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THE CAUCHY PROBLEM FOR THE LIQUID CRYSTALS SYSTEM IN THE CRITICAL BESOV SPACE WITH NEGATIVE INDEX

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Abstract. The local well-posedness for the Cauchy problem of the liquid crystals system in the critical Besov space $\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)\times\dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$ with $n<p<2n$ is established by using the heat semigroup theory and the Littlewood-Paley theory. The global well-posedness for the system is obtained with small initial datum by using the fixed point theorem. The blow-up results for strong solutions to the system are also analysed.

Keywords: liquid crystals system; critical Besov space; negative index; well-posedness; blow-up

MSC 2010: 35Q35, 76A15, 35B44

1. Introduction

We are concerned with the liquid crystals system

$$(1.1)\qquad \begin{cases} u_t+u\cdot\nabla u-\mu\Delta u+\nabla P=-\nabla\cdot(\nabla d\odot\nabla d), & t>0,\ x\in\mathbb{R}^n,\\ d_t+u\cdot\nabla d-\Delta d=|\nabla d|^2 d, & t>0,\ x\in\mathbb{R}^n,\\ \nabla\cdot u=0, & t>0,\\ u(0,x)=u_0(x),\quad d(0,x)=d_0(x), & x\in\mathbb{R}^n, \end{cases}$$

where $\mu>0$ is a constant, $u(t,x)=(u_1(t,x),u_2(t,x),\ldots,u_n(t,x))$ is the velocity, $d(t,x)=(d_1(t,x),d_2(t,x),\ldots,d_n(t,x))$ is the macroscopic average of molecular arrangement, $\nabla d\odot\nabla d$ is a matrix $(\nabla d)^T\nabla d$, whose (i,j)th entry is

$$\sum_{k=1}^{n}\frac{\partial d_k(t,x)}{\partial x_i}\frac{\partial d_k(t,x)}{\partial x_j}.$$

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The elements u and d satisfy the far field behavior $u \to 0$, $d \to \bar{d_0}$ as $|x| \to \infty$, where $\bar{d_0}$ is a constant vector with $|\bar{d_0}| = 1$. The initial value satisfies $|d_0(x)| = 1$, $(u_0, d_0) \in \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n) \times \dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$. The scalar function $P(t, x)$ is the pressure.

Before stating the main results, we give a brief overview of several related works in the literature. If d is a constant vector, system (1.1) becomes the classical incompressible Navier-Stokes equation

$$\begin{cases} u_t + u\cdot\nabla u - \mu\Delta u + \nabla P = 0, & t > 0,\ x \in \mathbb{R}^n, \\ \nabla\cdot u = 0, & t > 0, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

If $u(t, x)$ is a solution to problem (1.2) with initial datum $u_0(x)$, then $u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$ is also a solution to problem (1.2) with initial datum $u_{0\lambda}(x) = \lambda u_0(\lambda x)$. If the space norm is invariant under the scaling $u(x) = \lambda u(\lambda x)$, $\lambda > 0$, then it is a critical space for problem (1.2) defined in [5]. In 1934, Leray established the existence and uniqueness of global weak solutions to problem (1.2) in the critical Sobolev space $L^2(\mathbb{R}^2)$ with finite energy. However, the uniqueness of solutions to problem (1.2) in the Sobolev space $L^2(\mathbb{R}^n)$, $n \geqslant 3$, has not been proved. Fujita and Kato in [12] obtained the uniqueness of solutions with $u_0 \in \dot{H}^{(n/2)-1}(\mathbb{R}^n)$. The Sobolev space $\dot{H}^{(n/2)-1}(\mathbb{R}^n)$ is a critical space for problem (1.2), and $s = (n/2) - 1$ is the lowest index for which the uniqueness of solutions has been proved in the Sobolev space $\dot{H}^s$.

For the density-dependent Navier-Stokes system with variable viscosity

$$\begin{cases} \varrho_t + u\cdot\nabla\varrho = 0, & t > 0,\ x \in \mathbb{R}^n, \\ \varrho u_t + \varrho u\cdot\nabla u - \operatorname{div}[2\mu(\varrho)d] + \nabla P = 0, & t > 0,\ x \in \mathbb{R}^n, \\ \nabla\cdot u = 0, & t > 0, \\ \varrho(0, x) = \varrho_0(x), \quad u(0, x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

Abidi in [1] proved the global existence of solutions in the critical Besov space $\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n) \times \dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$ in general space dimension n with small initial datum. Danchin and Mucha in [9] proved the global well-posedness for system (1.3) with $\mu(\varrho) = \mu > 0$ in space dimension n in the critical space with small initial datum $(u_0, \varrho_0 - 1) \in \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n) \times \dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$, $1 \leqslant p < 2n$. For $n = 2$, Abidi and Zhang in [3] studied system (1.3) in the critical Besov space with initial datum $(u_0, \varrho_0 - 1) \in \dot{B}^{2/p-1}_{p,1}(\mathbb{R}^2) \times \dot{B}^{2/p}_{p,1}(\mathbb{R}^2)$, $1 < p < 4$. They established the global well-posedness for system (1.3) with weaker assumptions on initial datum as compared with that in [15]. We note that the obtained solution $u(t, x)$ belongs to the critical Besov space $\dot{B}^{2/p-1}_{p,1}(\mathbb{R}^2)$ with negative index when $2 < p < 4$ in [3]. One may check [3], [15], [24], [2] for more details in this direction and the references therein.

Roughly speaking, system (1.1) is a strongly coupled system between the incompressible Navier-Stokes equation and the transported heat flow of harmonic map. It is a macroscopic continuum description of the time evolution of materials, which are influenced by the flow field $u(t, x)$ and the orientation configurations of rod-like liquid crystals $d(t, x)$. The hydrodynamic theory of the nematic liquid crystals is established by Ericksen in [11]. Lin in [18] studied a simple model for the liquid crystals system. After that, many good results for the approximate liquid crystals system were derived in [20], [21]. Hong in [14] proved the global existence of solutions to the liquid crystals system with initial datum belonging to the space $L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ by energy estimates. The solutions were regular except for a finite number of singular times. Wang in [25] established global well-posedness for the heat flow of harmonic map in the space $\mathrm{BMO}(\mathbb{R}^n)$. Similarly to [25], Lin and Ding in [22] studied system (1.1) in the critical space $L^n(\mathbb{R}^n) \times \dot{W}^{1,n}(\mathbb{R}^n)$ by using the contraction mapping theory. They obtained the local well-posedness for the system and the global well-posedness with small initial datum. Xu and Zhang in [27] established the global existence and regularity of weak solutions to system (1.1) in space dimension $n = 2$ with large velocity. The uniqueness of weak solutions was proved by using the Littlewood-Paley theory. Chen and Miao in [7] proved the global well-posedness for the micropolar fluid system in the critical Besov space $\dot{B}^{3/p-1}_{p,\infty}(\mathbb{R}^3) \times \dot{B}^{3/p}_{p,\infty}(\mathbb{R}^3)$, $1 \leqslant p < 6$, by making a suitable transformation of solutions and using the Fourier localization method. Hao and Liu in [13] studied system (1.1) in the critical Besov space $\dot{B}^{(n/2)-1}_{2,1}(\mathbb{R}^n) \times \dot{B}^{n/2}_{2,1}(\mathbb{R}^n)$ with $n \geqslant 2$. They established the local well-posedness for the system and the global well-posedness with small initial datum. They also presented the blow-up criterion of solutions. For system (1.1) without the term $|\nabla d|^2 d$, Zhao, Liu and Cui in [28] established the local well-posedness for the system in the critical Besov space $\dot{B}^{n/p-1}_{p,r}(\mathbb{R}^n) \times \dot{B}^{n/p}_{p,r}(\mathbb{R}^n)$ with $2 \leqslant p < 2n$. The global well-posedness for the system was obtained with small initial datum. They also derived the blow-up criterion of solutions. Here we note that the obtained solution $u(t, x)$ belongs to the Besov space $\dot{B}^{n/p-1}_{p,r}(\mathbb{R}^n)$ with negative index when $n < p < 2n$. Xu, Hao and Yuan in [26] studied the well-posedness for the density-dependent incompressible flow of liquid crystals in the critical Besov space $\dot{B}^{n/p}_{p,r}(\mathbb{R}^n) \times \dot{B}^{n/p-1}_{p,r}(\mathbb{R}^n) \times \dot{B}^{n/p}_{p,r}(\mathbb{R}^n)$, $n \geqslant 2$, with negative regularity indices. They established the local well-posedness for the system with large initial velocity field and director field under the condition that the initial density was close to a positive constant. The global well-posedness for the system was obtained with small initial datum. We note that the blow-up criteria of solutions to the system have not been discussed yet. Jiang, Jiang and Wang in [16] considered a simplified Ericksen-Leslie system of the two dimensional compressible flow of nematic liquid crystals. They established the global existence of weak solutions under a restriction imposed on the initial energy including the case of small initial energy. For other methods of establishing the local well-posedness for the incompressible flow of liquid crystals systems and global existence of solutions, the reader is referred to [23], [10], [19], [6], [17] and the references therein.

