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HILBERT-SCHMIDT HANKEL OPERATORS WITH ANTI-HOLOMORPHIC SYMBOLS ON COMPLETE PSEUDOCONVEX REINHARDT DOMAINS

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Abstract. On complete pseudoconvex Reinhardt domains in $\mathbb{C}^2$, we show that there is no nonzero Hankel operator with anti-holomorphic symbol that is Hilbert-Schmidt. In the proof, we explicitly use the pseudoconvexity property of the domain. We also present two examples of unbounded non-pseudoconvex domains in $\mathbb{C}^2$ that admit nonzero Hilbert-Schmidt Hankel operators with anti-holomorphic symbols. In the first example the Bergman space is finite dimensional. However, in the second example the Bergman space is infinite dimensional and the Hankel operator $H_{\overline{z}_1\overline{z}_2}$ is Hilbert-Schmidt.

Keywords: canonical solution operator for $\overline{\partial}$-problem; Hankel operator; Hilbert-Schmidt operator

MSC 2010: 47B35, 32A36, 47B10

1. Introduction

1.1. Setup and problem. For a domain Ω in $\mathbb{C}^n$, we denote the space of square integrable functions and the space of square integrable holomorphic functions on Ω by $L^2(\Omega)$ and $A^2(\Omega)$ (the Bergman space of Ω), respectively. The Bergman projection operator, P, is the orthogonal projection from $L^2(\Omega)$ onto $A^2(\Omega)$. It is an integral operator with the kernel called the Bergman kernel, which is denoted by $B_\Omega(z,w)$. Moreover, if $\{e_n(z)\}_{n=0}^\infty$ is an orthonormal basis for $A^2(\Omega)$ then the Bergman kernel can be represented as

$$B_\Omega(z,w)=\sum_{n=0}^{\infty} e_n(z)\overline{e_n(w)}.$$

On complete Reinhardt domains the monomials $\{z^\gamma\}_{\gamma\in\mathbb{N}^n}$ (or a subset of them) constitute an orthogonal basis for $A^2(\Omega)$.

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For $f \in A^2(\Omega)$, the Hankel operator with the anti-holomorphic symbol $\overline{f}$ is formally defined on $A^2(\Omega)$ by

$$H_{\overline{f}}(g) = (I - P)(\overline{f}g).$$

Note that this (possibly unbounded) operator is densely defined on $A^2(\Omega)$.

For a multi-index $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}^n$, we set

$$c_\gamma^2 = \int_\Omega |z^\gamma|^2 \, \mathrm{d}V(z). \tag{1}$$

Then on complete Reinhardt domains the set $\{z^\gamma/c_\gamma\}_{\gamma\in\mathbb{N}^n}$ gives a complete orthonormal basis for $A^2(\Omega)$. Each $f \in A^2(\Omega)$ can be written in the form $f(z) = \sum\limits_{\gamma\in\mathbb{N}^n} f_\gamma z^\gamma/c_\gamma$ where the sum converges in $A^2(\Omega)$, but also uniformly on compact subsets of Ω. For the coefficients f_γ, we have $f_\gamma = \langle f(z), z^\gamma/c_\gamma\rangle_\Omega$.

Definition 1. A linear bounded operator T on a Hilbert space H is called a *Hilbert-Schmidt operator* if there is an orthonormal basis $\{\xi_j\}$ for H such that the sum $\sum\limits_{j=1}^{\infty} \|T(\xi_j)\|^2$ is finite.

The sum does not depend on the choice of the orthonormal basis $\{\xi_j\}$. For more on Hilbert-Schmidt operators see [10], Section X.

In this paper, we investigate the following problem. On a given Reinhardt domain in $\mathbb{C}^n$, characterize the symbols for which the corresponding Hankel operators are Hilbert-Schmidt. This question was first studied in $\mathbb{C}$ on the unit disc in [2]. The problem was studied on higher dimensional domains in [13], Theorem at page 2, where the author showed that when $n \geqslant 2$, on an n-dimensional complex ball there are no nonzero Hilbert-Schmidt Hankel operators (with anti-holomorphic symbols) on the Bergman space. The result was revisited in [11] with a more robust approach. On more general domains in higher dimensions, the problem was explored in [6], Theorem 1.1, where the authors extended the result [13], Theorem at page 2, to bounded pseudoconvex domains of finite type in $\mathbb{C}^2$ with smooth boundary. Moreover, the authors of the current article studied the same problem on complex ellipsoids [3], in $\mathbb{C}^2$ with not necessarily smooth boundary.

