Czechoslovak Mathematical Journal

Azam Babai; Zeinab Akhlaghi

A new characterization of symmetric group by NSE

Czechoslovak Mathematical Journal, Vol. 67 (2017), No. 2, 427–437

Persistent URL: `http://dml.cz/dmlcz/146766`

Czechoslovak Mathematical Journal, 67 (142) (2017), 427–437

A NEW CHARACTERIZATION OF SYMMETRIC GROUP BY NSE

Azam Babai, Qom, Zeinab Akhlaghi, Tehran

Received December 26, 2015. First published March 20, 2017.

Abstract. Let G be a group and $\omega(G)$ be the set of element orders of G. Let $k \in \omega(G)$ and $m_k(G)$ be the number of elements of order k in G. Let $\mathrm{nse}(G) = \{m_k(G) \colon k \in \omega(G)\}$. Assume r is a prime number and let G be a group such that $\mathrm{nse}(G) = \mathrm{nse}(S_r)$, where S_r is the symmetric group of degree r. In this paper we prove that $G \cong S_r$, if r divides the order of G and r^2 does not divide it. To get the conclusion we make use of some well-known results on the prime graphs of finite simple groups and their components.

Keywords: set of the numbers of elements of the same order; prime graph

MSC 2010: 20D06, 20D15

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a group. Denote by $\pi(G)$ the set of primes p such that G contains an element of order p. The set of element orders of G is denoted by $\omega(G)$.

We denote the set of numbers of elements of G of the same order by $\mathrm{nse}(G)$. Set $M_k(G) = \{g \in G \colon g^k = 1\}$. Groups G and H are said to be of the same order type if and only if $|M_k(G)| = |M_k(H)|$, $k = 1, 2, \ldots$, Thompson in 1987 posed a very interesting problem as follows (see [12]).

Thompson's problem. Suppose that groups G and H are of the same order type. If G is solvable, is it true that H is also necessarily solvable?

So far, nobody can solve it perfectly, or even give a counterexample.

We note that there are finite groups which are not characterizable by $\mathrm{nse}(G)$ and $|G|$. In 1987, Thompson gave an example as follows: Let $G_1 = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_7$ and $G_2 = L_3(4) \rtimes C_2$, where both G_1 and G_2 are maximal subgroups of M_{23}. Then $\mathrm{nse}(G_1) = \mathrm{nse}(G_2)$, but $G_1 \ncong G_2$.

DOI: 10.21136/CMJ.2017.0700-15

In [1], it is proved that if $\mathrm{nse}(G) = \mathrm{nse}(A_p)$, where p is a prime and p divides the order of $|G|$ but p^2 does not divide it, then $G \cong A_p$. Also it is proved that $\mathrm{PGL}(2, p)$ is characterizable by nse, where p is a prime and $p^2 \parallel |G|$ (see [2]). Also in [9], we see that $\mathrm{PSL}_2(q)$ can be determined by exactly the set $\mathrm{nse}(\mathrm{PSL}_2(q))$ if $q \leqslant 13$ is a prime power. In [3], Asboei proved that if G is a group such that $\mathrm{nse}(G) = \mathrm{nse}(S_r)$, where r is prime number and $|G| = |S_r|$, then $G \cong S_r$.

In this paper we follow up on these works and as a main result we proved the following theorem:

Main theorem. *Let G be a group such that* $\mathrm{nse}(G) = \mathrm{nse}(S_r)$, *where $r > 5$ is a prime divisor of $|G|$ but r^2 does not divide $|G|$. Then $G \cong S_r$.*

Throughout this paper, by prime graph of G, denoted by $\Gamma(G)$, we mean the graph with the vertex set $\pi(G)$, where two distinct primes r and s are joined by an edge if G contains an element of order rs. Let $s(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1(G), \ldots, \pi_{s(G)}(G)$ be the sets of vertices of connected components of $\Gamma(G)$. If $2 \in \omega(G)$, then $2 \in \pi_1(G)$. We denote by $\varphi(n)$ the Euler totient function. If G is a finite group, then we denote by P_q a Sylow q-subgroup of G and by $n_q(G) = |\mathrm{Syl}_q(G)|$ the number of Sylow q-subgroups of G. In this paper, we say $p^k \parallel n$, if $p^k \mid n$ and $p^{k+1} \nmid n$, and by $|G|_t$ we mean the t-part of $|G|$.

