Czechoslovak Mathematical Journal

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*Czechoslovak Mathematical Journal*, Vol. 67 (2017), No. 1, 123–141

Persistent URL: http://dml.cz/dmlcz/146044

**Czechoslovak Mathematical Journal, 67 (142) (2017), 123–141**

# PSEUDO ALMOST PERIODICITY OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH IMPULSIVE EFFECTS IN BANACH SPACES

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Received July 22, 2015. First published February 24, 2017.

*Abstract.* In this paper, for the impulsive fractional integro-differential equations involving Caputo fractional derivative in Banach space, we investigate the existence and uniqueness of a pseudo almost periodic $PC$-mild solution. The working tools are based on the fixed point theorems, the fractional powers of operators and fractional calculus. Some known results are improved and generalized. Finally, existence and uniqueness of a pseudo almost periodic $PC$-mild solution of a two-dimensional impulsive fractional predator-prey system with diffusion are investigated.

*Keywords*: impulsive fractional integro-differential equation; pseudo almost periodicity; probability density; fractional powers of operator

*MSC 2010*: 34A37, 26A33, 34C27

## 1. Introduction

The concept of a pseudo almost periodic function was introduced by Zhang [26], [27] in the early nineties. It is an important generalization of an almost periodic function. Since then, this pioneer work has attracted more and more attention and many authors have made important contributions to this theory. For more details on pseudo almost periodic functions and related topics, one can see [5], [8], [9], [14], [12], [17] and the references therein.

This research is supported by the National Natural Science Foundation of China (Grant Nos. 11501507, 11426201).

DOI: 10.21136/CMJ.2017.0398-15

In this paper, we investigate the existence and uniqueness of pseudo almost periodic mild solutions of impulsive fractional integro-differential equations

$$
{}^{c}D^{\alpha}u(t)+Au(t)=f(t,u(t))+(Ku)(t)+\sum_{k=-\infty}^{\infty}G_k(u(t))\delta(t-\tau_k), \tag{1.1}
$$

where

$$
(Ku)(t)=\int_{-\infty}^{t}k(t-s)g(s,u(s))\,\mathrm{d}s,
$$

$0<\alpha\leqslant 1$, $-A\colon \mathcal{D}(A)\subset X\to X$ is a linear infinitesimal operator of an analytic semigroup $S(t)$, $f,g$ are pseudo almost periodic in $t\in\mathbb{R}$ uniformly in the second variable, $G_k\colon \mathcal{D}(G_k)\subset X\to X$ are continuous impulsive operators, $\delta(\cdot)$ is Dirac's delta-function, $\{\tau_k\}\in T$, where $T$ will be defined later. Here the fractional derivative is understood in Caputo's sense. We notice that fractional order models have received much attention in recent years due to their extensive and efficient applications to nonlinear dynamics concerning fluid flows, electrical networks, viscoelasticity, biology and many other branches of science [1], [10], [20].

If (1.1) is without impulsive effects, then (1.1) becomes a fractional integro-differential equation. Existence of almost periodic mild solutions is studied in [7] by semigroup theory. By the Banach contraction mapping principle, pseudo almost periodic solutions are studied in [4].

If $(Ku)(t)=0$ and $\alpha=1$, then (1.1) becomes the impulsive differential equations

$$
u'(t)+Au(t)=f(t,u(t))+\sum_{k=-\infty}^{\infty}G_k(u(t))\delta(t-\tau_k). \tag{1.2}
$$

For (1.2), the existence and uniqueness of almost periodic solution is investigated under the condition that $A$ is the infinitesimal generator of an analytic semigroup by Stamov and Alzabout in [23]. Later, the results of [23] are generalized by Chérif in [6], where pseudo almost periodic solutions are studied. If $A$ is the infinitesimal generator of a $C_0$-semigroup, Liu and Zhang investigate the existence and uniqueness of almost periodic and pseudo almost periodic solutions in Banach space, see [15], [16], [18].

Notice that if $(Ku)(t)=0$, then (1.1) becomes an impulsive fractional differential equations and existence and uniqueness of almost periodic solutions are investigated in [24]. However, for fractional integro-differential equations with impulsive effects, i.e., (1.1), the study of asymptotic behavior of solutions is rare; particularly for the pseudo almost periodicity of (1.1), it is an untreated topic and this is the main motivation of this paper. We will make use of the fixed point theorems and the fractional powers of operators to derive some sufficient conditions guaranteeing the existence and uniqueness of pseudo almost periodic solution to (1.1).

The paper is organized as follows. In Section 2, we recall some fundamental results about the notion of piecewise pseudo almost periodic functions including the composition theorem. Sections 3 is devoted to the existence and uniqueness of pseudo almost periodic mild solution of (1.1) by fractional powers of operators and fixed point theorems. In Section 4, some interesting examples are presented to illustrate the main results.

## 2. Preliminaries and basic results

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be Banach spaces, $\Omega$ a subset of $X$ and let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ stand for the set of natural numbers, integers, real numbers, and complex numbers, respectively. For $A$ being a linear operator on $X$, $\mathcal{D}(A)$, $\varrho(A)$, $R(\lambda, A)$ and $\sigma(A)$ stand for the domain, the resolvent set, the resolvent and spectrum of $A$. Let $T$ be the set consisting of all real sequences $\{\tau_k\}_{k\in\mathbb{Z}}$ such that $\kappa = \inf_{k\in\mathbb{Z}}(\tau_{k+1} - \tau_k) > 0$. It is immediate that this condition implies that $\lim_{k\to\infty} \tau_k = \infty$ and $\lim_{k\to-\infty} \tau_k = -\infty$.

In order to facilitate the discussion below, we further introduce the following notations

- ▷ $C(\mathbb{R}, X)$ (or $C(\mathbb{R}\times\Omega, X)$): the set of continuous functions from $\mathbb{R}$ to $X$ (from $\mathbb{R}\times\Omega$ to $X$, respectively).
- ▷ $BC(\mathbb{R}, X)$ (or $BC(\mathbb{R}\times\Omega, X)$): the Banach space of bounded continuous functions from $\mathbb{R}$ to $X$ (from $\mathbb{R}\times\Omega$ to $X$, respectively) with the supremum norm.
- ▷ $PC(\mathbb{R}, X)$: the space formed by all piecewise continuous functions $f\colon \mathbb{R}\to X$ such that $f(\cdot)$ is continuous at $t$ for any $t \notin \{\tau_k\}_{k\in\mathbb{Z}}$, $f(\tau_k^+)$, $f(\tau_k^-)$ exist, and $f(\tau_k^-) = f(\tau_k)$ for all $k\in\mathbb{Z}$.
- ▷ $PC(\mathbb{R}\times\Omega, X)$: the space formed by all piecewise continuous functions $f\colon \mathbb{R}\times\Omega\to X$ such that for any $x\in\Omega$, $f(\cdot, x)\in PC(\mathbb{R}, X)$ and for any $t\in\mathbb{R}$, $f(t,\cdot)$ is continuous at $x\in\Omega$.