The same question was investigated on Cartan domains of tube type in [1], Section 2, and on strongly psuedoconvex domains in [8], [9]. Arazy studied the natural generalization of Hankel operators on Cartan domains (circular, convex, irreducible bounded symmetric domains in $\mathbb{C}^n$) of tube type and rank $r > 1$ in $\mathbb{C}^n$ for which n/r is an integer. He showed that there is no non-trivial Hilbert-Schmidt Hankel operator with anti-holomorphic symbols on those type of domains. Li and Peloso, independently, obtained the same result on strongly pseudoconvex domains with smooth boundary.

1.2. Results. Let

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 \colon\ z_1 \in \mathbb{D} \ \text{ and } \ |z_2| < \mathrm{e}^{-\varphi(z_1)}\}$$

$(\varphi(z_1) = \varphi(|z_1|))$ be a complete pseudoconvex Reinhardt domain where monomials $\{z^\alpha\}$ (or a subset of monomials) form a complete system for $A^2(\Omega)$. In this paper, we show that on complete pseudoconvex Reinhardt domains in $\mathbb{C}^2$ there are no nonzero Hilbert-Schmidt Hankel operator with anti-holomorphic symbols. Moreover, we also present examples of unbounded non-pseudoconvex domains on which there are nonzero Hilbert-Schmidt Hankel operators with anti-holomorphic symbols.

Theorem 1. *Let Ω be as above and $f \in A^2(\Omega)$. If the Hankel operator $H_{\overline{f}}$ is Hilbert-Schmidt on $A^2(\Omega)$ then f is constant.*

Remark 1. Theorem 1 generalizes Zhu's result on the unit ball in $\mathbb{C}^n$, see [13], Schnider's result on the unit ball in $\mathbb{C}^n$ and its variations, see [11]. Theorem 1 also generalizes the result in [6], Theorem 1.1, by dropping the finite type condition on complete pseudoconvex Reinhardt domains.

Remark 2. The new ingredient in the proof of Theorem 1 is the explicit use of the pseudoconvexity property of the domain Ω, see the assumption made at (6) and how it is used at (10). Additionally, we employ the key estimate (4) proven in [3].

Remark 3. After completing this note, the authors have learned that by using the estimate (4), Le obtained the same result on bounded complete Reinhardt domains without the pseudoconvexity assumption, see [7]. Although our statement requires pseudoconvexity, it also works on unbounded domains. The complex function theory on unbounded domains (and its relation to pseudoconvexity) has been investigated recently in [4], [5] and new phenomenas have been observed.

Wiegerinck in [12] constructed Reinhardt domains (unbounded but with finite volume) in $\mathbb{C}^2$ for which the Bergman spaces are k-dimensional. In fact, for these domains the Bergman spaces are spanned by monomials of the form $\{(z_1z_2)^j\}_{j=1}^{k-1}$. Therefore, Hankel operators with nontrivial anti-holomorphic symbols are Hilbert-Schmidt. We revisit these and similar domains in the last section to present examples of domains that admit nonzero Hilbert-Schmidt Hankel operators with anti-holomorphic symbols.

2. An identity and an estimate on Reinhardt domains

The set $\{z^\gamma/c_\gamma\}_{\gamma\in\mathbb{N}^n}$ is an orthonormal basis for $A^2(\Omega)$. In order to prove Theorem 1, we will look at the sum

$$\sum_\gamma \Big\| H_{\overline{f}}\Big(\frac{z^\gamma}{c_\gamma}\Big)\Big\|^2 = \sum_\alpha |f_\alpha|^2 \sum_\gamma \Big(\frac{c^2_{\alpha+\gamma}}{c^2_\gamma} - \frac{c^2_\gamma}{c^2_{\gamma-\alpha}}\Big) \tag{2}$$

for $f \in A^2(\Omega)$. For detailed computation of (2) and of the later estimate (4) we refer to [3].