2. Preliminary results

Lemma 2.1. (1) ([10], Lemma 1) *If $n \geqslant 6$ is a natural number, then there are at least $s'(n)$ prime numbers p_i such that $(n+1)/2 < p_i < n$. Here*

- ▷ $s'(n) = 6$ *for* $n \geqslant 48$;
- ▷ $s'(n) = 5$ *for* $42 \leqslant n \leqslant 47$;
- ▷ $s'(n) = 4$ *for* $38 \leqslant n \leqslant 41$;
- ▷ $s'(n) = 3$ *for* $18 \leqslant n \leqslant 37$;
- ▷ $s'(n) = 2$ *for* $14 \leqslant n \leqslant 17$;
- ▷ $s'(n) = 1$ *for* $6 \leqslant n \leqslant 13$.

(2) ([10], Lemma 6 (c)) *Let S be a finite simple group of Lie type with $s(S) \geqslant 2$ and there exists $2 \leqslant i \leqslant s(S)$ such that $k_i(S) = p$. If $S \ncong {}^2G_2(q)$, then for every $1 \leqslant j \leqslant s(S)$, $j \neq i$, there exists at most one prime number $s \in \pi_j(S)$ such that $(p+1)/2 < s < p$. If $S \cong {}^2G_2(q)$, then there exist at most three prime numbers $s \in \pi(S)$ such that $(p+1)/2 < s < p$.*

The next lemma summarizes the basic structural properties of a Frobenius group (see [5], [6], [8]):

Lemma 2.2. *Let G be a Frobenius group and let H, K be the Frobenius complement and Frobenius kernel of G, respectively. Then $s(G) = 2$, and the prime graph components of G are $\pi(H)$, $\pi(K)$. Also we know that K is nilpotent and $|H| \mid |K| - 1$.*

Lemma 2.3 ([4]). *Let G be a finite group and k be a positive integer dividing $|G|$. If $M_k(G) = \{g \in G\colon\ g^k = 1\}$, then $k \mid |M_k(G)|$.*

Let m_n be the number of elements of order n. We note that $m_n = k\varphi(n)$, where k is the number of cyclic subgroups of order n in G.

Lemma 2.4 ([11], Lemma 2.2). *Let G be a group and P be a cyclic Sylow p-subgroup of G of order p^a. If there is a prime r such that $p^a r \in \omega(G)$, then $m_{p^a r} = m_r(C_G(P))m_{p^a}$. In particular, $\varphi(r)m_{p^a} \mid m_{p^a r}$.*

3. Proof of the main theorem

In the following we assume that r is prime. Also let G be a group such that $r \parallel |G|$ and $\mathrm{nse}(G) = \mathrm{nse}(S_r)$. We are going to prove the main theorem using the following lemmas:

Lemma 3.1. *Let $k \in \omega(S_r)$ such that $r \nmid m_k(S_r)$. Then either $k = 1$ or $k = r$.*

P r o o f. We know that $m_k(S_r) = \sum\limits_{o(x_i)=k} |\mathrm{cl}_{S_r}(x_i)|$, where x_i belong to distinct conjugacy classes. Let the cyclic structure of x_i, for every i, be $1^{t_1}2^{t_2}\ldots l^{t_l}$ such that $t_1, t_2, \ldots, t_l$ and $1, 2, \ldots, l$ are not equal to r. Since $|\mathrm{cl}_{S_r}(x_i)| = r!/(1^{t_1}2^{t_2}\ldots l^{t_l}t_1!\,t_2!\ldots t_l!)$, by [7], we conclude that r divides $|\mathrm{cl}_{S_r}(x_i)|$ for every i. Hence $r \mid m_k(S_r)$, which is a contradiction. Consequently, either there exists j such that $t_j = r$, or $1 \leqslant r \leqslant l$. If $t_j = r$, then the cyclic structure of x_i is 1^r, hence $o(x_i) = 1$, so $k = 1$. If $1 \leqslant r \leqslant l$, then the cyclic structure of x_i is r^1, so $o(x_i) = r$, and so $k = r$. □

Lemma 3.2. $m_r(G) = m_r(S_r)$.

P r o o f. We know that $m_r(G) \in \mathrm{nse}(G)$, so there exists $k \in \omega(S_r)$ such that $m_r(G) = m_k(S_r)$. We have

$$r \mid |M_r(G)| = 1 + m_r(G) = 1 + m_k(S_r).$$

Therefore, r does not divide $m_k(S_r)$. By Lemma 3.1, we conclude that $k = r$ and so $m_r(G) = m_r(S_r)$. □

Lemma 3.3. $m_r(G) = (r-1)!$ *and* $n_r(G) = (r-2)!$.