For system (1.1), we introduce the scaling transformation

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x), \quad d_\lambda(t,x) = d(\lambda^2 t, \lambda x), \quad P_\lambda(t,x) = \lambda^2 P(\lambda^2 t, \lambda x), \tag{1.4}$$

where (u, d, P) is a solution to system (1.1). It follows that $(u_\lambda, d_\lambda, P_\lambda)$ still satisfies system (1.1) with initial data $u_{0\lambda} = \lambda u_0(\lambda x)$ and $d_{0\lambda} = d_0(\lambda x)$. In this paper, we mainly consider the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^n)$. We find

$$\|u_\lambda(t,x)\|_{\dot{B}^{n/p-1}_{p,1}} = \|u(t,x)\|_{\dot{B}^{n/p-1}_{p,1}}, \|d_\lambda(t,x)\|_{\dot{B}^{n/p}_{p,1}} = \|d(t,x)\|_{\dot{B}^{n/p}_{p,1}}.$$

The norm in $\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n) \times \dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$ is invariant under the scaling transformation (1.4). Motivated by [3], [13], [28], we study system (1.1) in the critical Besov space $\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n) \times \dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$. In addition, we suppose $|d(t,x)| = 1$ and note that $\dot{B}^{n/p}_{p,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$. Let $\widetilde{\mathcal{R}}$ be the usual Riesz transform. The Riesz transform $\widetilde{\mathcal{R}}$ maps continuously from the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^n)$ to $\dot{B}^s_{p,r}(\mathbb{R}^n)$ with the operator norm $\|\widetilde{\mathcal{R}}\|_{\mathcal{L}(\dot{B}^s_{p,r}, \dot{B}^s_{p,r})} \leqslant C_0$. We denote $\mathbb{Q} = \nabla(\Delta^{-1})\nabla\cdot$ and let $\mathbb{P} = \mathbb{I} - \mathbb{Q}$ be the Leray projection operator on the space of divergence free vector fields. Then the operator $\mathbb{P}$ maps continuously from the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^n)$ to $\dot{B}^s_{p,r}(\mathbb{R}^n)$. Using the operator $\mathbb{P}$, we project the first equation in system (1.1) onto the divergence free vector field. Then the term ∇P can be eliminated. Let $\tau(t,x) = d(t,x) - \bar{d_0}$, we have

$$\begin{cases} u_t - \mu\Delta u = -\mathbb{P}[u\cdot\nabla u + \nabla\cdot(\nabla\tau\odot\nabla\tau)], & t>0,\ x\in\mathbb{R}^n,\\ \tau_t - \Delta\tau = -u\cdot\nabla\tau + |\nabla\tau|^2\tau + |\nabla\tau|^2\bar{d_0}, & t>0,\ x\in\mathbb{R}^n,\\ \nabla\cdot u = 0, & t>0,\\ u|_{t=0} = u_0(x), \quad \tau|_{t=0} = \tau_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.5}$$

with conditions $d(0,x) = d_0(x)$, $\tau_0(x) = d_0(x) - \bar{d_0}$, $|d_0(x)| = 1$, $|\tau + \bar{d_0}| = 1$ and the far behaviors $u \to 0$, $\tau \to 0$ as $|x| \to \infty$. Motivated by the papers [23], [3], [25], [27], [13], [28], [26], we study the Cauchy problem for system (1.5) in the critical Besov space $\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n) \times \dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$ with negative index when $n < p < 2n$. This problem is meaningful and it has not been discussed yet. Because of the presence of nonlinear term $|\nabla\tau|^2\tau$ and the low regularity of the space, the conservation law of system (1.5) which plays an important role in studying the global well-posedness for system (1.5) in previous papers [14], [27], here is useless. However, this difficulty has been solved by using the Bony's paraproduct method.

Notation. Let $[A;B]=AB-BA$ be the commutator between the operators A and B. By $a\lesssim b$ we mean that there exists a uniform constant C, which may be different on different lines, such that $a\leqslant Cb$. Since functions in all the spaces are over $\mathbb{R}^n$, for simplicity, we drop $\mathbb{R}^n$ in our symbols if there is no ambiguity. We have $\widetilde{L}^\varrho_T\dot{B}^s_{p,r}=\widetilde{L}^\varrho([0,T];\dot{B}^s_{p,r})$, $\widetilde{L}^\varrho\dot{B}^s_{p,r}=\widetilde{L}^\varrho([0,\infty);\dot{B}^s_{p,r})$ and $\widetilde{L}^\varrho_{[T_1,T_2]}\dot{B}^s_{p,r}=\widetilde{L}^\varrho([T_1,T_2];\dot{B}^s_{p,r})$.

The main results of this paper are stated as follows.

Theorem 1.1. *Let $n\geqslant 2$, $n<p<2n$, $(u_0,\tau_0)\in\dot{B}^{n/p-1}_{p,1}\times\dot{B}^{n/p}_{p,1}$ and $\delta=\|u_0\|_{\dot{B}^{n/p-1}_{p,1}}+\|\tau_0\|_{\dot{B}^{n/p}_{p,1}}$. Then*

(I) [Local well-posedness.] *There exists $T>0$ such that system* (1.5) *has a unique local solution $(u,\tau)\in C([0,T];\dot{B}^{n/p-1}_{p,1})\cap\widetilde{L}^1_T\dot{B}^{n/p+1}_{p,1}\times C([0,T];\dot{B}^{n/p}_{p,1})\cap\widetilde{L}^1_T\dot{B}^{n/p+2}_{p,1}$.*

(II) [Global well-posedness.] *There exists $\delta_0>0$ such that if $\delta\leqslant\delta_0$, then system* (1.5) *has a unique global solution $(u,\tau)\in C([0,\infty);\dot{B}^{n/p-1}_{p,1})\cap\widetilde{L}^1\dot{B}^{n/p+1}_{p,1}\times C([0,\infty);\dot{B}^{n/p}_{p,1})\cap\widetilde{L}^1\dot{B}^{n/p+2}_{p,1}$.*

Remark 1.1. After having concluded the proof of Theorem 1.1 presented in this paper, we found that the paper [26], which was concerned with the well-posedness for the density-dependent incompressible flow of liquid crystals in the critical Besov spaces $\dot{B}^{n/p}_{p,1}(\mathbb{R}^n)\times\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)\times\dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$, $1<p\leqslant 2n$, $n\geqslant 2$, has been published recently. The system studied in [26] containes the density variable. For the case in which the density is a constant, we prove that the solution $u(t,x)$ to system (1.5) belongs to the Besov space $\dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)$ with negative index when $n<p<2n$ by establishing suitable a-priori estimates and using the heat semigroup theory. Thus, the proof procedure for the existence of solutions to the system is simplified significantly. We only present the main points of the proof of Theorem 1.1 for simplicity. We note that the blow-up mechanism of solutions to the system was not investigated in [26]. Moreover, the blow-up criteria of solutions to system (1.5) are also analysed in the present paper.

