or negated arrows, represent implications or non-implications, respectively, that are "known" or "straightforward". Also, in the diagram below, we abbreviate the statements "*every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has a non-trivial solution in $\mathbb{Z}$*" and "*every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution* *in $\mathbb{Z}$, has an uncountable subsystem with a non-trivial solution in $\mathbb{Z}$*" by $\mathsf{ULS}(\mathbb{Z})$ and $\mathsf{ULSubS}(\mathbb{Z})$, respectively.

Lastly, in the diagram, 'T.x' stands for 'Theorem x', 'C.x' stands for 'Corollary x', and '$\kappa$' in '$\mathsf{DC}_\kappa$' runs through the class of regular well-ordered cardinal numbers.




## 4. Problems

1. Does $\mathsf{AC}$ imply $\mathsf{ULS}(\mathbb{Z})$?
2. Does $\mathsf{ULSubS}(\mathbb{Z})$ imply $\mathsf{ULS}(\mathbb{Z})$ in $\mathsf{ZF}$?
3. Does $\mathsf{ULS}(\mathbb{Z})$ imply $\mathsf{AC}^{\mathsf{WO}}$, i.e., $\mathsf{AC}$ restricted to well-orderable families of non-empty sets? Note that if the answer is in the affirmative, then combined with Corollary 1(a), we would have that $\mathsf{ULS}(\mathbb{Z})$ is false in every FM model of $\mathsf{ZFA}$, since there is no FM model in which both $\mathsf{AC}^{\mathsf{WO}}$ and $\mathsf{AC}_{\mathsf{WO}}$ are true (see [4]).
4. Does $\mathsf{ULS}(\mathbb{Z})$ imply van Douwen's Choice Principle ($\mathsf{vDCP}$)?

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