P r o o f. It is clear that Sylow r-subgroups of S_r are cyclic. Therefore, we have $m_r(S_r) = (r-1)n_r(S_r)$. By [13], the number of Sylow r-subgroups of S_r is equal to $r!/(r(r-1)) = (r-2)!$. Then by Lemma 3.2, we get that $m_r(G) = (r-1)!$.

Also since $r \in \pi(G)$ and $r^2 \nmid |G|$, Sylow r-subgroups of G are cyclic. Then

$$(r-1)! = m_r(G) = \varphi(r)n_r(G) = (r-1)n_r(G),$$

which implies that $n_r(G) = (r-2)!$. □

Lemma 3.4. *For every* $t \in \pi(G)$, *we have* $tr \notin \omega(G)$.

P r o o f. On the contrary, assume that $tr \in \omega(G)$ for some $t \in \pi(G)$. Since Sylow r-subgroups of G are cyclic, we conclude $m_{rt}(G) = m_r(G)(t-1)k$, where k is the number of cyclic subgroups of order t in $C_G(R)$ and $R \in \mathrm{Syl}_r(G)$, by Lemma 2.4. If $r \mid t-1k$, then $m_{rt}(G) \geqslant |S_r|$, which is impossible. Thus $r \nmid (t-1)k$ and so $r \nmid m_{rt}(G)$. It follows that $m_r(G) = m_{rt}(G)$, by Lemma 3.1. It follows that $t = 2$. We know that

$$2r \mid |M_{2r}(G)| = 1 + m_2(G) + m_r(G) + m_{2r}(G) = 1 + m_2(G) + 2m_r(G).$$

Since $m_2(G)$ is odd, $m_2(G) \neq m_r(G)$, hence $r \mid m_2(G)$, by Lemma 3.1. Also we have $r \mid 1 + m_r(G)$, so $r \mid m_r(G)$, which is a contradiction. □

Remark 3.5. By Lemma 3.4, r is an isolated vertex in $\Gamma(G)$ and so $s(G) \geqslant 2$ and $\Gamma(G)$ is disconnected.

Lemma 3.6. *If* $t \in \pi(G)$ *and* $t \neq r$, *then* $|G|_t \leqslant (m_r(G))_t$.

P r o o f. Let $x \in G \setminus \{1\}$ and $o(x) = r$. We have that $C_G(x)$ is a r-group, by the previous lemma. Since $|G : C_G(x)| = |\mathrm{cl}_G(x)|$, $|\mathrm{cl}_G(x)|$ is a r'-number. Therefore, $|G|_t = |\mathrm{cl}_G(x)|_t$ for every $t \in \pi(G)$. We recall that $m_r(G) = \sum\limits_{o(x)=r} |\mathrm{cl}_G(x)|$, where the x's belong to distinct conjugacy classes, so the lemma follows. □

Lemma 3.7. $r(r-2)! \mid |G|$ *and* $|G| \mid r!$ *and hence* $\pi(G) = \pi(S_r)$.

P r o o f. By Lemma 3.6, $|G| \mid r[((r-1)!)_2((r-1)!)_3\ldots] = r!$, therefore $|G| \mid r!$. On the other hand, we have $(r-2)! \mid |G|$, by Lemma 3.3 and by the fact that $|G|_r = r$, then the statement of the lemma follows. □

Lemma 3.8. *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a non-abelian simple group, $s(K/H) \geqslant 2$ and $t \in \pi(K/H)$ for every prime $(r+1)/2 \leqslant t \leqslant r$.*

Proof. Let K/H be a chief factor of G, whose order is divisible by r. So $K/H \cong (S)^k$, where S is a simple group. We know that $r^2 \nmid |K/H|$, hence $k = 1$ and so K/H is a simple group. It follows that either K/H is a non-abelian simple group or $|K/H| = r$.

Let $R \in \mathrm{Syl}_r(K)$. Since $r^2 \nmid |G|$, we have $r \nmid |H|$. Consequently, $H \rtimes R$ is a Frobenius group, by Lemma 3.4. Hence H is nilpotent, by Lemma 2.2. Let $t \in \pi(G)$ be a prime number such that $(r+1)/2 \leqslant t < r$. We claim that $t \in \pi(K/H)$.