We obtain a blow-up result of solutions to system (1.5).

Theorem 1.2. *Let $n\geqslant 2$, $n<p<2n$ and $2<\varrho_1$, $\varrho_2<\infty$, $n/p+2/\varrho_1>3$, $n/p+2/\varrho_2>2$. Assume $(u_0,\tau_0)\in\dot{B}^{n/p-1}_{p,1}\times\dot{B}^{n/p}_{p,1}$ and let T^* be the maximal existence time of solutions to system* (1.5). *If $T^*<\infty$, then*

$$\lim_{t\to T^*}\left[\|u\|_{\widetilde{L}^{\varrho_1}_{T^*}\dot{B}^{2/\varrho_1-1}_{\infty,\infty}}+\|\tau\|_{\widetilde{L}^{\varrho_2}_{T^*}\dot{B}^{2/\varrho_2}_{\infty,\infty}}+\|\tau\|_{\widetilde{L}^{\varrho_1}_{T^*}\dot{B}^{n/p+2/\varrho_1}_{p,\infty}}\right]=\infty. \tag{1.6}$$

Remark 1.2. We obtain the blow-up mechanism of the solutions to system (1.5) under the condition that $u_0(x) \in \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n)$ with negative index when $n < p < 2n$. Theorem 1.2 improves the results in [13], where $p = 2$, $p \leqslant n$. Theorem 1.1 also improves the results in [28], where the nonlinear term $|\nabla d|^2 d$ was omitted in the system. It is worth pointing out that for the case $2 \leqslant p \leqslant n$, the proof is similar to [13], hence we omit it.

The remainder of this paper is organized as follows. In Section 2, the definition and some properties of the Besov space are reviewed. Section 3 is devoted to the proof of Theorem 1.1. The proof of Theorem 1.2 is presented in Section 4.

2. Preliminary

We recall some basic facts related to the Besov spaces. One may check [4], [5] for more details.

We write the definition of Besov spaces.

Definition 2.1 ([4])**.** Let $s \in \mathbb{R}$, $(p, r) \in [1, \infty]^2$, $u \in S'_h(\mathbb{R}^n)$ and we give the definition of the Besov spaces as follows:

$$\|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^n)} = \begin{cases} \left(\sum\limits_{j=-\infty}^{\infty} 2^{jrs} \|\dot{\Delta}_j f\|^r_{L^p} \right)^{1/r}, & r < \infty, \\ \sup\limits_{j\in\mathbb{Z}} 2^{js} \|\dot{\Delta}_j f\|_{L^p}, & r = \infty. \end{cases}$$

The properties of Besov spaces are presented as follows.

Proposition 2.1 ([4])**.** *Let $s \in \mathbb{R}$, $(p, r, p_1, p_2, r_1, r_2, \varrho) \in [1, \infty]^7$. Then:*

(1) $\dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^{s-n(1/p_1-1/p_2)}_{p_2,r_2}$, *if* $p_1 \leqslant p_2$, $r_1 \leqslant r_2$. $\dot{B}^{s_2}_{p,r_2} \hookrightarrow \dot{B}^{s_1}_{p,r_1}$ *is locally compact if* $s_1 \leqslant s_2$.

(2) *For all* $s > 0$, $\dot{B}^s_{p,r} \cap L^\infty$ *is an algebra.* $\dot{B}^s_{p,r}$ *is an algebra* $\Leftrightarrow \dot{B}^s_{p,r} \hookrightarrow L^\infty \Leftrightarrow s > n/p$ *or* $s \geqslant n/p$, $r = 1$.

(3) $\|u\|_{\widetilde{L}^\varrho_T \dot{B}^s_{p,r}} = \Big[\sum\limits_{j\in\mathbb{Z}} (2^{js}\|\dot{\Delta}_j u\|_{L^\varrho_T L^p})^r\Big]^{1/r}$ *and* $\|u\|_{\widetilde{L}^\varrho_T \dot{B}^s_{p,r}} \leqslant \|u\|_{L^\varrho_T \dot{B}^s_{p,r}}$ *if* $\varrho \leqslant r$; $\|u\|_{\widetilde{L}^\varrho_T \dot{B}^s_{p,r}} > \|u\|_{L^\varrho_T \dot{B}^s_{p,r}}$ *if* $\varrho > r$.

We present Bony's paraproduct decomposition in the homogeneous Besov spaces. We define $\dot{S}_j u = \sum\limits_{q=-\infty}^{j-1} \dot{\Delta}_q u$, $\mathcal{T}_u v = \sum\limits_{j\in\mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v$, $\mathcal{R}(u, v) = \sum\limits_{|j-j'|\leqslant 1} \dot{\Delta}_j u \dot{\Delta}_{j'} v$. For $u, v \in S'_h(\mathbb{R}^n)$, $uv = \mathcal{T}_u v + \mathcal{T}_v u + \mathcal{R}(u, v)$. The terms $\mathcal{T}_u v, \mathcal{T}_v u, \mathcal{R}(u, v)$ are estimated by the following two lemmas.

Lemma 2.1 ([4]). *There exists a constant C such that for any couple of real numbers (s,σ) with $\sigma>0$ and $(p,r,r_1,r_2)\in[1,\infty]^4$, $1/r=\min\{1,1/r_1+1/r_2\}$,*

$$\|\mathcal{T}\|_{\mathcal{L}(L^\infty\times\dot B^s_{p,r};\dot B^s_{p,r})}\leqslant C,\quad s<\frac{n}{p}\quad\text{or}\quad s=\frac{n}{p},\quad r=1;$$
$$\|\mathcal{T}\|_{\mathcal{L}(\dot B^{-\sigma}_{\infty,r_1}\times\dot B^s_{p,r_2};\dot B^{s-\sigma}_{p,r})}\leqslant C,\quad s-\sigma<\frac{n}{p}\quad\text{or}\quad s-\sigma=\frac{n}{p},\quad r=1.$$

Lemma 2.2 ([4]). *There exists a constant C such that for any couple of real numbers (s_1,s_2) and $(p,p_1,p_2,r,r_1,r_2)\in[1,\infty]^6$ with $s_1+s_2>0$, $1/p\leqslant 1/p_1+1/p_2\leqslant 1$, $1/r\leqslant 1/r_1+1/r_2\leqslant 1$, $\sigma-n/p=s_1-n/p_1+s_2-n/p_2$, we have*

$$\|\mathcal{R}\|_{\mathcal{L}(\dot B^{s_1}_{p_1,r_1}\times\dot B^{s_2}_{p_2,r_2};\dot B^{\sigma}_{p,r})}\leqslant C,\quad \sigma<\frac{n}{p}\quad\text{or}\quad \sigma=\frac{n}{p},\quad r=1.$$

For the heat equation

$$\begin{cases} u_t-a\Delta u=f(t,x), & t>0,\ x\in\mathbb{R}^n,\\ u|_{t=0}=u_0(x), & x\in\mathbb{R}^n,\end{cases}\tag{2.1}$$

we have the following two lemmas.