The term $\sum\limits_\gamma (c^2_{\gamma+\alpha}/c^2_\gamma - c^2_\gamma/c^2_{\gamma-\alpha})$ in the identity (2) plays an essential role in the rest of the proof, and we label it as,

$$S_\alpha := \sum_\gamma \Big(\frac{c^2_{\gamma+\alpha}}{c^2_\gamma} - \frac{c^2_\gamma}{c^2_{\gamma-\alpha}}\Big). \tag{3}$$

Note that the Cauchy-Schwarz inequality guarantees that $c^2_{\gamma+\alpha}/c^2_\gamma - c^2_\gamma/c^2_{\gamma-\alpha} \geqslant 0$ for all α and γ.

The computations above hold on any domains where the monomials (or a subset of monomials) form an orthonormal basis for the Bergman space.

Now, we estimate the term S_α on complete pseudoconvex Reinhardt domains. Our goal is to show that S_α diverges for all nonzero α on these domains. By (2), this will be sufficient to conclude Theorem 1.

In earlier results, S_α's were computed explicitly to obtain the divergence. Here we obtain the divergence by using the estimate (4):

For any sufficiently large N, we have

$$S_\alpha \geqslant \sum_{|\gamma|=N} \frac{c^2_{\gamma+\alpha}}{c^2_\gamma} \tag{4}$$

for any nonzero α, see [3].

3. Computations on complete pseudoconvex Reinhardt domains, proof of Theorem 1

Let $\varphi(r) \in C^2([0,1))$, define the complete Reinhardt domain

$$\Omega = \{(z_1,z_2)\in\mathbb{C}^2\colon\ z_1\in\mathbb{D} \quad \text{and} \quad |z_2| < \mathrm{e}^{-\varphi(z_1)}\}.$$

Note that $\varphi(z_1) = \varphi(|z_1|)$.

If $\limsup_{r\to 1^-}\varphi(r)$ is finite then there exists $c>0$ such that for any $z_1\in\mathbb{D}$ the fiber in the z_2 direction contains a disc of radius c. Hence, Ω contains a polydisc $\mathbb{D}\times c\,\mathbb{D}$. This indicates that there are no nonzero Hilbert-Schmidt Hankel operators with anti-holomorphic symbols on Ω. This also indicates that there are no compact Hankel operators with anti-holomorphic symbols.

Therefore, from this point we assume

$$\limsup_{r\to 1^-}\varphi(r)=\infty.$$

In fact, the later assumption (6) made on the domain forces $\varphi(r)$ not to oscillate, so we can assume

$$\lim_{r\to 1^-}\varphi(r)=\infty. \tag{5}$$

On the other hand, Ω is pseudoconvex if and only if $z_1\to\varphi(|z_1|)$ is a subharmonic function on $\mathbb{D}$. A simple calculation gives $\Delta\varphi(z_1)=\varphi''(r)+\varphi'(r)/r$. We assume Ω is pseudoconvex hence we have

$$\varphi''(r)+\frac{1}{r}\varphi'(r)\geqslant 0 \quad \text{on } (0,1). \tag{6}$$

Our goal is to show that the sum $\sum_{|\gamma|=N} c^2_{\gamma+\alpha}/c^2_\gamma$ diverges for any nonzero α on a complete pseudoconvex Reinhardt domain Ω. We start with computing c_γ's.

We have

$$\begin{aligned} c_\gamma^2 &= \int_\Omega |z^\gamma|^2\,\mathrm{d}V(z)=\int_{\mathbb{D}}|z_1|^{2\gamma_1}\int_{|z_2|<\mathrm{e}^{-\varphi(|z_1|)}}|z_2|^{2\gamma_2}\,\mathrm{d}A(z_2)\,\mathrm{d}A(z_1)\\ &= \int_{\mathbb{D}}\left\{|z_1|^{2\gamma_1}\frac{2\pi}{2\gamma_2+2}\,\mathrm{e}^{-(2\gamma_2+2)\varphi(|z_1|)}\right\}\mathrm{d}A(z_1)=\frac{2\pi^2}{\gamma_2+1}\int_0^1 r^{2\gamma_1+1}\mathrm{e}^{-(2\gamma_2+2)\varphi(r)}\,\mathrm{d}r. \end{aligned}$$