On the contrary, we consider the two following cases:

Case 1. Let $t \in \pi(H)$. If $T \in \mathrm{Syl}_t(G)$, then $|T| = t$, by Lemma 3.7. Since H is nilpotent, $T \trianglelefteq K$. Similar to the above discussion, $T \rtimes R$ is a Frobenius group and so $r \mid t-1$, which is a contradiction.

Case 2. Let $t \in \pi(G/K)$. By Frattini's argument, we have $G/K \cong N_G(R)/N_K(R)$. It follows that $N_G(R)$ has a Sylow t-subgroup T and $|T| = t$. Similar to the above discussion, $R \rtimes T$ is a Frobenius group and so $t \mid r-1$, which is a contradiction.

Consequently, for every prime number t such that $(r+1)/2 \leqslant t \leqslant r$, we have $t \in \pi(K/H)$. Therefore, K/H is a non-abelian simple group and $s(K/H) \geqslant 2$, by Lemma 3.4. Moreover, there exists $j \geqslant 2$ such that $\pi_j(K/H) = \{r\}$. $\square$

Theorem 3.9. *K/H is isomorphic neither to any finite simple group of Lie type nor any sporadic simple group.*

Proof. First, on the contrary, assume that K/H is isomorphic to a finite simple group of Lie type. It is well-known that the number of connected components of a finite simple group of Lie type is at most 5, so $s(K/H) \leqslant 5$. By Lemma 3.8, we have that $\{r\}$ is one of the components of $\Gamma(K/H)$. From Lemma 2.1, we obtain that if $K/H \cong {}^2G_2(q)$, then $s'(r) \leqslant 3$ and if $K/H \ncong {}^2G_2(q)$, then by part (2) of Lemma 2.1, we have that every connected component of $\Gamma(K/H)$, except for the component that has r as its single element, has at most one element which lies between $(r+1)/2$ and r. Also we know that every prime p such that $(r+1)/2 \leqslant p \leqslant r$ divides $|K/H|$. So $s'(r) \leqslant s(K/H) - 1$. In the sequel we consider each possibility for $s(K/H)$.

(1) Let $s(K/H) = 2$.

Hence $s'(r) \leqslant 1$, so $r \in \{7, 11, 13\}$, by Lemma 2.1. In this case, we have the following cases:

Case 1-1: Let $K/H \cong B_n(q)$, where $n = 2^m \geqslant 4$.

According to Lemma 3.8, we have $\pi_2(B_n(q)) = \{r\}$, so $(q^n + 1)/2 = r^\alpha$, where α is a natural number. Since $r^2 \nmid |G|$, so $\alpha = 1$ and $(q^n + 1)/2 = r$. Let $r = 7$, hence