Following [20], we recall the fractional integral of order $\alpha > 0$ as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s,$$

and the fractional Caputo's derivative of the function $f$ of order $0<\alpha<1$ as

$$^cD^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{f'(s)}{(t-s)^\alpha}\,\mathrm{d}s,$$

where $\Gamma(\alpha)$ is the classical Gamma function given by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t.$$

**Definition 2.1** ([11]). A function $f\colon \mathbb{R} \to X$ is said to be almost periodic if for each $\varepsilon > 0$ there exists an $l(\varepsilon) > 0$ such that every interval $J$ of length $l(\varepsilon)$ contains a number $\tau$ with the property that $\|f(t+\tau) - f(t)\| < \varepsilon$ for all $t \in \mathbb{R}$. Denote by $AP(\mathbb{R}, X)$ the set of such functions.

**Definition 2.2** ([21]). A sequence $\{x_n\}$ is called almost periodic if for any $\varepsilon > 0$ there exists a relatively dense set of its $\varepsilon$-periods, i.e., there exists a natural number $l = l(\varepsilon)$ such that for $k \in \mathbb{Z}$ there is at least one number $p$ in $[k, k+l]$ for which inequality $\|x_{n+p} - x_n\| < \varepsilon$ holds for all $n \in \mathbb{N}$. Denote by $AP(\mathbb{Z}, X)$ the set of such sequences.

For $\{\tau_k\}_{k\in\mathbb{Z}} \in T$, $\{\tau_k^j\}$ is defined by

$$\{\tau_k^j = \tau_{k+j} - \tau_k\}, \quad k \in \mathbb{Z},\ j \in \mathbb{Z}.$$

It is easy to verify that the numbers $\tau_k^j$ satisfy

$$\tau_{k+i}^j - \tau_k^j = \tau_{k+j}^i - \tau_k^i, \quad \tau_k^j - \tau_k^i = \tau_{k+i}^{j-i} \quad \text{for } i, j, k \in \mathbb{Z}.$$

**Definition 2.3** ([21]). A function $f \in PC(\mathbb{R}, X)$ is said to be piecewise almost periodic if the following conditions are fulfilled:

(1) $\{\tau_k^j = \tau_{k+j} - \tau_k\}$, $k, j \in \mathbb{Z}$ are equipotentially almost periodic, that is, for any $\varepsilon > 0$ there exists a relatively dense set in $\mathbb{R}$ of $\varepsilon$-almost periods common for all of the sequences $\{\tau_k^j\}$.

(2) For any $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that if the points $t'$ and $t''$ belong to the same interval of continuity of $f$ and $|t' - t''| < \delta$, then $\|f(t') - f(t'')\| < \varepsilon$.

(3) For any $\varepsilon > 0$ there exists a relatively dense set $\Omega_\varepsilon$ in $\mathbb{R}$ such that if $\tau \in \Omega_\varepsilon$, then

$$\|f(t+\tau) - f(t)\| < \varepsilon$$

for all $t \in \mathbb{R}$ which satisfy the condition $|t - \tau_k| > \varepsilon$, $k \in \mathbb{Z}$.

We denote by $AP_T(\mathbb{R}, X)$ the space of all piecewise almost periodic functions. Obviously, $AP_T(\mathbb{R}, X)$ endowed with the supremum norm is a Banach space. Throughout the rest of this paper, we always assume that $\{\tau_k^j\}$ are equipotentially almost periodic. Let $\mathcal{U}PC(\mathbb{R}, X)$ be the space of all functions $f \in PC(\mathbb{R}, X)$ such that $f$ satisfies the condition (2) in Definition 2.3.

**Definition 2.4.** A function $f \in PC(\mathbb{R} \times \Omega, X)$ is said to be piecewise almost periodic in $t$ uniformly in $x \in \Omega$ if for each compact set $K \subseteq \Omega$, $\{f(\cdot, x)\colon x \in K\}$ is uniformly bounded, and given $\varepsilon > 0$, there exists a relatively dense set $\Omega_\varepsilon$ such that $\|f(t+\tau, x) - f(t, x)\| \leqslant \varepsilon$ for all $x \in K$, $\tau \in \Omega_\varepsilon$ and $t \in \mathbb{R}$, $|t - \tau_k| > \varepsilon$. Denote by $AP_T(\mathbb{R} \times \Omega, X)$ the set of all such functions.

**Lemma 2.1** ([21])**.** *If the sequences $\{\tau_k^j\}$ are equipotentially almost periodic, then for each $j > 0$ there exists a positive integer $N$ such that on each interval of length $j$ there are no more than $N$ elements of the sequence $\{\tau_k\}$, i.e.,*

$$i(t, s) \leqslant N(t - s) + N,$$

*where $i(t, s)$ is the number of the points $\{\tau_k\}$ in the interval $[s, t]$.*

**Lemma 2.2** ([21])**.** *Assume that $f \in AP_T(\mathbb{R}, X)$, $\{x_k\}_{k\in\mathbb{Z}} \in AP(\mathbb{Z}, X)$, and $\{\tau_k^j\}$, $j \in \mathbb{Z}$ are equipotentially almost periodic. Then for each $\varepsilon > 0$ there exist relatively dense sets $\Omega_\varepsilon$ of $\mathbb{R}$ and $Q_\varepsilon$ of $\mathbb{Z}$ such that*

(i) $\|f(t+\tau) - f(t)\| < \varepsilon$ *for all* $t \in \mathbb{R}$, $|t - \tau_k| > \varepsilon$, $\tau \in \Omega_\varepsilon$ *and* $k \in \mathbb{Z}$*;*
(ii) $\|x_{k+q} - x_k\| < \varepsilon$ *for all* $q \in Q_\varepsilon$ *and* $k \in \mathbb{Z}$*;*
(iii) $|\tau_k^q - \tau| < \varepsilon$ *for all* $q \in Q_\varepsilon$, $\tau \in \Omega_\varepsilon$ *and* $k \in \mathbb{Z}$*.*

Define

$$\begin{aligned} PAP_T^0(\mathbb{R}, X) &= \Big\{ f \in PC(\mathbb{R}, X)\colon \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(t)\| \,\mathrm{d}t = 0 \Big\}, \\ PAP_T^0(\mathbb{R} \times \Omega, X) &= \Big\{ f \in PC(\mathbb{R} \times \Omega, X)\colon \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(t, x)\| \,\mathrm{d}t = 0 \\ &\qquad \text{uniformly with respect to } x \in K, \\ &\qquad \text{where } K \text{ is an arbitrary compact subset of } \Omega \Big\}. \end{aligned}$$

**Definition 2.5** ([16])**.** A function $f \in PC(\mathbb{R}, X)$ is said to be piecewise pseudo almost periodic if it can be decomposed as $f = g + \varphi$, where $g \in AP_T(\mathbb{R}, X)$ and $\varphi \in PAP_T^0(\mathbb{R}, X)$. Denote by $PAP_T(\mathbb{R}, X)$ the set of all such functions. $PAP_T(\mathbb{R}, X)$ is a Banach space when endowed with the supremum norm.

**Definition 2.6** ([16])**.** Let $PAP_T(\mathbb{R} \times \Omega, X)$ consist of all functions $f \in PC(\mathbb{R} \times \Omega, X)$ such that $f = g + \varphi$, where $g \in AP_T(\mathbb{R} \times \Omega, X)$ and $\varphi \in PAP_T^0(\mathbb{R} \times \Omega, X)$.

**Remark 2.1.** The set $PAP_T^0(\mathbb{R}, X)$ is a translation invariant subset of $PC(\mathbb{R}, X)$.

The following composition theorem holds for piecewise pseudo almost periodic functions.