Lemma 2.3 ([4]). *Let $T>0$, $s\in\mathbb{R}$, $(\varrho,p,r)\in[1,\infty]^3$. Assume $u_0\in\dot B^s_{p,r}$, $f\in\widetilde L^{\varrho_1}_T\dot B^{s+(2/\varrho_1)-2}_{p,r}$. Then system (2.1) has a unique solution $u\in\widetilde L^\infty_T\dot B^s_{p,r}\cap\widetilde L^1_T\dot B^{s+2}_{p,r}$ satisfying*

$$a^{1/\varrho}\|u\|_{\widetilde L^\varrho_T\dot B^{s+2/\varrho}_{p,r}}\leqslant C(\|u_0\|_{\dot B^s_{p,r}}+a^{1/\varrho_1-1}\|f\|_{\widetilde L^{\varrho_1}_T\dot B^{s+2/\varrho_1-2}_{p,r}}),\quad 1\leqslant\varrho_1\leqslant\varrho\leqslant\infty.$$

In addition, if $1\leqslant r<\infty$, then $u\in C([0,T];\dot B^s_{p,r})$.

Lemma 2.4 ([13]). *Let $u(t)=\mathrm{e}^{at\Delta}u_0$, $u_0\in\dot B^s_{p,1}$, $(p,\varrho)\in[1,\infty]^2$, $s\in\mathbb{R}$ and let ε_0 be a constant. Assume $\|u_0\|_{\dot B^s_{p,1}}\leqslant c_0$. Then, for any small $\varepsilon_0>0$, there exists $T_0>0$ such that the estimate $\|u\|_{\widetilde L^\varrho_T\dot B^{s+2/\varrho}_{p,1}}\leqslant\varepsilon_0$, $0<T\leqslant T_0$ holds; for any small constant $\varepsilon_1>0$ there exists δ_1 such that if $\|u_0\|_{\dot B^s_{p,1}}\leqslant\delta_1$, then $\|u\|_{\widetilde L^\varrho\dot B^{s+2/\varrho}_{p,1}}\leqslant\varepsilon_1$.*

Lemma 2.5 ([4]). *Let $(s,p,r)\in(0,\infty)\times[1,\infty]^2$. Then there exists a constant C depending only on the dimension n, such that*

$$\|uv\|_{\dot B^s_{p,r}}\leqslant C(\|u\|_{L^\infty}\|v\|_{\dot B^s_{p,r}}+\|u\|_{\dot B^s_{p,r}}\|v\|_{L^\infty}).$$

Lemma 2.6 ([5]). *Let X be a Banach space with the norm $\|\cdot\|_X$ and let $B\colon X\times X\to X$ be a bilinear operator, $\eta>0$. For any $x_1,\ x_2\in X$ we have $\|B(x_1,x_2)\|_X\leqslant \eta\|x_1\|_X\|x_2\|_X$. Then for any $x_0\in X$ and $\|x_0\|_X<1/(4\eta)$, the equation $x=x_0+B(x,x)$ has a solution $x\in X$. In particular, the solution x satisfies $\|x\|_X\leqslant 2\|x_0\|_X$ and x is the unique solution satisfying $\|x\|_X<1/(2\eta)$.*

Lemma 2.7 ([4]). *Let $(p,q,q_1,q_2,r)\in[1,\infty]^5$, $s_1<n/p$, $s_2<n/p$, $s_1+s_2>0$ and $1/q=1/q_1+1/q_2$. Then there exists a constant $C>0$ depending only on p,q,q_1,q_2,r,s_1,s_2,n such that*

$$\|uv\|_{\widetilde{L}^q_T\dot{B}^{s_1+s_2-n/p}_{p,r}}\leqslant C\|u\|_{\widetilde{L}^{q_1}_T\dot{B}^{s_1}_{p,r}}\|v\|_{\widetilde{L}^{q_2}_T\dot{B}^{s_2}_{p,r}}.$$

3. The proof of Theorem 1.1

3.1. Local well-posedness. In order to prove Theorem 1.1, we use a standard iterative process to construct the approximate solutions to system (1.5). Assume $i\in N^*$ and let $u_1(t)=\mathrm{e}^{\mu t\Delta}u_0$, $\tau_1(t)=\mathrm{e}^{t\Delta}\tau_0$ for $i=1$. Then $u_1|_{t=0}=u_0$, $\tau_1|_{t=0}=\tau_0$. For $i\geqslant 2$, we define by induction a sequence of smooth functions $(u_i,\tau_i)_{i\in N^*}$ satisfying the system

$$\begin{cases}\partial_t u_i-\mu\Delta u_i=G_1, & t>0,\ x\in\mathbb{R}^n,\\ \partial_t\tau_i-\Delta\tau_i=G_2, & t>0,\ x\in\mathbb{R}^n,\\ \nabla\cdot u_i=0, & t>0,\\ u_i|_{t=0}=u_0,\quad \tau_i|_{t=0}=\tau_0, & x\in\mathbb{R}^n,\end{cases}\tag{3.1}$$

where

$$G_1=-\mathbb{P}[u_{i-1}\cdot\nabla u_{i-1}+\nabla\cdot(\nabla\tau_{i-1}\odot\nabla\tau_{i-1})],$$
$$G_2=-u_{i-1}\cdot\nabla\tau_{i-1}+|\nabla\tau_{i-1}|^2\tau_{i-1}+|\nabla\tau_{i-1}|^2\bar{d}_0.$$

First step: Uniform boundedness. We claim that the following estimates hold for some $T>0$ and there exists an absolute constant C such that

$$\|u_i\|_{\widetilde{L}^4_T\dot{B}^{n/p-1/2}_{p,1}}+\|\tau_i\|_{\widetilde{L}^4_T\dot{B}^{n/p+1/2}_{p,1}}\leqslant(C+1)\varepsilon_0,\tag{3.2}$$

$$\|\tau_i\|_{\widetilde{L}^\infty_T\dot{B}^{n/p}_{p,1}}\leqslant(C+1)\delta.\tag{3.3}$$

By induction, for $i=1$, $u_1=\mathrm{e}^{\mu t\Delta}u_0$, $\tau_1=\mathrm{e}^{t\Delta}\tau_0$. Using Lemma 2.4, we deduce that (3.2) and (3.3) hold for $i=1$. Now we assume that (3.2) and (3.3) hold for $i-1$, i.e.,

$$\|u_{i-1}\|_{\widetilde{L}^4_T\dot{B}^{n/p-1/2}_{p,1}}+\|\tau_{i-1}\|_{\widetilde{L}^4_T\dot{B}^{n/p+1/2}_{p,1}}\leqslant(C+1)\varepsilon_0,$$
$$\|\tau_{i-1}\|_{\widetilde{L}^\infty_T\dot{B}^{n/p}_{p,1}}\leqslant(C+1)\delta.$$