G	$\pi_1(G)$	n_2
A_n, $6 < n = p, p+1, p+2$, n or $n-2$ is not prime	$\pi((n-3)!)$	p
$A_{p-1}(q)$, $(p,q) \neq (3,2), (3,4)$	$\pi\Big(q \prod_{i=1}^{p-1}(q^i - 1)\Big)$	$\dfrac{q^p - 1}{(q-1)(p, q-1)}$
$A_p(q)$, $q - 1 \mid p + 1$	$\pi\Big(q(q^{p+1} - 1)\prod_{i=2}^{p-1}(q^i - 1)\Big)$	$\dfrac{q^p - 1}{q - 1}$
${}^2A_{p-1}(q)$	$\pi\Big(q \prod_{i=1}^{p-1}(q^i - (-1)^i)\Big)$	$\dfrac{(q^p + 1)}{(q+1)(p, q+1)}$
${}^2A_p(q)$, $q + 1 \mid p + 1$, $(p,q) \neq (3,3), (5,2)$	$\pi\Big(q(q^{p+1} - 1)\prod_{i=2}^{p-1}(q^i - (-1)^i)\Big)$	$\dfrac{q^p + 1}{q + 1}$
${}^2A_3(2)$	$\{2, 3\}$	5
$B_n(q)$, $2 \nmid q$, $n = 2^m \geqslant 4$	$\pi\Big(q(q^n - 1)\prod_{i=1}^{n-1}(q^{2i} - 1)\Big)$	$(q^n + 1)/2$
$B_p(3)$	$\pi\Big(3(3^p + 1)\prod_{i=1}^{p-1}(3^{2i} - 1)\Big)$	$(3^p - 1)/2$
$C_n(q)$, $n = 2^m \geqslant 2$	$\pi\Big(q(q^n - 1)\prod_{i=1}^{n-1}(q^{2i} - 1)\Big)$	$\dfrac{q^n + 1}{(2, q-1)}$
$C_p(q)$, $q = 2, 3$	$\pi\Big(q(q^p + 1)\prod_{i=1}^{p-1}(q^{2i} - 1)\Big)$	$\dfrac{q^p - 1}{(2, q-1)}$
$D_p(q)$, $p \geqslant 5$, $q = 2, 3, 5$	$\pi\Big(q \prod_{i=1}^{p-1}(q^{2i} - 1)\Big)$	$\dfrac{q^p - 1}{q - 1}$
$D_{p+1}(q)$, $q = 2, 3$	$\pi\Big(q(q^p + 1)(q^{p+1} - 1)\prod_{i=1}^{p-1}(q^{2i} - 1)\Big)$	$\dfrac{q^p - 1}{(2, q-1)}$
${}^2D_n(q)$, $n = 2^m \geqslant 4$	$\pi\Big(q \prod_{i=1}^{n-1}(q^{2i} - 1)\Big)$	$\dfrac{q^n + 1}{(2, q+1)}$
${}^2D_n(2)$, $n = 2^m + 1 \geqslant 5$	$\pi\Big(2(2^n + 1)(2^{n-1} - 1)\prod_{i=1}^{n-2}(2^{2i} - 1)\Big)$	$2^{n-1} + 1$
${}^2D_p(3)$, $p \neq 2^m + 1$, $p \geqslant 5$	$\pi\Big(3 \prod_{i=1}^{p-1}(3^{2i} - 1)\Big)$	$(3^p + 1)/4$
${}^2D_n(3)$, $n = 2^m + 1 \neq p$, $m \geqslant 2$	$\pi\Big(3(3^n + 1)(3^{n-1} - 1)\prod_{i=1}^{n-2}(3^{2i} - 1)\Big)$	$(3^{n-1} + 1)/2$
$G_2(q)$, $q \equiv \varepsilon \pmod 3$, $\varepsilon = \pm 1$, $q > 2$	$\pi(q(q^3 - \varepsilon)(q^2 - 1)(q + \varepsilon))$	$q^2 - \varepsilon q + 1$
${}^3D_4(q)$	$\pi(q(q^6 - 1)(q^2 - 1)(q^4 + q^2 + 1))$	$q^4 - q^2 + 1$
$F_4(q), 2 \nmid q$	$\pi(q(q^8 - 1)(q^6 - 1)(q^2 - 1))$	$q^4 - q^2 + 1$
${}^2F_4(2)'$, $2 \nmid q$	$\{2, 3, 5\}$	13
$E_6(q)$	$\pi(q(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^5 - 1))$	$\dfrac{q^6 + q^3 + 1}{(3, q-1)}$
${}^2E_6(q)$	$\pi(q(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^5 + 1))$	$\dfrac{q^6 - q^3 + 1}{(3, q+1)}$

Table 1. The odd order components of the finite simple group K/H, where $s(K/H) = 2$.

$(q^n+1)/2=7$ and so $q^n=13$. It follows that $n=1$, which is not possible by our assumption.

Similarly, for $r=11$ and $r=13$, we get a contradiction.

Case 1-2: Let $K/H \cong B_p(3)$, where p is prime.

Similar to the above case, we have $(3^p-1)/2=r$. Then $r=13$ and $p=3$. But $11 \nmid |B_3(3)|$, which is a contradiction by Lemma 3.8.

Completely similar to the above two cases, K/H cannot be isomorphic to the groups below:

- $C_n(q)$, where $n=2^m \geqslant 2$;
- $C_p(q)$, where p is prime and $q \in \{2,3\}$;
- $D_p(q)$, where $q \in \{2,3,5\}$, $p \geqslant 5$;
- $D_{p+1}(q)$, where p is odd prime and $q \in \{2,3\}$;
- ${}^2D_n(q)$, where $n=2^m \geqslant 4$;
- ${}^2D_n(2)$, where $n=2^m+1 \geqslant 5$;
- ${}^2D_p(3)$, where $p \neq 2^m+1$ and $p \geqslant 5$ is prime;
- ${}^2D_n(3)$, where $n=2^m+1$, n is not prime and $m \geqslant 2$;
- $G_2(q)$, where $q \equiv \pm 1 \pmod 3$ and $q>2$;
- ${}^3D_4(q)$;
- $F_4(q)$, where q is odd;
- $E_6(q)$;
- ${}^2E_6(q)$;
- ${}^2F_4(2)'$.