**Theorem 2.3** ([16]). *Let $f \in PAP_T(\mathbb{R} \times \Omega, X)$, $\varphi \in PAP_T(\mathbb{R}, X)$ and $\mathcal{R}(\varphi) \subset \Omega$. Assume that there exists a constant $L_f > 0$ such that*

$$\|f(t,u) - f(t,v)\| \leqslant L_f \|u - v\|, \quad t \in \mathbb{R},\ u, v \in \Omega.$$

*Then $f(\cdot, \varphi) \in PAP_T(\mathbb{R}, X)$.*

Next, we introduce the concept of a generalized pseudo almost periodic function (sequence) which is more general than a pseudo almost periodic function (sequence), see [2], [13].

Define

$$\widetilde{P}AP_0(\mathbb{Z}, X) = \left\{x\colon \lim_{n\to\infty} \frac{1}{2n} \sum_{k=-n}^{n} \|x_k\| = 0\right\}.$$

$$\widetilde{P}AP_0(\mathbb{R}, X) = \left\{f\colon\ \mathbb{R} \to X \quad \text{is measure and } \lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|f(t)\|\,\mathrm{d}t = 0\right\}.$$

**Definition 2.7** ([2]). A measurable function $f\colon\ \mathbb{R} \to X$ is called generalized pseudo-almost periodic if $f = g + \varphi$, where $g \in AP(\mathbb{R}, X)$, $\varphi \in \widetilde{P}AP_0(\mathbb{R}, X)$. Denote by $\widetilde{P}AP(\mathbb{R}, X)$ the set of all such functions.

**Definition 2.8** ([13]). A sequence $\{x_n\}_{n\in\mathbb{Z}}$ is called generalized pseudo almost periodic if $x_n = x_n^1 + x_n^2$, where $x_n^1 \in AP(\mathbb{Z}, X)$, $x_n^2 \in \widetilde{P}AP_0(\mathbb{Z}, X)$. Denote by $\widetilde{P}AP(\mathbb{Z}, X)$ the set of such sequences.

**Lemma 2.4** ([13]). *If $\{x_n\}_{n\in\mathbb{Z}}$ is a $\widetilde{P}AP_0(\mathbb{Z}, X)$ sequence, then there exists a function $g \in \widetilde{P}AP_0(\mathbb{R}, X)$ such that $g(n) = x_n$, $n \in \mathbb{Z}$.*

Similarly to the proof in [16], one has

**Theorem 2.5.** *Assume that a sequence of vector-valued functions $\{G_k\}_{k\in\mathbb{Z}}$ is generalized pseudo almost periodic, and there exists a constant $L_1 > 0$ such that*

$$\|G_k(u) - G_k(v)\| \leqslant L_1 \|u - v\|, \quad u, v \in \Omega,\ k \in \mathbb{Z}.$$

*If $\varphi \in \widetilde{P}AP(\mathbb{R}, X)$, then $G_k(\varphi(\tau_k))$ is generalized pseudo almost periodic.*

## 3. Impulsive fractional integro-differential equations

In this section, we investigate the existence and uniqueness of piecewise pseudo almost periodic mild solutions of (1.1).

Let $t_0 \in \mathbb{R}$, denote by $u(t) = u(t, t_0, u_0), u_0 \in X$, the solution of (1.1) with an initial condition

$$u(t_0) = u_0. \tag{3.1}$$

The solution $u(t) = u(t, t_0, u_0)$ of problem (1.1) and (3.1) is a piecewise continuous function with points of discontinuity at the moments $\tau_k$, $k \in \mathbb{Z}$, at which it is continuous from the left, i.e. the following relations hold:

$$u(\tau_k^-) = u(\tau_k), \quad u(\tau_k^+) = u(\tau_k) + G_k(u(\tau_k)), \quad k \in \mathbb{Z},$$

that is $u \in PC(\mathbb{R}, X)$. With respect to the norm $\|u\| = \sup\limits_{t\in\mathbb{R}}\|u(t)\|$, one can easily see that $PC(\mathbb{R}, X)$ is a Banach space.

First, we make the following assumptions:

($\mathrm{H}_1$) $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$ such that

$$\|S(t)\| \leqslant M\mathrm{e}^{-\omega t} \quad \text{for } t \geqslant 0,$$

where $\omega > 0$.

($\mathrm{H}_2$) $k \in C(\mathbb{R}^+, \mathbb{R})$ and $|k(t)| \leqslant C_k\mathrm{e}^{-\eta t}$ for some positive constants $C_k, \eta$.

($\mathrm{H}_3$) $f \in PAP_T(\mathbb{R} \times X_\beta, X)$ and there exist constants $L_f > 0$, $0 < \beta < 1$ such that

$$\|f(t, u) - f(t, v)\| \leqslant L_f\|u - v\|_\beta, \quad t \in \mathbb{R},\ u, v \in X_\beta,$$

where $X_\beta$, $\|\cdot\|_\beta$ are defined later.

($\mathrm{H}_4$) $g \in PAP_T(\mathbb{R} \times X_\beta, X)$ and there exists a constant $L_g > 0$ such that

$$\|g(t, u) - g(t, v)\| \leqslant L_g\|u - v\|_\beta, \quad t \in \mathbb{R},\ u, v \in X_\beta.$$

($\mathrm{H}_5$) $G_k \in \widetilde{PAP}(\mathbb{Z}, X)$ and there exists a constant $L_1 > 0$ such that

$$\|G_k(u) - G_k(v)\| \leqslant L_1\|u - v\|_\beta, \quad t \in \mathbb{R},\ u, v \in X_\beta,\ k \in \mathbb{Z}.$$

**Definition 3.1** ([25]). By a $PC$-mild solution of (1.1) and (3.1) we mean a function $u \in PC(\mathbb{R}, X)$ which satisfies the following integral equation:

$$
(3.2) \qquad u(t) = \begin{cases} \mathscr{T}(t-t_0)u_0 + \displaystyle\int_{t_0}^{t} (t-s)^{\alpha-1}\mathscr{S}(t-s)(f(s,u(s)) + (Ku)(s))\,\mathrm{d}s \\ \hfill \text{for } t \in [t_0, \tau_1], \\ \mathscr{T}(t-t_0)u_0 + \displaystyle\int_{t_0}^{t} (t-s)^{\alpha-1}\mathscr{S}(t-s)(f(s,u(s)) + (Ku)(s))\,\mathrm{d}s \\ \quad +(Ku)(s))\,\mathrm{d}s + \mathscr{T}(t-\tau_1)y_1 \hfill \text{for } t \in (\tau_1, \tau_2], \\ \vdots \\ \mathscr{T}(t-t_0)u_0 + \displaystyle\int_{t_0}^{t} (t-s)^{\alpha-1}\mathscr{S}(t-s)(f(s,u(s)) + (Ku)(s))\,\mathrm{d}s \\ \quad + \displaystyle\sum_{t_0<\tau_k<t} \mathscr{T}(t-\tau_k)y_k \hfill \text{for } t \in (\tau_k, \tau_{k+1}], \end{cases}
$$

where

$$
\begin{aligned}
y_k &= G_k(u(\tau_k)), \quad (Ku)(t) = \int_{-\infty}^{t} k(t-s)g(s,u(s))\,\mathrm{d}s, \\
\mathscr{T}(t) &= \int_0^{\infty} \xi_\alpha(\theta)S(t^\alpha\theta)\,\mathrm{d}\theta, \quad \mathscr{S}(t) = \alpha\int_0^{\infty} \theta\xi_\alpha(\theta)S(t^\alpha\theta)\,\mathrm{d}\theta, \\
\xi_\alpha(\theta) &= \frac{1}{\alpha}\theta^{-1-1/\alpha}\varpi_\alpha(\theta^{-1/\alpha}) \geqslant 0, \\
\varpi_\alpha(\theta) &= \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta \in (0,\infty),
\end{aligned}
$$