Applying Lemma 2.3 to the second equation in (3.1) and taking $\varrho = \infty$, $s = n/p$, $\varrho_1 = 2$ yields

$$\|\tau_i\|_{\widetilde{L}^\infty_T \dot{B}^{n/p}_{p,1}} \lesssim \|\tau_0\|_{\dot{B}^{n/p}_{p,1}} + \|G_2(t,x)\|_{\widetilde{L}^2_T \dot{B}^{n/p-1}_{p,1}}. \tag{3.4}$$

Using Minkowski's inequality, we derive

$$\begin{aligned} \|G_2\|_{\widetilde{L}^2_T \dot{B}^{n/p-1}_{p,1}} \leqslant \|u_{i-1} \cdot \nabla\tau_{i-1}\|_{\widetilde{L}^2_T \dot{B}^{n/p-1}_{p,1}} + \||\nabla\tau_{i-1}|^2\tau_{i-1}\|_{\widetilde{L}^2_T \dot{B}^{n/p-1}_{p,1}} \\ + \||\nabla\tau_{i-1}|^2\bar{d}_0\|_{\widetilde{L}^2_T \dot{B}^{n/p-1}_{p,1}} = I_1 + I_2 + I_3. \end{aligned} \tag{3.5}$$

From Bony's paraproduct decomposition, we have

$$I_1 = \| - u_{i-1} \cdot \nabla\tau_{i-1}\|_{\widetilde{L}^2_T \dot{B}^{n/p-1}_{p,1}} \lesssim \|u_{i-1}\|_{\widetilde{L}^4_T \dot{B}^{n/p-1/2}_{p,1}} \|\tau_{i-1}\|_{\widetilde{L}^4_T \dot{B}^{n/p+1/2}_{p,1}}. \tag{3.6}$$

Similarly to the derivation of (3.6), one gets

$$\begin{aligned} I_2 &= \||\nabla\tau_{i-1}|^2\tau_{i-1}\|_{\widetilde{L}^2_T \dot{B}^{n/p-1}_{p,1}} \\ &\lesssim \|\tau_{i-1}\|_{\widetilde{L}^\infty_T \dot{B}^{n/p}_{p,1}} \|\tau_{i-1}\|_{\widetilde{L}^4_T \dot{B}^{n/p+1/2}_{p,1}} \|\tau_{i-1}\|_{\widetilde{L}^4_T \dot{B}^{n/p+1/2}_{p,1}}, \end{aligned} \tag{3.7}$$

where we have used Lemma 2.5. From (3.7) we obtain

$$I_3 = \||\nabla\tau_{i-1}|^2\bar{d}_0\|_{\widetilde{L}^2_T \dot{B}^{n/p-1}_{p,1}} \lesssim \|\tau_{i-1}\|_{\widetilde{L}^4_T \dot{B}^{n/p+1/2}_{p,1}} \|\tau_{i-1}\|_{\widetilde{L}^4_T \dot{B}^{n/p+1/2}_{p,1}}. \tag{3.8}$$

One deduces from (3.4)–(3.8) that

$$\begin{aligned} \|\tau_i\|_{\widetilde{L}^\infty_T \dot{B}^{n/p}_{p,1}} &\lesssim \|\tau_0\|_{\dot{B}^{n/p}_{p,1}} + \|u_{i-1}\|_{\widetilde{L}^4_T \dot{B}^{n/p-1/2}_{p,1}} \|\tau_{i-1}\|_{\widetilde{L}^4_T \dot{B}^{n/p+1/2}_{p,1}} \\ &\quad + \|\tau_{i-1}\|_{\widetilde{L}^\infty_T \dot{B}^{n/p}_{p,1}} \|\tau_{i-1}\|^2_{\widetilde{L}^4_T \dot{B}^{n/p+1/2}_{p,1}} + \|\tau_{i-1}\|^2_{\widetilde{L}^4_T \dot{B}^{n/p+1/2}_{p,1}} \\ &\lesssim C\delta + [(C+1)\varepsilon_0]^2 + (C+1)\delta[(C+1)\varepsilon_0]^2 + [(C+1)\varepsilon_0]^2. \end{aligned} \tag{3.9}$$

Taking $\varepsilon_0 \leqslant ((C+1)\sqrt{C+1+2/\delta}\,)^{-1}$ yields $\|\tau_i\|_{\widetilde{L}^\infty_T \dot{B}^{n/p}_{p,1}} \leqslant (C+1)\delta$.

We turn to present the estimates for $\|\tau_i\|_{\widetilde{L}^4_T \dot{B}^{n/p+1/2}_{p,1}}$. Applying Lemma 2.3 and taking $\varrho = 4$, $s = n/p$, $\varrho_1 = 2$, we obtain

$$\begin{aligned} \|\tau_i\|_{\widetilde{L}^4_T \dot{B}^{n/p+1/2}_{p,1}} &\lesssim \|\tau_0\|_{\dot{B}^{n/p}_{p,1}} + \|G_2\|_{\widetilde{L}^2_T \dot{B}^{n/p-1}_{p,1}} \\ &\lesssim \|\tau_0\|_{\dot{B}^{n/p}_{p,1}} + [(C+1)\varepsilon_0]^2 \\ &\qquad + (C+1)\delta[(C+1)\varepsilon_0]^2 + [(C+1)\varepsilon_0]^2. \end{aligned} \tag{3.10}$$

Now we present the estimates for $\|u_i\|_{\widetilde{L}^4_T\dot{B}^{n/p-1/2}_{p,1}}$. Applying Lemma 2.3 and taking $\varrho=4$, $s=n/p-1$, $\varrho_1=2$, we obtain

$$\|u_i\|_{\widetilde{L}^4_T\dot{B}^{n/p-1/2}_{p,1}} \lesssim \|u_0\|_{\dot{B}^{n/p-1}_{p,1}} + \|G_1\|_{\widetilde{L}^2_T\dot{B}^{n/p-2}_{p,1}}. \tag{3.11}$$

It follows that

$$\begin{aligned}\|G_1\|_{\widetilde{L}^2_T\dot{B}^{n/p-2}_{p,1}} &\lesssim \|u_{i-1}\cdot\nabla u_{i-1}\|_{\widetilde{L}^2_T\dot{B}^{n/p-2}_{p,1}} + \|\nabla\tau_{i-1}\odot\nabla\tau_{i-1}\|_{\widetilde{L}^2_T\dot{B}^{n/p-1}_{p,1}}\\ &= I_4 + I_5.\end{aligned} \tag{3.12}$$

Using Lemmas 2.1 and 2.2, we have

$$I_4 \lesssim \|u_{i-1}\|^2_{\widetilde{L}^4_T\dot{B}^{n/p-1/2}_{p,1}} \tag{3.13}$$

and

$$I_5 \lesssim \|\tau_{i-1}\|_{\widetilde{L}^4_T\dot{B}^{n/p+1/2}_{p,1}}\|\tau_{i-1}\|_{\widetilde{L}^4_T\dot{B}^{n/p+1/2}_{p,1}}. \tag{3.14}$$

From (3.11)–(3.14) we obtain

$$\begin{aligned}\|u_i\|_{\widetilde{L}^4_T\dot{B}^{n/p-1/2}_{p,1}} &\lesssim \|u_0\|_{\dot{B}^{n/p-1}_{p,1}} + \|u_{i-1}\|^2_{\widetilde{L}^4_T\dot{B}^{n/p-1/2}_{p,1}} + \|\tau_{i-1}\|^2_{\widetilde{L}^4_T\dot{B}^{n/p+1/2}_{p,1}}\\ &\lesssim \|u_0\|_{\dot{B}^{n/p-1}_{p,1}} + [(C+1)\varepsilon_0]^2 + [(C+1)\varepsilon_0]^2,\end{aligned} \tag{3.15}$$

which together with (3.10) gives

$$\|u_i\|_{\widetilde{L}^4_T\dot{B}^{n/p-1/2}_{p,1}} + \|\tau_i\|_{\widetilde{L}^4_T\dot{B}^{n/p+1/2}_{p,1}} \leqslant C\varepsilon_0 + 4(C+1)^2\varepsilon_0^2 + (C+1)^3\varepsilon_0^2\delta \leqslant (C+1)\varepsilon_0, \tag{3.16}$$

where $\varepsilon_0 \leqslant \min(((C+1)\sqrt{C+1+2/\delta}\,)^{-1}, ((C+1)^2[4+(C+1)\delta])^{-1})$.