(2) Let $s(K/H)=3$.

Then $s'(r) \leqslant 2$ and $r \in \{7,11,13,17\}$, by Lemma 2.1. In this case, $\pi_j(K/H)=\{r\}$ for some $j \in \{2,3\}$. We consider the following cases:

Case 2-1: Let $K/H \cong A_1(q)$, where $q \equiv 1 \pmod 4$.

Then similar to Case 1-1, we get that either $q=r$ or $(q+1)/2=r$.

Let $r=13$. If $q=r$, then $|A_1(q)|=156$. So $11 \nmid |A_1(q)|$, which is a contradiction by Lemma 3.8. Otherwise, $(q+1)/2=r$, which implies that $q=25$ and $11 \nmid |A_1(q)|$, which is a contradiction.

Similarly for $r \in \{7,11,17\}$, we get a contradiction.

Similarly we get that $K/H \ncong A_1(q)$, where $q \equiv -1 \pmod 4$ and $K/H \ncong A_1(q)$, where $q=2^m$.

Case 2-2: Let $K/H \cong {}^2A_5(2)$.

Then $r=11$. By comparing the orders of G and K/H we get a contradiction, since $13 \in \pi(G)$ and $13 \notin \pi(K/H)$.

Case 2-3: Let K/H be isomorphic to one of the following groups:

- ${}^2D_p(3)$, where $p=2^n+1$ and $n \geqslant 2$;

- ▷ $F_4(q)$, where $2 \mid q$;
- ▷ ${}^2F_4(q)$, where $q = 2^{2m+1} > 2$;
- ▷ $G_2(q)$, where $3 \mid q$;
- ▷ ${}^2G_2(q)$;
- ▷ $E_7(2)$;
- ▷ $E_7(3)$;
- ▷ A_p, where p and $p-2$ are prime.

In all of the above cases we get a contradiction by comparing the orders of K/H and G, using Lemma 3.8 and also the fact that $\pi_j(K/H) = \{r\}$ for some $j \in \{2, 3\}$.

G	$\pi_1(G)$	m_2	m_3
A_p, p and $p-2$ are prime	$\pi((p-3)!(p-1))$	$p-2$	p
$A_1(q)$, $4 \mid q+1$	$q+1$	q	$\frac{1}{2}(q-1)$
$A_1(q)$, $4 \mid q-1$	$q-1$	q	$\frac{1}{2}(q+1)$
$A_1(q)$, $2 \mid q$	q	$q+1$	$q-1$
$A_2(2)$	$\{2\}$	3	7
${}^2A_5(2)$	$\{2, 3, 5\}$	7	11
${}^2D_p(3)$, $p = 2^n+1 \geqslant 5$	$\pi\Big(2\cdot 3(3^{p-1}-1)\prod_{i=1}^{p-2}(3^{2i}-1)\Big)$	$\frac{1}{2}(3^{p-1}+1)$	$\frac{1}{4}(3^p+1)$
$E_7(2)$	$\{2, 3, 5, 7, 11, 13, 17, 19, 31, 43\}$	73	127
$F_4(q)$, $q = 2^n > 2$	$\pi(q(q^6-1)(q^4-1))$	q^4+1	q^4-q^2+1
${}^2F_4(q)$, $q = 2^{2n+1} > 2$	$\pi(q(q^4-1)(q^3+1))$	$q^2-\sqrt{2q^3}+q-\sqrt{2q}+1$	$q^2+\sqrt{2q^3}+q+\sqrt{2q}+1$
$G_2(q)$, $3 \mid q$	$\pi(q(q^2-1))$	q^2+q+1	q^2-q+1
${}^2G_2(q)$, $q = 3^{2n+1}$	$\pi(q(q^2-1))$	$q-\sqrt{3q}+1$	$q+\sqrt{3q}+1$
$E_7(3)$	$\{2, 3, 5, 7, 11, 13, 19, 37, 41, 61, 73, 547\}$	757	1093

Table 2. The odd order components of the finite simple group K/H, where $s(K/H) = 3$.

(3) Let $s(K/H) = 4$ or $K/H \cong {}^2G_2(q)$.