$\xi_\alpha$ is a probability density function defined on $(0,\infty)$, that is

$$
\xi_\alpha \geqslant 0, \quad \theta \in (0,\infty) \quad \text{and} \quad \int_0^{\infty} \xi_\alpha(\theta)\,\mathrm{d}\theta = 1.
$$

Note that when ($\mathrm{H}_1$) holds, we deduce that if $u(t)$ is a bounded $PC$-mild solution of (1.1) on $\mathbb{R}$, then we take the limit as $t_0 \to -\infty$ and using (3.2), we obtain

$$
(3.3) \quad u(t) = \int_{-\infty}^{t} (t-s)^{\alpha-1}\mathscr{S}(t-s)\big(f(s,u(s)) + (Ku)(s)\big)\,\mathrm{d}s + \sum_{\tau_k<t} \mathscr{T}(t-\tau_k)y_k.
$$

Let the operator $-A$ in (1.1) and (3.1) be an infinitesimal operator of an analytic semigroup $S(t)$ in the Banach space $X$ and $0 \in \varrho(A)$. For any $\beta > 0$, we define the fractional power $A^{-\beta}$ of the operator $A$ by

$$
A^{-\beta} = \frac{1}{\Gamma(\beta)}\int_0^{\infty} t^{\beta-1}S(t)\,\mathrm{d}t.
$$

$A^{-\beta}$ is bounded, bijective and $A^{\beta}=(A^{-\beta})^{-1}$, $\beta>0$ is a closed linear operator such that $\mathcal{D}(A^{\beta})=\mathcal{R}(A^{-\beta})$. The operator $A^0$ is the identity operator in $X$ and for $0\leqslant\beta\leqslant 1$, the space $X_\beta=\mathcal{D}(A^{\beta})$ with the norm $\|x\|_\beta=\|A^{\beta}x\|$ is a Banach space.

**Lemma 3.1** ([19]). *Let $-A$ be an infinitesimal operator of an analytic semigroup $S(t)$. Then*

(i) *$S(t)\colon X\to\mathcal{D}(A^{\beta})$ for every $t>0$ and $\beta\geqslant 0$;*
(ii) *for every $x\in\mathcal{D}(A^{\beta})$, it follows that $S(t)A^{\beta}x=A^{\beta}S(t)x$;*
(iii) *for every $t>0$, the operator $A^{\beta}S(t)$ is bounded and*

$$\|A^{\beta}S(t)\|\leqslant M_\beta t^{-\beta}\mathrm{e}^{-\lambda t},\quad M_\beta>0,\ \lambda>0; \tag{3.4}$$

(iv) *for $0<\beta\leqslant 1$ and $x\in\mathcal{D}(A^{\beta})$, we have*

$$\|S(t)x-x\|\leqslant C_\beta t^{\beta}\|A^{\beta}x\|,\quad C_\beta>0.$$

**Lemma 3.2.** *Assume that* (H$_1$), (H$_2$), (H$_4$) *hold. If $u\in PAP_T(\mathbb{R},X_\beta)$, then*

$$(Ku)(t)=\int_{-\infty}^{t}k(t-s)g(s,u(s))\,\mathrm{d}s\in PAP_T(\mathbb{R},X).$$

P r o o f. For $u\in PAP_T(\mathbb{R},X_\beta)$, it is not difficult to see that $\varphi(\cdot)=g(\cdot,u(\cdot))\in PAP_T(\mathbb{R},X)$ by Theorem 2.3. Let $\varphi=\varphi_1+\varphi_2$, where $\varphi_1\in AP_T(\mathbb{R},X)$, $\varphi_2\in PAP_T^0(\mathbb{R},X)$, then

$$\int_{-\infty}^{t}k(t-s)g(s,u(s))\,\mathrm{d}s=\int_{-\infty}^{t}k(t-s)\varphi_1(s)\,\mathrm{d}s+\int_{-\infty}^{t}k(t-s)\varphi_2(s)\,\mathrm{d}s:=\Psi_1(t)+\Psi_2(t),$$

where

$$\Psi_1(t)=\int_{-\infty}^{t}k(t-s)\varphi_1(s)\,\mathrm{d}s,\quad \Psi_2(t)=\int_{-\infty}^{t}k(t-s)\varphi_2(s)\,\mathrm{d}s.$$

(i) $\Psi_1\in AP_T(\mathbb{R},X)$. It is not difficult to see that $\Psi_1\in\mathcal{UPC}(\mathbb{R},X)$. Since $\varphi_1\in AP_T(\mathbb{R},X)$, for $\varepsilon>0$, let $\Omega_\varepsilon$ be a relatively dense subset of $\mathbb{R}$ formed by the $\varepsilon$-periods of $\varphi_1$. If $\tau\in\Omega_\varepsilon$, $t\in\mathbb{R}$, $|t-t_i|>\varepsilon$, $i\in\mathbb{Z}$, then

$$\|\varphi_1(t+\tau)-\varphi_1(t)\|<\varepsilon.$$

Hence, by ($\mathrm{H}_2$), for $t \in \mathbb{R}$, $|t - t_i| > \varepsilon$, $i \in \mathbb{Z}$, one has

$$\begin{aligned}
\|\Psi_1(t+\tau) - \Psi_1(t)\| &= \left\| \int_{-\infty}^{t+\tau} k(t+\tau-s)\varphi_1(s)\,\mathrm{d}s - \int_{-\infty}^{t} k(t-s)\varphi_1(s)\,\mathrm{d}s \right\| \\
&= \left\| \int_{-\infty}^{t} k(t-s)(\varphi_1(s+\tau) - \varphi_1(s))\,\mathrm{d}s \right\| \\
&\leqslant \int_{-\infty}^{t} C_k \mathrm{e}^{-\eta(t-s)} \|\varphi_1(s+\tau) - \varphi_1(s)\|\,\mathrm{d}s < \frac{C_k}{\eta}\varepsilon,
\end{aligned}$$

which implies that $\Psi_1 \in AP_T(\mathbb{R}, X)$.