In what follows we turn to prove the inequality

$$\|u_i\|_{\widetilde{L}^\infty_T\dot{B}^{n/p-1}_{p,1}\cap\widetilde{L}^1_T\dot{B}^{n/p+1}_{p,1}} + \|\tau_i\|_{\widetilde{L}^\infty_T\dot{B}^{n/p}_{p,1}\cap\widetilde{L}^1_T\dot{B}^{n/p+2}_{p,1}} \leqslant C(\delta). \tag{3.17}$$

Bearing in mind that $\|\tau_i\|_{\widetilde{L}^\infty_T\dot{B}^{n/p}_{p,1}} \leqslant C(\delta)$ has been proved, we only need to prove

$$\|u_i\|_{\widetilde{L}^\infty_T\dot{B}^{n/p-1}_{p,1}},\quad \|u_i\|_{\widetilde{L}^1_T\dot{B}^{n/p+1}_{p,1}},\quad \|\tau_i\|_{\widetilde{L}^1_T\dot{B}^{n/p+2}_{p,1}} \leqslant C(\delta). \tag{3.18}$$

By induction and Lemma 2.4 we obtain that (3.18) holds for $i=1$. Assume that (3.18) holds for $i-1$, i.e.,

$$\|u_{i-1}\|_{\widetilde{L}^\infty_T\dot{B}^{n/p-1}_{p,1}},\quad \|u_{i-1}\|_{\widetilde{L}^1_T\dot{B}^{n/p+1}_{p,1}},\quad \|\tau_{i-1}\|_{\widetilde{L}^1_T\dot{B}^{n/p+2}_{p,1}} \leqslant C(\delta). \tag{3.19}$$

Applying Lemma 2.3 and taking $\varrho = 1$, $s = n/p$, $\varrho_1 = 1$, we obtain

$$\|\tau_i\|_{\widetilde{L}^1_T \dot{B}^{n/p+2}_{p,1}} \lesssim \|\tau_0\|_{\dot{B}^{n/p}_{p,1}} + \|G_2\|_{\widetilde{L}^1_T \dot{B}^{n/p}_{p,1}}. \tag{3.20}$$

From Lemmas 2.1 and 2.2 we have

$$\|u_{i-1} \cdot \nabla \tau_{i-1}\|_{\widetilde{L}^1_T \dot{B}^{n/p}_{p,1}} \leqslant C(\delta). \tag{3.21}$$

Simple calculations give rise to

$$\||\nabla \tau_{i-1}|^2 \tau_{i-1}\|_{\widetilde{L}^1_T \dot{B}^{n/p}_{p,1}} \leqslant C(\delta) \quad \text{and} \quad \||\nabla \tau_{i-1}|^2 \bar{d}_0\|_{\widetilde{L}^1_T \dot{B}^{n/p}_{p,1}} \leqslant C(\delta).$$

Combining this with (3.20), we deduce $\|\tau_i\|_{\widetilde{L}^1_T \dot{B}^{n/p+2}_{p,1}} \leqslant C(\delta)$.

Using Lemma 2.3 and taking $\varrho = \infty$, $s = n/p - 1$, $\varrho_1 = 1$ yields

$$\|u_i\|_{\widetilde{L}^\infty_T \dot{B}^{n/p-1}_{p,1}} \lesssim \|u_0\|_{\dot{B}^{n/p-1}_{p,1}} + \|G_1\|_{\widetilde{L}^1_T \dot{B}^{n/p-1}_{p,1}}. \tag{3.22}$$

It follows by some calculations that

$$\|u_i\|_{\widetilde{L}^\infty_T \dot{B}^{n/p-1}_{p,1}} \leqslant C(\delta). \tag{3.23}$$

Using Lemma 2.3 and taking $\varrho_1 = 1$, $s = n/p - 1$, $\varrho = 1$, we derive

$$\|u_i\|_{\widetilde{L}^1_T \dot{B}^{n/p+1}_{p,1}} \lesssim \|u_0\|_{\dot{B}^{n/p-1}_{p,1}} + \|G_1\|_{\widetilde{L}^1_T \dot{B}^{n/p-1}_{p,1}}. \tag{3.24}$$

Thanks to (3.22) and (3.23), we have

$$\|u_i\|_{\widetilde{L}^1_T \dot{B}^{n/p+1}_{p,1}} \leqslant C(\delta).$$

Consequently, (3.17) is true.

Second step: Convergence. We demonstrate that $(u_i, \tau_i)_{i \in N^*}$ is a Cauchy sequence in $\widetilde{L}^\infty_T \dot{B}^{n/p-1}_{p,1}(\mathbb{R}^n) \times \widetilde{L}^\infty_T \dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$. From system (3.1) we have

$$\begin{cases} (\partial_t - \mu\Delta)(u_{i+j+1} - u_{i+1}) = G_3, & t > 0,\ x \in \mathbb{R}^n, \\ (\partial_t - \Delta)(\tau_{i+j+1} - \tau_{i+1}) = G_4, & t > 0,\ x \in \mathbb{R}^n, \\ \nabla \cdot u_{i+j+1} = 0, \quad \nabla \cdot u_{i+1} = 0, & t > 0, \\ (u_{i+j+1} - u_{i+1})|_{t=0} = 0, & x \in \mathbb{R}^n, \\ (\tau_{i+j+1} - \tau_{i+1})|_{t=0} = 0, & x \in \mathbb{R}^n, \end{cases} \tag{3.25}$$

where

$$G_3 = -\mathbb{P}[u_{i+j} \cdot \nabla u_{i+j} - u_i \cdot \nabla u_i + \nabla \cdot (\nabla \tau_{i+j} \odot \nabla \tau_{i+j}) - \nabla \cdot (\nabla \tau_i \odot \nabla \tau_i)],$$
$$G_4 = -(u_{i+j} \cdot \nabla \tau_{i+j} - u_i \cdot \nabla \tau_i) + |\nabla \tau_{i+j}|^2 \tau_{i+j} - |\nabla \tau_i|^2 \tau_i + |\nabla \tau_{i+j}|^2 \bar{d}_0 - |\nabla \tau_i|^2 \bar{d}_0.$$

Using the heat semigroup theory, we obtain

$$(u_{i+j+1} - u_{i+1})(t,x) = \int_0^t \mathrm{e}^{\mu(t-s)\Delta} G_3(s,x)\,\mathrm{d}s, \tag{3.26}$$

$$(\tau_{i+j+1} - \tau_{i+1})(t,x) = \int_0^t \mathrm{e}^{(t-s)\Delta} G_4(s,x)\,\mathrm{d}s. \tag{3.27}$$