For sufficiently large x and y, consider the ratio

$$R_{x,y}:=\frac{\int_0^1 r^{x+2\alpha_1}\mathrm{e}^{-(y+2\alpha_2)\varphi(r)}\,\mathrm{d}r}{\int_0^1 r^x\mathrm{e}^{-y\varphi(r)}\,\mathrm{d}r}, \tag{7}$$

and define

$$\Phi_{x,y}(r):=\frac{r^x\mathrm{e}^{-y\varphi(r)}}{\int_0^1 r^x\mathrm{e}^{-y\varphi(r)}\,\mathrm{d}r}.$$

Note that $\Phi_{x,y}(0)=0$, $\Phi_{x,y}(1)=0$, and $\int_0^1\Phi_{x,y}(r)\,\mathrm{d}r=1$.

Also, define

$$g_\alpha(r) = r^{2\alpha_1}\mathrm{e}^{-2\alpha_2\varphi(r)}. \tag{8}$$

Note that $g_\alpha(r)$ does not vanish inside the interval $(0,1)$, but may vanish at $r=0$ and $r=1$ depending on α. Now, we can rewrite the ratio $R_{x,y}$ as

$$R_{x,y} = \int_0^1 \Phi_{x,y}(r) r^{2\alpha_1}\mathrm{e}^{-2\alpha_2\varphi(r)}\,\mathrm{d}r = \int_0^1 \Phi_{x,y}(r) g_\alpha(r)\,\mathrm{d}r. \tag{9}$$

Our goal is to find a sub-interval $(a,b) \subset\subset (0,1)$ such that for sufficiently large x and y

$$\int_a^b \Phi_{x,y}(r)\,\mathrm{d}r \geqslant \frac{1}{2}.$$

For this purpose, we analyse $\Phi_{x,y}(r)$ further on $(0,1)$ and locate the local maximum of $\Phi_{x,y}(r)$. We have

$$\frac{\mathrm{d}}{\mathrm{d}r}\Phi_{x,y}(r) = (x - y\varphi'(r)r)(r^{x-1}\mathrm{e}^{-y\varphi(r)})\left(\int_0^1 r^x \mathrm{e}^{-y\varphi(r)}\,\mathrm{d}r\right)^{-1}.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}r}\Phi_{x,y}(r) = 0 \quad \text{on } (0,1) \text{ when } x - y\varphi'(r)r = 0.$$

We label $f_{x,y}(r) := x - y\varphi'(r)r$. Note that $f_{x,y}(r)$ controls the sign of $\frac{\mathrm{d}}{\mathrm{d}r}\Phi_{x,y}(r)$, since the rest of the terms in $\frac{\mathrm{d}}{\mathrm{d}r}\Phi_{x,y}(r)$ is positive. Furthermore,

$$f_{x,y}(0) = x \ > 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}r} f_{x,y}(r) = -y(\varphi'(r) + r\varphi''(r)) \ < 0 \text{ (by the assumption (6))}. \tag{10}$$

Hence, $f_{x,y}(r)$ decreases on $(0,1)$ and can vanish at a point. We will show that by choosing x,y appropriately we can guarantee that $f_{x,y}(r)$ vanishes on $(0,1)$. All we need is a point $s \in (0,1)$ such that

$$s\varphi'(s) > 0.$$

However, this is possible by the assumption (5). If there were no such point $s \in (0,1)$, then $\varphi(r)$ would not grow up to infinity. Moreover, if there exists $s \in (0,1)$ such that $s\varphi'(s) > 0$ then since $r\varphi'(r) > 0$ is an increasing function we have

$$r\varphi'(r) > 0, \quad r \in [s,1).$$

Therefore, there exists a relatively compact subinterval (a, b) of $(0, 1)$ such that

$$a\varphi'(a) > 0$$

and hence $r\varphi'(r) > 0$ on (a, b). Moreover, by choosing x and y appropriately we can make

$$f_{x,y}(a) > 0 \quad \text{and} \quad f_{x,y}(b) < 0.$$

That is,

$$x - ya\varphi'(a) > 0 \quad \text{and} \quad x - yb\varphi'(b) < 0.$$

Equivalently,

$$a\varphi'(a) < \frac{x}{y} \quad \text{and} \quad \frac{x}{y} < b\varphi'(b).$$