(ii) $\Psi_2 \in PAP_T^0(\mathbb{R}, X)$. In fact, for $r > 0$, one has

$$\begin{aligned}
\frac{1}{2r}\int_{-r}^{r} \|\Psi_2(t)\|\,\mathrm{d}t &= \frac{1}{2r}\int_{-r}^{r} \left\| \int_{-\infty}^{t} k(t-s)\varphi_2(s)\,\mathrm{d}s \right\| \mathrm{d}t \\
&= \frac{1}{2r}\int_{-r}^{r} \left\| \int_{0}^{\infty} k(s)\varphi_2(t-s)\,\mathrm{d}s \right\| \mathrm{d}t \\
&\leqslant \frac{1}{2r}\int_{-r}^{r}\int_{0}^{\infty} C_k \mathrm{e}^{-\eta s}\|\varphi_2(t-s)\|\,\mathrm{d}s\,\mathrm{d}t \\
&\leqslant \int_{0}^{\infty} C_k \mathrm{e}^{-\eta s}\Phi_r(s)\,\mathrm{d}s,
\end{aligned}$$

where

$$\Phi_r(s) = \frac{1}{2r}\int_{-r}^{r} \|\varphi_2(t-s)\|\,\mathrm{d}t.$$

Since $\varphi_2 \in PAP_T^0(\mathbb{R}, X)$, it follows that $\varphi_2(\cdot - s) \in PAP_T^0(\mathbb{R}, X)$ for each $s \in \mathbb{R}$ by Remark 2.1, hence $\lim\limits_{r\to\infty} \Phi_r(s) = 0$ for all $s \in \mathbb{R}$. By using the Lebesgue dominated convergence theorem, we have $\Psi_2 \in PAP_T^0(\mathbb{R}, X)$. This completes the proof. $\square$

**Lemma 3.3.** *Assume that* ($\mathrm{H}_1$)–($\mathrm{H}_4$) *hold. If* $u \in PAP_T(\mathbb{R}, X_\beta)$, *then*

$$(\Lambda u)(t) := \int_{-\infty}^{t} (t-s)^{\alpha-1}\mathscr{S}(t-s)(f(s,u(s)) + (Ku)(s))\,\mathrm{d}s$$

*lies in* $PAP_T(\mathbb{R}, X_\beta)$.

Proof. If $u \in PAP_T(\mathbb{R}, X_\beta)$, $Ku \in PAP_T(\mathbb{R}, X)$ by Lemma 3.2, and $f(\cdot, u(\cdot)) \in PAP_T(\mathbb{R}, X)$ by Theorem 2.3. Hence $h(\cdot) = (Ku)(\cdot) + f(\cdot, u(\cdot)) \in PAP_T(\mathbb{R}, X)$. Let $h = h_1 + h_2$, where $h_1 \in AP_T(\mathbb{R}, X)$, $h_2 \in PAP_T^0(\mathbb{R}, X)$, then

$$(\Lambda u)(t) = \int_{-\infty}^{t} (t-s)^{\alpha-1}\mathscr{S}(t-s)h(s)\,\mathrm{d}s := \Lambda_1(t) + \Lambda_2(t),$$

where

$$\Lambda_1(t) = \int_{-\infty}^{t} (t-s)^{\alpha-1}\mathscr{S}(t-s)h_1(s)\,\mathrm{d}s,$$
$$\Lambda_2(t) = \int_{-\infty}^{t} (t-s)^{\alpha-1}\mathscr{S}(t-s)h_2(s)\,\mathrm{d}s.$$

(i) $\Lambda_1 \in AP_T(\mathbb{R}, X_\beta)$. It is not difficult to see that $\Lambda_1 \in \mathcal{UPC}(\mathbb{R}, X)$. Since $h_1 \in AP_T(\mathbb{R}, X)$, for $\varepsilon > 0$ there exists a relatively dense set $\Omega_\varepsilon$ such that for $\tau \in \Omega_\varepsilon$, $t \in \mathbb{R}$, $|t - t_i| > \varepsilon$, $i \in \mathbb{Z}$,

$$\|h_1(t+\tau) - h_1(t)\| < \varepsilon.$$

Hence, by Lemma 3.1, for $t \in \mathbb{R}$, $|t - t_i| > \varepsilon$, $i \in \mathbb{Z}$, one has

$$\begin{aligned}
\|\Lambda_1(t+\tau) - \Lambda_1(t)\|_\beta &= \|A^\beta(\Lambda_1(t+\tau) - \Lambda_1(t))\| \\
&\leqslant \int_{-\infty}^{t} (t-s)^{\alpha-1}\|A^\beta \mathscr{S}(t-s)\|\|h_1(s+\tau) - h_1(s)\|\,\mathrm{d}s \\
&\leqslant \alpha\varepsilon M_\beta \int_{-\infty}^{t}\int_0^\infty \theta^{1-\beta}\xi_\alpha(\theta)(t-s)^{-\alpha\beta+\alpha-1}\mathrm{e}^{-\lambda\theta(t-s)^\alpha}\,\mathrm{d}\theta\,\mathrm{d}s \\
&= \alpha\varepsilon M_\beta \int_0^\infty\int_0^\infty \theta^{1-\beta}\xi_\alpha(\theta)\sigma^{-\alpha\beta+\alpha-1}\mathrm{e}^{-\lambda\theta\sigma^\alpha}\,\mathrm{d}\theta\,\mathrm{d}\sigma,
\end{aligned}$$

where $\sigma = t - s$. Note that

$$\begin{aligned}
&\alpha\int_0^\infty\int_0^\infty \theta^{1-\beta}\xi_\alpha(\theta)\sigma^{-\alpha\beta+\alpha-1}\mathrm{e}^{-\lambda\theta\sigma^\alpha}\,\mathrm{d}\theta\,\mathrm{d}\sigma \\
&\qquad = \alpha\int_0^\infty \xi_\alpha(\theta)\int_0^\infty \theta^{1-\beta}\sigma^{-\alpha\beta+\alpha-1}\mathrm{e}^{-\lambda\theta\sigma^\alpha}\,\mathrm{d}\sigma\,\mathrm{d}\theta \\
&\qquad = \frac{1}{\lambda^{1-\beta}}\int_0^\infty \xi_\alpha(\theta)\int_0^\infty (\lambda\theta\sigma^\alpha)^{-\beta}\mathrm{e}^{-\lambda\theta\sigma^\alpha}\,\mathrm{d}(\lambda\theta\sigma^\alpha)\,\mathrm{d}\theta = \frac{\Gamma(1-\beta)}{\lambda^{1-\beta}}.
\end{aligned}$$

Hence, one has

$$\|\Lambda_1(t+\tau) - \Lambda_1(t)\|_\beta \leqslant \frac{\Gamma(1-\beta)M_\beta\varepsilon}{\lambda^{1-\beta}},$$

which implies that $\Lambda_1 \in AP_T(\mathbb{R}, X_\beta)$.

(ii) $\Lambda_2 \in PAP_T^0(\mathbb{R}, X_\beta)$. In fact, for $r > 0$ one has

$$\begin{aligned}\frac{1}{2r}\int_{-r}^{r}\|\Lambda_2(t)\|_\beta\,\mathrm{d}t &= \frac{1}{2r}\int_{-r}^{r}\|A^\beta\Lambda_2(t)\|\,\mathrm{d}t \\ &\leqslant \frac{1}{2r}\int_{-r}^{r}\int_{-\infty}^{t}(t-s)^{\alpha-1}\|A^\beta\mathscr{S}(t-s)\|\|h_2(s)\|\,\mathrm{d}s\,\mathrm{d}t \\ &\leqslant \frac{\alpha M_\beta}{2r}\int_{-r}^{r}\int_{-\infty}^{t}\int_0^\infty \theta^{1-\beta}\xi_\alpha(\theta)(t-s)^{-\alpha\beta+\alpha-1}\mathrm{e}^{-\lambda\theta(t-s)^\alpha}\|h_2(s)\|\,\mathrm{d}\theta\,\mathrm{d}s\,\mathrm{d}t \\ &= \frac{\alpha M_\beta}{2r}\int_{-r}^{r}\int_0^\infty\int_0^\infty \theta^{1-\beta}\xi_\alpha(\theta)\sigma^{-\alpha\beta+\alpha-1}\mathrm{e}^{-\lambda\theta\sigma^\alpha}\|h_2(t-\sigma)\|\,\mathrm{d}\theta\,\mathrm{d}\sigma\,\mathrm{d}t \\ &= \alpha M_\beta\int_0^\infty\int_0^\infty \theta^{1-\beta}\xi_\alpha(\theta)\sigma^{-\alpha\beta+\alpha-1}\mathrm{e}^{-\lambda\theta\sigma^\alpha}H_r(\sigma)\,\mathrm{d}\theta\,\mathrm{d}\sigma,\end{aligned}$$