We denote

$$I_1(t,x) = -\int_0^t \mathrm{e}^{\mu(t-s)\Delta} \mathbb{P}[u_{i+j} \cdot \nabla u_{i+j} - u_i \cdot \nabla u_i]\,\mathrm{d}s, \tag{3.28}$$

$$I_2(t,x) = -\int_0^t \mathrm{e}^{\mu(t-s)\Delta} \mathbb{P}[\nabla \cdot (\nabla \tau_{i+j} \odot \nabla \tau_{i+j}) - \nabla \cdot (\nabla \tau_i \odot \nabla \tau_i)]\,\mathrm{d}s, \tag{3.29}$$

$$I_3(t,x) = -\int_0^t \mathrm{e}^{(t-s)\Delta} [u_{i+j} \cdot \nabla \tau_{i+j} - u_i \cdot \nabla \tau_i]\,\mathrm{d}s, \tag{3.30}$$

$$I_4(t,x) = \int_0^t \mathrm{e}^{(t-s)\Delta} [|\nabla \tau_{i+j}|^2 \tau_{i+j} - |\nabla \tau_i|^2 \tau_i]\,\mathrm{d}s, \tag{3.31}$$

$$I_5(t,x) = \int_0^t \mathrm{e}^{(t-s)\Delta} [|\nabla \tau_{i+j}|^2 \bar{d}_0 - |\nabla \tau_i|^2 \bar{d}_0]\,\mathrm{d}s. \tag{3.32}$$

Using Lemma 2.3 in the first equation of (3.25) and taking $\varrho = \infty$, $s = n/p - 1$, $\varrho_1 = 2$, we obtain

$$\begin{aligned} &\|u_{i+j+1} - u_{i+1}\|_{\widetilde{L}^\infty_T \dot{B}^{n/p-1}_{p,1}} \\ &\quad\lesssim \|u_{i+j} \cdot \nabla u_{i+j} - u_i \cdot \nabla u_i + \nabla \cdot (\nabla \tau_{i+j} \odot \nabla \tau_{i+j}) - \nabla \cdot (\nabla \tau_i \odot \nabla \tau_i)\|_{\widetilde{L}^2_T \dot{B}^{n/p-2}_{p,1}} \\ &\qquad + \|(\nabla \tau_{i+j} - \nabla \tau_i) \odot \nabla \tau_{i+j}\|_{\widetilde{L}^2_T \dot{B}^{n/p-1}_{p,1}} + \|\nabla \tau_i \odot (\nabla \tau_{i+j} - \nabla \tau_i)\|_{\widetilde{L}^2_T \dot{B}^{n/p-1}_{p,1}}. \end{aligned}$$

Using Lemmas 2.1 and 2.2, we have

$$\|I_1(t,x)\|_{\widetilde{L}^2_T \dot{B}^{n/p-2}_{p,1}} \lesssim \|u_{i+j} - u_i\|_{\widetilde{L}^\infty_T \dot{B}^{n/p-1}_{p,1}} (\|u_{i+j}\|_{\widetilde{L}^2_T \dot{B}^{n/p}_{p,1}} + \|u_i\|_{\widetilde{L}^2_T \dot{B}^{n/p}_{p,1}}) \tag{3.33}$$

and

$$\|I_2(t,x)\|_{\widetilde{L}^2_T \dot{B}^{n/p-1}_{p,1}} \lesssim \|\tau_{i+j} - \tau_i\|_{\widetilde{L}^\infty_T \dot{B}^{n/p}_{p,1}} (\|\tau_{i+j}\|_{\widetilde{L}^2_T \dot{B}^{n/p+1}_{p,1}} + \|\tau_i\|_{\widetilde{L}^2_T \dot{B}^{n/p+1}_{p,1}}). \tag{3.34}$$

Applying Lemma 2.3 to the second equation in (3.25) and taking $\varrho=\infty$, $s=n/p$, $\varrho_1=2$, we derive

$$\|\tau_{i+j+1}-\tau_{i+1}\|_{\widetilde{L}^\infty_T\dot B^{n/p}_{p,1}}\lesssim\|G_4\|_{\widetilde{L}^2_T\dot B^{n/p-1}_{p,1}}. \tag{3.35}$$

Simple calculations give rise to

$$\begin{aligned}\|I_3(t,x)\|_{\widetilde{L}^2_T\dot B^{n/p-1}_{p,1}}\lesssim{}&\|\tau_{i+j}-\tau_i\|_{\widetilde{L}^\infty_T\dot B^{n/p}_{p,1}}\|u_{i+j}\|_{\widetilde{L}^2_T\dot B^{n/p}_{p,1}}\\&+\|u_{i+j}-u_i\|_{\widetilde{L}^\infty_T\dot B^{n/p-1}_{p,1}}\|\tau_i\|_{\widetilde{L}^2_T\dot B^{n/p+1}_{p,1}}.\end{aligned} \tag{3.36}$$

Similarly, we have

$$\begin{aligned}&\|I_4(t,x)\|_{\widetilde{L}^2_T\dot B^{n/p-1}_{p,1}}\\&\quad\lesssim\|\tau_{i+j}-\tau_i\|_{\widetilde{L}^\infty_T\dot B^{n/p}_{p,1}}(\|\tau_{i+j}\|_{\widetilde{L}^2_T\dot B^{n/p+1}_{p,1}}+\|\tau_i\|_{\widetilde{L}^2_T\dot B^{n/p+1}_{p,1}})\|\tau_{i+j}\|_{\widetilde{L}^\infty_T\dot B^{n/p}_{p,1}}\\&\qquad+\|\tau_{i+j}-\tau_i\|_{\widetilde{L}^\infty_T\dot B^{n/p}_{p,1}}\|\tau_i\|^2_{\widetilde{L}^4_T\dot B^{n/p+1/2}_{p,1}}\end{aligned} \tag{3.37}$$

and

$$\|I_5(t,x)\|_{\widetilde{L}^2_T\dot B^{n/p-1}_{p,1}}\lesssim\|\tau_{i+j}-\tau_i\|_{\widetilde{L}^\infty_T\dot B^{n/p}_{p,1}}(\|\tau_{i+j}\|_{\widetilde{L}^2_T\dot B^{n/p+1}_{p,1}}+\|\tau_i\|_{\widetilde{L}^2_T\dot B^{n/p+1}_{p,1}}). \tag{3.38}$$

Now we have obtained the estimates for $I_1(t,x)-I_5(t,x)$. We deduce that there exists $\theta\in(0,1)$ such that

$$\begin{aligned}\|u_{i+j+1}-u_{i+1}\|_{\widetilde{L}^\infty_T\dot B^{n/p-1}_{p,1}}&+\|\tau_{i+j+1}-\tau_{i+1}\|_{\widetilde{L}^\infty_T\dot B^{n/p}_{p,1}}\\&\leqslant\theta[\|u_{i+j}-u_i\|_{\widetilde{L}^\infty_T\dot B^{n/p-1}_{p,1}}+\|\tau_{i+j}-\tau_i\|_{\widetilde{L}^\infty_T\dot B^{n/p}_{p,1}}]\end{aligned} \tag{3.39}$$

by choosing ε_0 small enough. Hence, $(u_i,\tau_i)_{i\in N^*}$ is a Cauchy sequence in the space $\widetilde{L}^\infty_T\dot B^{n/p-1}_{p,1}\times\widetilde{L}^\infty_T\dot B^{n/p}_{p,1}$. Let $u_i\to u$ in $\widetilde{L}^\infty_T\dot B^{n/p-1}_{p,1}$ and $\tau_i\to\tau$ in $\widetilde{L}^\infty_T\dot B^{n/p}_{p,1}$ as $i\to\infty$. Then (u,τ) is a solution to system (1.5) on $[0,T]\times\mathbb{R}^n$. For $r<\infty$ and using Lemma 2.3, we have $(u,\tau)\in C([0,T];\dot B^{n/p-1}_{p,1})\times C([0,T];\dot B^{n/p}_{p,1})$. From (3.17), we deduce $(u,\tau)\in\widetilde{L}^1_T\dot B^{n/p+1}_{p,1}\times\widetilde{L}^1_T\dot B^{n/p+2}_{p,1}$. Therefore, $(u,\tau)\in C([0,T];\dot B^{n/p-1}_{p,1})\cap\widetilde{L}^1_T\dot B^{n/p+1}_{p,1}\times C([0,T];\dot B^{n/p}_{p,1})\cap\widetilde{L}^1_T\dot B^{n/p+2}_{p,1}$.