Therefore, as long as we keep

$$a\varphi'(a) < \frac{x}{y} < b\varphi'(b) \tag{11}$$

there exist a solution to $x - yr\varphi'(r) = 0$ on the interval $(a, b) \subset\subset (0, 1)$, and so we guarantee that the function $\Phi_{x,y}(r)$ assumes its maximum somewhere inside (a, b). Let us take the point $\varrho_{xy} \in (a, b)$ where $\Phi_{x,y}(r)$ takes its maximum value . We have

$$\int_0^{a/2} \Phi_{x,y}(r)\,\mathrm{d}r \leqslant \int_{a/2}^{\varrho_{xy}} \Phi_{x,y}(r)\,\mathrm{d}r \quad \text{and} \quad \int_{(1+b)/2}^{1} \Phi_{x,y}(r)\,\mathrm{d}r \leqslant \int_{\varrho_{xy}}^{(1+b)/2} \Phi_{x,y}(r)\,\mathrm{d}r.$$

Hence, we deduce that

$$\int_{a/2}^{(1+b)/2} \Phi_{x,y}(r)\,\mathrm{d}r \geqslant \int_0^1 \Phi_{x,y}(r)\,\mathrm{d}r \geqslant \frac{1}{2} \tag{12}$$

as long as $a\varphi'(a) < x/y < b\varphi'(b)$. The inequality at (12) is the crucial step for the rest of the proof. It guarantees that the integral of $\Phi_{x,y}(r)$ is located somewhere in the middle, i.e. does not lean towards any of the end points.

For a multi-index $\gamma = (\gamma_1, \gamma_2)$, let us write $\Phi_\gamma(r) = \Phi_{\gamma_1,\gamma_2}(r)$. Then

$$\begin{aligned} \frac{c^2_{\gamma+\alpha}}{c^2_\gamma} &= \frac{\gamma_2 + 1}{\gamma_2 + \alpha_2 + 1} \frac{\int_0^1 r^{2\gamma_1+2\alpha_1+1} \mathrm{e}^{-(2\gamma_2+2+2\alpha_2)\varphi(r)}\,\mathrm{d}r}{\int_0^1 r^{2\gamma_1+1} \mathrm{e}^{-(2\gamma_2+2)\varphi(r)}\,\mathrm{d}r} \\ &= \frac{\gamma_2 + 1}{\gamma_2 + \alpha_2 + 1} \int_0^1 \Phi_{2\gamma_1+1,2\gamma_2+2}(r) g_\alpha(r)\,\mathrm{d}r. \end{aligned} \tag{13}$$

Then

$$S_\alpha \geqslant \sum_{|\gamma|=N} \frac{c^2_{\gamma+\alpha}}{c^2_\gamma} = \sum_{k=0}^{N} \frac{c^2_{\alpha+(k,N-k)}}{c^2_{(k,N-k)}} = \sum_{k=0}^{N} \frac{c^2_{(k+\alpha_1,N-k+\alpha_2)}}{c^2_{(k,N-k)}} \tag{14}$$
$$= \sum_{k=0}^{N} \frac{N-k+1}{N-k+\alpha_2+1} \int_0^1 \Phi_{2k+1,2(N-k)+2}(r) g_\alpha(r)\,\mathrm{d}r.$$

We want to keep

$$\frac{2k+1}{2N-2k+2} \in (a\varphi'(a), b\varphi'(b)),$$

see (11). This is equivalent to asking k to be in the interval

$$\frac{2a\varphi'(a)}{2a\varphi'(a)+2}N + \frac{2a\varphi'(a)-1}{2a\varphi'(a)+2} < k < \frac{2b\varphi'(b)}{2b\varphi'(b)+2}N + \frac{2b\varphi'(b)-1}{2b\varphi'(b)+2}.$$

We further restrict k to the interval

$$I_N := \Big(\frac{2a\varphi'(a)}{2a\varphi'(a)+2}N + \frac{2a\varphi'(a)-1}{2a\varphi'(a)+2}, \frac{2b\varphi'(b)}{2b\varphi'(b)+2}N + \frac{2b\varphi'(b)-1}{2b\varphi'(b)+2}\Big) \cap (0,N).$$