Hence $s'(r) \leqslant 3$ and so $r \in T = \{7, 11, 13, 17, 19, 23, 29, 31, 37\}$. In this case, $\pi_j(K/H) = \{r\}$ for some $2 \leqslant j \leqslant 4$.

Let $K/H \cong {}^2B_2(q)$, where $q = 2^{2n+1} > 2$.

Then either $q-1 = r$, $q+\sqrt{2q}+1 = r$, or $q-\sqrt{2q}+1 = r$ and using the fact that $r \in T$, we get a contradiction.

By a similar method we get a contradiction when K/H is isomorphic to either $A_2(4)$, ${}^2E_6(2)$, or $E_8(q)$, where $q \equiv 2, 3 \pmod 5$.

G	$\pi_1(G)$	m_2	m_3	m_4
$A_2(4)$	$\{2\}$	5	7	9
${}^2B_2(q)$, $q = 2^{2n+1} > 2$	$\{2\}$	$q - \sqrt{2q} + 1$	$q + \sqrt{2q} + 1$	$q - 1$
${}^2E_6(2)$	$\{2, 3, 5, 7, 11\}$	13	17	19

Table 3. The odd order components of the finite simple group K/H, where $s(K/H) = 4$.

	If $q \equiv 0, 1, 4 \pmod 5$
m_1	$q^{120}(q^{18} - 1)(q^{14} - 1)(q^{12} - 1)(q^{10} - 1)(q^8 - 1)(q^4 + q^2 + 1)$
m_2	$(q^8 + q^7 - q^5 - q^4 - q^3 + q + 1)$
m_3	$(q^8 - q^7 + q^5 - q^4 + q^3 - q + 1)$
m_4	$(q^8 - q^6 + q^4 - q^2 + 1)$
m_5	$(q^8 - q^4 + 1)$

	If $q \equiv 2, 3 \pmod 5$
m_1	$q^{120}(q^{20} - 1)(q^{18} - 1)(q^{14} - 1)(q^{12} - 1)(q^{10} - 1)(q^8 - 1)(q^4 + 1)(q^4 + q^2 + 1)$
m_2	$(q^8 + q^7 - q^5 - q^4 - q^3 + q + 1)$
m_3	$(q^8 - q^7 + q^5 - q^4 + q^3 - q + 1)$
m_4	$(q^8 - q^4 + 1)$

Table 4. The odd order components of $E_8(q)$.

(4) Let $s(K/H) = 5$.

So $K/H \cong E_8(q)$, where $q \equiv 0, 1, 4 \pmod 5$. In this case, since $s(K/H) = 5$, we have, by Lemma 2.1, $s'(K/H) \leqslant 4$ and $r \in \{5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41\}$, which can be excluded analogously to the above cases.

Consequently, K/H is not isomorphic to any simple group of Lie type.

Now assuming that K/H is a sporadic group, we consider the following cases:

Case 1: Let $K/H \cong M_{12}$.

We have $r = 11$, but we can see that $7 \nmid |K/H|$, which contradicts Lemma 3.8.

If K/H is isomorphic to HN, Ru, He, Co_1, Co_3, Co_2, M_{11}, M_{23}, M_{24}, J_1, J_3, J_4, F_{23}, F_1, F_2, F_3, ON, Ly, F'_{24}, then we produce a contradiction similarly.

Case 2: Let $K/H \cong J_2$.

We have $r = 7$. By comparing the orders of K/H and G we get a contradiction.

The method for excluding $M\mathrm{cl}$, Fi_{22}, HS, SZ is the same.

Case 3: Let $K/H \cong M_{22}$.

By considering the order of K/H and G, we see that $|H|_3 = 3^2$. Let P be the Sylow 3-subgroup of H, which is normal in G. Then we see that a Sylow 11-subgroup of G acts fixed point freely on P and so G has a Frobenius subgroup of order 99, which is impossible, since $11 \nmid 3^2 - 1$. □

Corollary 3.10. *K/H is isomorphic to A_r.*

P r o o f. By Lemma 3.8, we get that K/H is isomorphic to a non-abelian simple group. By Theorem 3.9, it follows that $K/H \cong A_n$ for some integer n. By considering the orders of G and K/H it is easy to see that $K/H \cong A_r$. $\square$