where

$$H_r(\sigma) = \frac{1}{2r}\int_{-r}^{r}\|h_2(t-\sigma)\|\,\mathrm{d}t.$$

Since $h_2 \in PAP_T^0(\mathbb{R}, X)$, it follows that $h_2(\cdot - \sigma) \in PAP_T^0(\mathbb{R}, X)$ for each $\sigma \in \mathbb{R}$ by Remark 2.1, then $\lim_{r\to\infty} H_r(\sigma) = 0$ for all $\sigma \in \mathbb{R}$. Hence $\Lambda_2 \in PAP_T^0(\mathbb{R}, X_\beta)$. $\square$

**Theorem 3.4.** *Assume that* (H$_1$)–(H$_5$) *hold. If* $\Theta < 1$, *where*

$$\Theta = M_\beta(L_g C_k \eta^{-1} + L_f)\frac{\Gamma(1-\beta)}{\lambda^{1-\beta}} + 2L_1 N M_\beta\Big(\frac{1}{m^\beta} + \frac{1}{\mathrm{e}^\lambda - 1}\Big),$$

*then* (1.1) *has a unique PC-mild solution* $u \in PAP_T(\mathbb{R}, X_\beta)$.

Proof. Let $\mathcal{F}\colon PAP_T(\mathbb{R}, X_\beta) \to PC(\mathbb{R}, X_\beta)$ be the operator defined by

$$(3.5)\qquad \begin{aligned}(\mathcal{F}u)(t) = \int_{-\infty}^{t}&(t-s)^{\alpha-1}\mathscr{S}(t-s)(f(s,u(s)) + (Ku)(s))\,\mathrm{d}s \\ &+ \sum_{\tau_k<t}\mathscr{T}(t-\tau_k)G_k(u(\tau_k)).\end{aligned}$$

We will show that $\mathcal{F}$ has a fixed point in $PAP_T(\mathbb{R}, X_\beta)$ and divide the proof into several steps.

(i) $\mathcal{F}u \in PAP_T(\mathbb{R}, X_\beta)$. For $u \in PAP_T(\mathbb{R}, X_\beta)$, by Lemma 3.3, one has

$$(\Lambda u)(t) = \int_{-\infty}^{t}(t-s)^{\alpha-1}\mathscr{S}(t-s)(f(s,u(s)) + (Ku)(s))\,\mathrm{d}s \in PAP_T(\mathbb{R}, X_\beta).$$

It remains to show that

$$(3.6)\qquad \sum_{\tau_k<t}\mathscr{T}(t-\tau_k)G_k(u(\tau_k)) \in PAP_T(\mathbb{R}, X_\beta).$$

By Theorem 2.5, $G_k(u(\tau_k)) \in \widetilde{P}AP(\mathbb{Z},X)$. Let $G_k(u(\tau_k)) = \beta_k + \gamma_k$, where $\beta_k \in AP(\mathbb{Z},X)$ and $\gamma_k \in \widetilde{P}AP_0(\mathbb{Z},X)$, then

$$\sum_{\tau_k<t} \mathscr{T}(t-\tau_k)G_k(u(\tau_k)) = \sum_{\tau_k<t} \mathscr{T}(t-\tau_k)\beta_k + \sum_{\tau_k<t} \mathscr{T}(t-\tau_k)\gamma_k := \Phi_1(t)+\Phi_2(t).$$

Since $\{\tau_k^j\}$, $k,j\in\mathbb{Z}$ are equipotentially almost periodic, hence by Lemma 2.2, for any $\varepsilon>0$ there exist relative dense sets of real numbers $\Omega_\varepsilon$ and integers $Q_\varepsilon$ such that for $\tau_k < t \leqslant \tau_{k+1}$, $\tau\in\Omega_\varepsilon$, $q\in Q_\varepsilon$, $|t-\tau_k|>\varepsilon$, $|t-\tau_{k+1}|>\varepsilon$, $j\in\mathbb{Z}$, one has

$$t+\tau > \tau_k+\varepsilon+\tau > \tau_{k+q},$$

and

$$\tau_{k+q+1} > \tau_{k+1}+\tau-\varepsilon > t+\tau,$$

that is $\tau_{k+q} < t+\tau < \tau_{k+q+1}$. Then

$$\begin{aligned}
\|\Phi_1(t+\tau)-\Phi_1(t)\|_\beta &= \Big\| \sum_{\tau_k<t+\tau} \mathscr{T}(t+\tau-\tau_k)\beta_k - \sum_{\tau_k<t} \mathscr{T}(t-\tau_k)\beta_k \Big\|_\beta \\
&\leqslant \sum_{\tau_k<t} \|\mathscr{T}(t-\tau_k)(\beta_{k+q}-\beta_k)\|_\beta \\
&= \sum_{\tau_k<t} \|A^\beta \mathscr{T}(t-\tau_k)\| \|(\beta_{k+q}-\beta_k)\| \\
&\leqslant M_\beta \varepsilon \sum_{\tau_k<t} \int_0^\infty \theta^{-\beta}\xi_\alpha(\theta)(t-\tau_k)^{-\alpha\beta} \mathrm{e}^{-\lambda\theta(t-\tau_k)^\alpha}\,\mathrm{d}\theta \\
&\leqslant M_\beta\varepsilon \int_0^\infty \xi_\alpha(\theta) \Big( \sum_{0<\theta(t-\tau_k)^\alpha \leqslant 1} (\theta(t-\tau_k)^\alpha)^{-\beta} \mathrm{e}^{-\lambda\theta(t-\tau_k)^\alpha} \\
&\qquad + \sum_{j=1}^\infty \sum_{j<\theta(t-\tau_k)^\alpha\leqslant j+1} (\theta(t-\tau_k)^\alpha)^{-\beta} \mathrm{e}^{-\lambda\theta(t-\tau_k)^\alpha} \Big)\,\mathrm{d}\theta \\
&\leqslant M_\beta\varepsilon \int_0^\infty \xi_\alpha(\theta)\Big(\frac{2N}{m^\beta}+\frac{2N}{\mathrm{e}^\lambda-1}\Big)\,\mathrm{d}\theta \\
&= 2M_\beta N\varepsilon\Big(\frac{1}{m^\beta}+\frac{1}{\mathrm{e}^\lambda-1}\Big),
\end{aligned}$$

where $m=\min\{\theta(t-\tau_k)^\alpha\colon\ 0<\theta(t-\tau_k)^\alpha\leqslant 1\}$. Hence $\Phi_1\in AP_T(\mathbb{R},X_\beta)$.