Therefore, the estimate (14) can be rewritten as

$$S_\alpha \geqslant \sum_{k\in I_N} \frac{N-k+1}{N-k+\alpha_2+1} \int_0^1 \Phi_{2k+1,2(N-k)+2}(r) g_\alpha(r)\,\mathrm{d}r. \tag{15}$$

When $k \in I_N$ we have

$$\frac{N-k+1}{N-k+\alpha_2+1} \int_0^1 \Phi_{2k+1,2(N-k)+2}(r) g_\alpha(r)\,\mathrm{d}r$$
$$\geqslant \frac{1}{1+\alpha_2} \int_{a/2}^{(1+b)/2} \Phi_{2k+1,2(N-k)+2}(r) g_\alpha(r)\,\mathrm{d}r$$
$$\geqslant \frac{1}{1+\alpha_2} \Big(\min_{a/2\leqslant r\leqslant(1+b)/2} \{g_\alpha(r)\}\Big) \int_{a/2}^{(1+b)/2} \Phi_{2k+1,2(N-k)+2}(r)\,\mathrm{d}r$$
$$\text{by (12)} \quad \geqslant \frac{1}{1+\alpha_2} \Big(\min_{a/2\leqslant r\leqslant(1+b)/2} \{g_\alpha(r)\}\Big) \frac{1}{2}.$$

Let $\lambda_\alpha := \Big(\min_{a/2\leqslant r\leqslant(1+b)/2} \{g_\alpha(r)\}\Big)/(2(1+\alpha_2))$. Note that $\lambda_\alpha > 0$ since $g_\alpha(r)$ is strictly positive on $(a/2,(1+b)/2)$, see (8). This gives us

$$S_\alpha \geqslant \sum_{k\in I_N} \frac{c^2_{\gamma+\alpha}}{c^2_\gamma} \geqslant \sum_{k\in I_N} \frac{N-k+1}{N-k+\alpha_2+1} \int_0^1 \Phi_{2k+1,2(N-k)+2}(r) g_\alpha(r)\,\mathrm{d}r$$
$$\geqslant \sum_{k\in I_N} \lambda_\alpha = |I_N|\lambda_\alpha.$$

Note that the number of integers in I_N is comparable to N. Therefore, $S_\alpha \gtrsim N$ and this suffices to conclude S_α diverges for nonzero α.

4. Examples of unbounded non-pseudoconvexs domain with nonzero Hilbert-Schmidt Hankel operators

In this section, we present two examples of domains that admit nonzero Hilbert-Schmidt Hankel operators with anti-holomorphic symbols. In the first example, the Bergman space is finite dimensional and the claim holds for trivial reasons. In the second example, the Bergman space is infinite dimensional; however, some of the terms S_α's are bounded.

We start with defining the following domains from [12]:

$$\begin{aligned}
X_1 &= \Big\{(z_1, z_2) \in \mathbb{C}^2\colon\ |z_1| > \mathrm{e},\ |z_2| < \frac{1}{|z_1| \log |z_1|}\Big\},\\
X_2 &= \Big\{(z_1, z_2) \in \mathbb{C}^2\colon\ |z_2| > \mathrm{e},\ |z_1| < \frac{1}{|z_2| \log |z_2|}\Big\},\\
X_3 &= \{(z_1, z_2) \in \mathbb{C}^2\colon\ |z_1| \leqslant \mathrm{e},\ |z_2| \leqslant \mathrm{e}\},\\
\Omega_0 &= X_1 \cup X_2 \cup X_3,\\
B_m &= \Big\{(z_1, z_2) \in \mathbb{C}^2\colon\ |z_1|, |z_2| > 1,\ \big||z_1| - |z_2|\big| < \frac{1}{(|z_1| + |z_2|)^m}\Big\},\\
\Omega_k &= \Omega_0 \cup B_{4k}.
\end{aligned}$$

Note that Ω_0 and Ω_k are unbounded non-pseudoconvex complete Reinhardt domains with finite volume. The following proposition is also from [12].