Theorem 3.11. *G is isomorphic to S_r.*

P r o o f. Let $\overline{G} = G/H$ and $\overline{K} = K/H$. We know that $A_r \cong \overline{K} \cong \overline{K}C_{\overline{G}}(\overline{K})/C_{\overline{G}}(\overline{K}) \leqslant \overline{G}/C_{\overline{G}}(\overline{K}) \cong N_{\overline{G}}(\overline{K})/C_{\overline{G}}(\overline{K}) \leqslant \mathrm{Aut}(A_r) \cong S_r$. On the other hand, G has a normal subgroup, say M, such that $G/M \cong \overline{G}/C_{\overline{G}}(\overline{K})$. So we have either $G/M \cong A_r$ or $G/M \cong S_r$. Let the first case occur. If $M = 1$, then $G \cong A_r$, which is a contradiction, since $(r-1)! \notin \mathrm{nse}(A_r)$ and so $\mathrm{nse}(A_r) \neq \mathrm{nse}(S_r)$. As $|G| \mid |S_r|$, we conclude that $|M| = 2$. So M is a normal subgroup of order 2 of G and then $M \subseteq \mathbf{Z}(G)$. It follows that there is an element of order $2r$ in G, which is a contradiction. Now assume $G/M = S_r$. Since $|G| \mid |S_r|$, we have $M = 1$ and $G \cong S_r$ as we wanted. $\square$

Corollary 3.12 follows immediately from the main theorem.

Corollary 3.12. *Let G be a finite group and r be a prime number. If* $\mathrm{nse}(G) = \mathrm{nse}(S_r)$ *and $|G| = |S_r|$, then $G \cong S_r$.*

In view of the results obtained in the paper, we propose the following conjecture:

Conjecture. Let G be a finite group and r be prime. If $\mathrm{nse}(G)=\mathrm{nse}(S_r)$, then $G \cong S_r$.

References

[1] *N. Ahanjideh, B. Asadian*: NSE characterization of some alternating groups. J. Algebra Appl. *14* (2015), Article ID 1550012, 14 pages.
[2] *A. K. Asboei*: A new characterization of $\mathrm{PGL}(2,p)$. J. Algebra Appl. *12* (2013), Article ID 1350040, 5 pages.
[3] *A. K. Asboei, S. S. S. Amiri, A. Iranmanesh, A. Tehranian*: A characterization of symmetric group S_r, where r is prime number. Ann. Math. Inform. *40* (2012), 13–23.
[4] *G. Frobenius*: Verallgemeinerung des Sylow'schen Satzes. Berl. Ber. (1895), 981–993. (In German.)
[5] *D. Gorenstein*: Finite Groups. Harper's Series in Modern Mathematics, Harper and Row, Publishers, New York, 1968.
[6] *K. W. Gruenberg, K. W. Roggenkamp*: Decomposition of the augmentation ideal and of the relation modules of a finite group. Proc. Lond. Math. Soc., III. Ser. *31* (1975), 149–166.
[7] *M. Hall,* Jr.: The Theory of Groups. The Macmillan Company, New York, 1959.
[8] *B. Huppert*: Endliche Gruppen. I. Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen 134, Springer, Berlin, 1967. (In German.)

[9] *M. Khatami, B. Khosravi, Z. Akhlaghi*: A new characterization for some linear groups. Monatsh. Math. *163* (2011), 39–50.
[10] *A. S. Kondrat'ev, V. D. Mazurov*: Recognition of alternating groups of prime degree from their element orders. Sib. Math. J. *41* (2000), 294–302; translation from Sib. Mat. Zh. *41* (2000), 359–369. (In Russian.)
[11] *C. Shao, Q. Jiang*: A new characterization of some linear groups by nse. J. Algebra Appl. *13* (2014), Article ID 1350094, 9 pages.
[12] *W. J. Shi*: A new characterization of the sporadic simple groups. Group Theory. Proc. Conf., Singapore, 1987, Walter de Gruyter, Berlin, 1989, pp. 531–540.
[13] *L. Weisner*: On the Sylow subgroups of the symmetric and alternating groups. Am. J. Math. *47* (1925), 121–124.

Authors' addresses: Azam Babai, Department of Mathematics, University of Qom, Alghadir Blvd., Qom, P.O. Box 37185-3766, Iran, e-mail: a_babai@aut.ac.ir; Zeinab Akhlaghi, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, District 6, Hafez Avenue No. 424, 15914 Tehran, Iran, e-mail: z_akhlaghi@aut.ac.ir.