Next, we show that $\Phi_2\in PAP_T^0(\mathbb{R},X_\beta)$. Since $\gamma_k\in\widetilde{P}AP_0(\mathbb{Z},X)$, by Lemma 2.4 and [13] there exists $g(t)=\gamma_k$, $t\in[k,k+1)$ such that $g\in\widetilde{P}AP_0(\mathbb{R},X)$ and $g(k)=\gamma_k$, $k\in\mathbb{Z}$.

By Lemma 3.1, one has

$$\begin{aligned}
\frac{1}{2r}\int_{-r}^{r}\|\Phi_2(t)\|_\beta\,\mathrm{d}t &= \frac{1}{2r}\int_{-r}^{r}\Big\|\sum_{\tau_k<t}\mathscr{T}(t-\tau_k)\gamma_k\Big\|_\beta\,\mathrm{d}t\\
&\leqslant \frac{1}{2r}\int_{-r}^{r}\sum_{\tau_k<t}\|A^\beta\mathscr{T}(t-\tau_k)\|\|\gamma_k\|\\
&\leqslant \frac{M_\beta}{2r}\int_{-r}^{r}\sum_{\tau_k<t}\int_0^\infty \theta^{-\beta}\xi_\alpha(\theta)(t-\tau_k)^{-\alpha\beta}\mathrm{e}^{-\lambda\theta(t-\tau_k)^\alpha}\|g(k)\|\,\mathrm{d}\theta\,\mathrm{d}t\\
&\leqslant \frac{M_\beta}{2r}\int_{-r}^{r}\sum_{\tau_k<t}\int_0^\infty \xi_\alpha(\theta)(\theta(t-\tau_k)^\alpha)^{-\beta}\mathrm{e}^{-\lambda\theta(t-\tau_k)^\alpha}\|g(t)\|\,\mathrm{d}\theta\,\mathrm{d}t\\
&\leqslant \frac{M_\beta}{2r}\int_{-r}^{r}\int_0^\infty \xi_\alpha(\theta)\Big(\sum_{0<\theta(t-\tau_k)^\alpha\leqslant 1}(\theta(t-\tau_k)^\alpha)^{-\beta}\mathrm{e}^{-\lambda\theta(t-\tau_k)^\alpha}\\
&\quad+\sum_{j=1}^{\infty}\sum_{j<\theta(t-\tau_k)^\alpha\leqslant j+1}(\theta(t-\tau_k)^\alpha)^{-\beta}\mathrm{e}^{-\lambda\theta(t-\tau_k)^\alpha}\Big)\|g(t)\|\,\mathrm{d}\theta\,\mathrm{d}t\\
&\leqslant \frac{M_\beta}{2r}\int_{-r}^{r}\int_0^\infty \xi_\alpha(\theta)\Big(\frac{2N}{m^\beta}+\frac{2N}{\mathrm{e}^\lambda-1}\Big)\|g(t)\|\,\mathrm{d}\theta\,\mathrm{d}t\\
&\leqslant 2NM_\beta\Big(\frac{1}{m^\beta}+\frac{1}{\mathrm{e}^\lambda-1}\Big)\frac{1}{2r}\int_{-r}^{r}\|g(t)\|\,\mathrm{d}t,
\end{aligned}$$

where $N$ is the constant in Lemma 2.1. Hence

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\|\Phi_2(t)\|_\beta\,\mathrm{d}t=0,$$

then $\Phi_2\in PAP_T^0(\mathbb{R},X_\beta)$. So $\mathcal{F}u\in PAP_T(\mathbb{R},X_\beta)$.

(ii) $\mathcal{F}$ is a contraction. For $u,v\in PAP_T(\mathbb{R},X_\beta)$,

$$\begin{aligned}
\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\|_\beta &\leqslant \int_{-\infty}^{t}\|(t-s)^{\alpha-1}\mathscr{S}(t-s)[((Ku)(s)+f(s,u(s)))\\
&\quad-((Kv)(s)+f(s,v(s)))]\|_\beta\,\mathrm{d}s\\
&\quad+\sum_{\tau_k<t}\|\mathscr{T}(t-\tau_k)[G_k(u(\tau_k))-G_k(v(\tau_k))]\|_\beta\\
&\leqslant \int_{-\infty}^{t}(t-s)^{\alpha-1}\|A^\beta\mathscr{S}(t-s)\|\|[((Ku)(s)+f(s,u(s)))\\
&\quad-((Kv)(s)+f(s,v(s)))]\|\,\mathrm{d}s\\
&\quad+\sum_{\tau_k<t}\|A^\beta\mathscr{T}(t-\tau_k)\|\|[G_k(u(\tau_k))-G_k(v(\tau_k))]\|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \alpha M_\beta(L_g C_k \eta^{-1} + L_f)\|u-v\|_\beta \\
&\quad \times \int_{-\infty}^{t}\int_0^\infty \theta^{1-\beta}\xi_\alpha(\theta)(t-s)^{-\alpha\beta+\alpha-1}\mathrm{e}^{-\lambda\theta(t-s)^\alpha}\,\mathrm{d}\theta\,\mathrm{d}s \\
&\quad + L_1 M_\beta\|u-v\|_\beta \sum_{\tau_k<t}\int_0^\infty \theta^{-\beta}\xi_\alpha(\theta)(t-\tau_k)^{-\alpha\beta}\mathrm{e}^{-\lambda\theta(t-\tau_k)^\alpha}\,\mathrm{d}\theta \\
&\leqslant M_\beta(L_g C_k\eta^{-1}+L_f)\frac{\Gamma(1-\beta)}{\lambda^{1-\beta}}\|u-v\|_\beta \\
&\quad + 2L_1 N M_\beta\Big(\frac{1}{m^\beta}+\frac{1}{\mathrm{e}^\lambda-1}\Big)\|u-v\|_\beta \\
&= \Theta\|u-v\|_\beta.
\end{aligned}
$$

Since $\Theta < 1$, $\mathcal{F}$ is a contraction.

By (i), $\mathcal{F}(PAP_T(\mathbb{R}, X_\beta)) \subset PAP_T(\mathbb{R}, X_\beta)$. Since (ii) holds, by the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $PAP_T(\mathbb{R}, X_\beta)$, which is the unique piecewise pseudo almost periodic $PC$-mild solution of (1.1). □

If $(Ku)(t) = 0$, then (1.1) is an impulsive fractional differential equation

$$
{}^cD^\alpha u(t) + Au(t) = f(t, u(t)) + \sum_{k=-\infty}^{\infty} G_k(u(t))\delta(t-\tau_k). \tag{3.7}
$$

By Theorem 3.4, one has the following result:

**Corollary 3.5.** *Assume that* (H$_1$), (H$_3$), (H$_5$) *hold. If* $M_\beta L_f\Gamma(1-\beta)/\lambda^{1-\beta} + 2L_1NM_\beta(1/m^\beta + 1/(\mathrm{e}^\lambda - 1)) < 1$, *then* (3.7) *has a unique $PC$-mild solution* $u \in PAP_T(\mathbb{R}, X_\beta)$.

## 4. Examples

In this section, we provide some examples to illustrate our main results.