Proposition 1. *Let k be a positive integer.*

(i) *The Bergman space $A^2(\Omega_k)$ is spanned by the monomials $\{(z_1 z_2)^j\}_{j=0}^{k}$.*
(ii) *The Bergman space $A^2(\Omega_0)$ is spanned by the monomials $\{(z_1 z_2)^j\}_{j=0}^{\infty}$.*

Next, we look at the Hankel operators on the Bergman spaces of Ω_0 and Ω_k.

Example 1. We start with Ω_k. Since $A^2(\Omega_k)$ is finite dimensional, for any multi-index of the form (j, j) for $j = 1, \ldots, k$, the term $S_{(j,j)}$ is a finite sum and consequently finite when restricted to the subspace of $A^2(\Omega_k)$ where the multiplication operator with the symbol $\overline{f}$ is bounded. Hence, for any $f \in A^2(\Omega_k)$, the Hankel operator with the symbol $\overline{f}$ is Hilbert-Schmidt on the subspace of $A^2(\Omega_k)$ where the operator is bounded.

Example 2. Next, we look at Ω_0 and observe that the terms S_α take a simpler form. Namely, for a multi-index (j,j),

$$S_{(j,j)} = \sum_{k=0}^{\infty}\Big(\frac{c^2_{(k+j,k+j)}}{c^2_{(k,k)}} - \frac{c^2_{(k,k)}}{c^2_{(k-j,k-j)}}\Big),$$

where

$$c^2_{(k,k)} = \int_{\Omega_0} |z_1 z_2|^{2k}\, \mathrm{d}V(z_1,z_2).$$

We will particularly compute $S_{(1,1)}$. A simple integration indicates

$$c^2_{(k,k)} = 4\pi^2\Big(\frac{2}{2k+1} + \frac{\mathrm{e}^{4k+4}}{(2k+2)^2}\Big)$$

and by simple algebra we obtain

$$\frac{c^2_{(k+1,k+1)}}{c^2_{(k,k)}} - \frac{c^2_{(k,k)}}{c^2_{(k-1,k-1)}} = \frac{\mathrm{e}^{8k+8}\frac{(2k+2)^4-(2k+4)^2(2k)^2}{(2k+4)^2(2k)^2(2k+2)^4} + \mathrm{e}^{4k}\frac{p_1(k)}{p_2(k)} + \frac{p_3(k)}{p_4(k)}}{\mathrm{e}^{8k+8}\frac{1}{(2k)^2(2k+2)^2} + \mathrm{e}^{4k}\frac{p_5(k)}{p_6(k)} + \frac{p_7(k)}{p_8(k)}}$$

where $p_1(k),\dots,p_8(k)$ are polynomials in k. For large values of k, the first terms at the numerator and the denominator dominate and we obtain

$$\frac{c^2_{(k+1,k+1)}}{c^2_{(k,k)}} - \frac{c^2_{(k,k)}}{c^2_{(k-1,k-1)}} \approx \frac{\frac{(2k+2)^4-(2k+4)^2(2k)^2}{(2k+4)^2(2k)^2(2k+2)^4}}{\frac{1}{(2k)^2(2k+2)^2}} \approx \frac{1}{k^2}.$$

Therefore, $S_{(1,1)}$ is finite and the Hankel operator $H_{\overline{z_1 z_2}}$ is Hilbert-Schmidt on $A^2(\Omega_0)$.

5. Remarks

5.1. Canonical solution operator for $\overline{\partial}$-problem:. The canonical solution operator for $\overline{\partial}$-problem restricted to $(0,1)$-forms with holomorphic coefficients is not a Hilbert-Schmidt operator on complete pseudoconvex Reinhardt domains because the canonical solution operator for $\overline{\partial}$-problem restricted to $(0,1)$-forms with holomorphic coefficients is a sum of Hankel operators with $\{\overline{z}_j\}_{j=1}^n$ as symbols (by Theorem 1 such Hankel operators are not Hilbert-Schmidt):

$$\overline{\partial}^* N_1(g) = \overline{\partial}^* N_1\Big(\sum_{j=1}^n g_j d\overline{z}_j\Big) = \sum_{j=1}^n H_{\overline{z}_j}(g_j)$$

for any $(0,1)$-form g with holomorphic coefficients.

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