**Example 4.1.** Consider the fractional partial differential equation with impulsive effects

$$
\begin{cases}
{}^cD^\alpha w(t, x(t)) - \dfrac{\partial^2 w(t,x)}{\partial x^2} \\
\quad = \displaystyle\int_{-\infty}^{t} k(t-s)g(s,x,w(s,x))\,\mathrm{d}s + \gamma F(t)\cos(w(t,x)), \\
\qquad\qquad\qquad t\in\mathbb{R},\ t\neq\tau_k,\ k\in\mathbb{Z},\ x\in(0,1), \\
w(\tau_k^+, x) = (\beta_k+1)w(\tau_k, x), \quad k\in\mathbb{Z},\ x\in[0,1], \\
w(t,0) = w(t,1) = 0,
\end{cases} \tag{4.1}
$$

where $1 < \alpha \leqslant 1$, $\tau_k = k + |\sin k + \sin\sqrt{2}k|/4$, $F \in PAP_T(\mathbb{R}, X)$, $\beta_k \in PAP(\mathbb{Z}, \mathbb{R})$. Note that integero $\{\tau_k^j\}$, $k \in \mathbb{Z}$, $j \in \mathbb{Z}$ are equipotentially almost periodic and $\kappa = \inf_{i\in\mathbb{Z}}(\tau_{k+1} - \tau_k) > 0$; one can see [16], [21] for more details.

Let $X = (L^2[0,1], \|\cdot\|_{L^2})$, define the linear operator $-A$ by

$$D(-A) = \{u \in X \colon u'' \in X,\ u(0) = u(1) = 0\} \quad \text{and} \quad -Au = \Delta u = u'',\ u \in D(-A).$$

It is well known, see [19] that $-A$ is the infinitesimal generator of a semigroup $S(t)$ on $X$ with $\|S(t)\|_{L^2} \leqslant \mathrm{e}^{-t}$ for $t \geqslant 0$, hence ($\mathrm{H}_1$) holds. Let $u(t)x = w(t,x)$, $t \in \mathbb{R}$, $x \in [0,1]$, then (4.8) can be rewritten in the abstract form (1.1). Since $G_k(u) = \beta_k u$ and $\beta_k \in PAP(\mathbb{Z}, \mathbb{R})$, ($\mathrm{H}_5$) holds with $L_1 = \sup_{k\in\mathbb{Z}} \|\beta_k\|$. By Theorem 3.4, one has

**Theorem 4.1.** *Under assumptions* ($\mathrm{H}_2$), ($\mathrm{H}_4$), *if* $L = \max\{\gamma, L_g, L_1\}$ *is sufficiently small, then* (4.8) *has a unique PC-mild* $PAP_T$ *solution.*

**Example 4.2.** Consider a two-dimensional impulsive fractional predator-prey system with diffusion

$$\tag{4.2} \begin{cases} {}^{c}D^{\alpha}u(t,x(t)) = \mu_1 \Delta u + u\Big[a_1(t,x) - b(t,x)u - \dfrac{c_1(t,x)v}{r(t,x)v + u}\Big], \\ \hfill t \in \mathbb{R},\ t \neq \tau_k,\ k \in \mathbb{Z}, \\ {}^{c}D^{\alpha}v(t,x(t)) = \mu_2 \Delta v + v\Big[-a_2(t,x) + \dfrac{c_2(t,x)u}{r(t,x)u + v}\Big], \\ \hfill t \in \mathbb{R},\ t \neq \tau_k,\ k \in \mathbb{Z}, \\ u(\tau_k^+, x) = u(\tau_k, x) I_k(x, u(\tau_k, x), v(\tau_k, x)), \quad k \in \mathbb{Z}, \\ v(\tau_k^+, x) = v(\tau_k, x) J_k(x, u(\tau_k, x), v(\tau_k, x)), \quad k \in \mathbb{Z}, \\ \dfrac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0,\ \dfrac{\partial v}{\partial n}\Big|_{\partial\Omega} = 0, \end{cases}$$

where $0 < \alpha \leqslant 1$, in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, nonuniformly distributed in the domain $\overline{\Omega} = \Omega \times \partial\Omega$; $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \ldots + \partial^2/\partial x_n^2$ is the Laplace operator and $\partial/\partial n$ is the outward normal derivative. $\mu_1 > 0$, $\mu_2 > 0$ are diffusion coefficients, the positive functions $a_1, a_2, c_1$ and $c_2$ stand for prey intrinsic growth rate, capturing rate of the predator, death rate of the predator and conversion rate, respectively; one can see [3] for more details.

Let

$$\tau_k = k + \alpha_k, \quad k \in \mathbb{Z},$$

where $\{\alpha_k\}$, $\alpha_k \in \mathbb{R}$, $k \in \mathbb{Z}$ is an almost periodic sequence such that

$$\sup_{k\in\mathbb{Z}} |\alpha_k| = \alpha < \frac{1}{2},$$

then $\{\tau_k^j\}$, $k, j \in \mathbb{Z}$ are equipotentially almost periodic and $\kappa = \inf_{k \in \mathbb{Z}} (\tau_{k+1} - \tau_k) > 0$; one can see [23], [24] for more details.

Let $w = (u, v)$ and

$$A = \begin{bmatrix} \lambda - \mu_1 \Delta & 0 \\ 0 & \lambda - \mu_2 \Delta \end{bmatrix},$$

$$f(t, w) = \begin{bmatrix} u\Big[a_1(t, x) - b(t, x)u - \dfrac{c_1(t, x)v}{r(t, x)v + u}\Big] + \lambda u \\ v\Big[-a_2(t, x) + \dfrac{c_2(t, x)u}{r(t, x)u + v}\Big] + \lambda v \end{bmatrix},$$

$$G_k(w(\tau_k)) = \begin{bmatrix} u(\tau_k, x)I_k(x, u(\tau_k, x), v(\tau_k, x)) - u(\tau_k, x) \\ v(\tau_k, x)J_k(x, u(\tau_k, x), v(\tau_k, x)) - v(\tau_k, x) \end{bmatrix},$$

where $\lambda > 0$, then (4.9) can be rewritten in the form (3.7):

$$^{c}D^{\alpha} w(t) + Aw(t) = f(t, w(t)) + \sum_{k=-\infty}^{\infty} G_k(w(t))\delta(t - \tau_k).$$

It is well known [22] that the operator $A$ is sectorial and $\operatorname{Re} \sigma(A) \leqslant -\lambda$, the analytic semigroup of the operator $A$ is $\mathrm{e}^{-At}$ and

$$A^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^{\infty} t^{\beta-1} \mathrm{e}^{-At} \, \mathrm{d}t.$$

Assume that

($\mathrm{A}_1$) $a_i(t, x)$, $c_i(t, x)$, $i = 1, 2$, $b(t, x)$ and $r(t, x)$ are piecewise pseudo almost periodic functions with respect to $t$, uniformly for $x \in \overline{\Omega}$, and positive-valued on $\mathbb{R} \times \overline{\Omega}$.

($\mathrm{A}_2$) The sequences of functions $\{I_k(x, u, v)\}$, $\{J_k(x, u, v)\}$, $k \in \mathbb{Z}$ are generalized pseudo almost periodic with respect to $k$, uniformly for $x, u, v \in \overline{\Omega}$.

By Corollary 3.5, one has

**Theorem 4.2.** *Under assumptions* ($\mathrm{A}_1$), ($\mathrm{A}_2$), ($\mathrm{H}_3$), ($\mathrm{H}_5$), *if* $L = \max\{L_f, L_1\}$ *is sufficiently small, then* (4.9) *has a unique PC-mild* $PAP_T$ *solution.*

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