Third step: Uniqueness. Let (u_1,τ_1), (u_2,τ_2) be two solutions to system (1.5), $\delta_1=u_1-u_2$ and $\delta_2=\tau_1-\tau_2$. Using the procedure of proving that $(u_i,\tau_i)_{i\in N^*}$ is a Cauchy sequence in the second step on the time interval $[0,T_0]$, where T_0 is a small positive constant, we deduce $\delta_1=0$, $\delta_2=0$. □

3.2. Global well-posedness. We prove the global well-posedness for system (1.5). We rewrite system (1.5) into the form

$$
\begin{cases}
u_t + \mathbb{P}[u\cdot\nabla u] - \mu\Delta u = -\mathbb{P}[\nabla\cdot(\nabla\tau\odot\nabla\tau)], & t>0,\ x\in\mathbb{R}^n,\\
\tau_t + u\cdot\nabla\tau - \Delta\tau = |\nabla\tau|^2\tau + |\nabla\tau|^2\bar{d}_0, & t>0,\ x\in\mathbb{R}^n,\\
\nabla\cdot u = 0, & t>0,\\
u(0,x)=u_0(x),\quad \tau|_{t=0}=\tau_0(x), & x\in\mathbb{R}^n.
\end{cases}\tag{3.40}
$$

Applying the operator $\dot{\Delta}_j$ to the first equation in (3.40), noting that $\mathbb{P}u = u$ and using a standard commutator's process, we derive

$$
\partial_t\dot{\Delta}_j u + u\cdot\nabla\dot{\Delta}_j u - \mu\Delta\dot{\Delta}_j u = -\dot{\Delta}_j\mathbb{P}[\nabla\cdot(\nabla\tau\odot\nabla\tau)] - [\dot{\Delta}_j\mathbb{P}; u\cdot\nabla]u.\tag{3.41}
$$

Taking $L^2(\mathbb{R}^n)$ inner product of (3.41) with $|\dot{\Delta}_j u|^{p-2}\dot{\Delta}_j u$, we obtain

$$
\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{dt}}\|\dot{\Delta}_j u\|_{L^p}^p &+ \int_{\mathbb{R}^n} u\cdot\nabla\dot{\Delta}_j u|\dot{\Delta}_j u|^{p-2}\dot{\Delta}_j u\,\mathrm{d}x - \mu\int_{\mathbb{R}^n}\Delta\dot{\Delta}_j u|\dot{\Delta}_j u|^{p-2}\dot{\Delta}_j u\,\mathrm{d}x\\
&= \int_{\mathbb{R}^n}(-\dot{\Delta}_j\mathbb{P}[\nabla\cdot(\nabla\tau\odot\nabla\tau)] - [\dot{\Delta}_j\mathbb{P}; u\cdot\nabla]u)|\dot{\Delta}_j u|^{p-2}\dot{\Delta}_j u\,\mathrm{d}x.
\end{aligned}\tag{3.42}
$$

Integrating by parts and using the condition $\nabla\cdot u = 0$, we get

$$
\int_{\mathbb{R}^n} u\cdot\nabla\dot{\Delta}_j u|\dot{\Delta}_j u|^{p-2}\dot{\Delta}_j u\,\mathrm{d}x = 0.
$$

Applying Lemmas 2.5 and A.5 in [8] yields

$$
-\mu\int_{\mathbb{R}^n}\Delta\dot{\Delta}_j u|\dot{\Delta}_j u|^{p-2}\dot{\Delta}_j u\,\mathrm{d}x \geqslant c2^{2j}\|\dot{\Delta}_j u\|_{L^p}^p,\quad c>0,
$$

which together with (3.42), Holder's inequality and Minkowski's inequality gives

$$
\frac{\mathrm{d}}{\mathrm{dt}}\|\dot{\Delta}_j u\|_{L^p} + c2^{2j}\|\dot{\Delta}_j u\|_{L^p} \leqslant \|\dot{\Delta}_j\mathbb{P}[\nabla\cdot(\nabla\tau\odot\nabla\tau)]\|_{L^p} + \|[\dot{\Delta}_j\mathbb{P}; u\cdot\nabla]u\|_{L^p}.\tag{3.43}
$$

Integrating both sides of (3.43) with respect to time from 0 to t and applying Definition 2.1, Proposition 2.1, we deduce for $n<p<2n$ and $t\in[0,T^*]$, $T^*<\infty$,

$$
\begin{aligned}
\|u\|_{\widetilde{L}_t^\infty\dot{B}_{p,1}^{n/p-1}} + c\|u\|_{\widetilde{L}_t^1\dot{B}_{p,1}^{n/p+1}} \leqslant \|u_0\|_{\dot{B}_{p,1}^{n/p-1}} &+ \|[\Delta_j\mathbb{P}; u\cdot\nabla]u\|_{\widetilde{L}_t^1\dot{B}_{p,1}^{n/p-1}}\\
&+ \|\mathbb{P}[\nabla\cdot(\nabla\tau\odot\nabla\tau)]\|_{\widetilde{L}_t^1\dot{B}_{p,1}^{n/p-1}}.
\end{aligned}\tag{3.44}
$$

Using the commutator estimates, we obtain

$$\begin{aligned}(3.45)\qquad &\|u\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{n/p-1}} + c\|u\|_{\widetilde{L}_t^1 \dot{B}_{p,1}^{n/p+1}} \\ &\lesssim \|u_0\|_{\dot{B}_{p,1}^{n/p-1}} + \|u\|_{\widetilde{L}_t^1 \dot{B}_{p,1}^{n/p+1}} \|u\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{n/p-1}} \\ &\quad + \|\tau\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{n/p}} \|\tau\|_{\widetilde{L}_t^1 \dot{B}_{p,1}^{n/p+2}}.\end{aligned}$$

For the second equation in (3.40), using a similar process, we deduce

$$\begin{aligned}(3.46)\qquad &\|\tau\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{n/p}} + c_1\|\tau\|_{\widetilde{L}_t^1 \dot{B}_{p,1}^{n/p+2}} \\ &\lesssim \|\tau_0\|_{\dot{B}_{p,1}^{n/p}} + \|u\|_{\widetilde{L}_t^1 \dot{B}_{p,1}^{n/p+1}} \|\tau\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{n/p}} \\ &\quad + \|\tau\|^2_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{n/p}} \|\tau\|_{\widetilde{L}_t^1 \dot{B}_{p,1}^{n/p+2}} + \|\tau\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{n/p}} \|\tau\|_{\widetilde{L}_t^1 \dot{B}_{p,1}^{n/p+2}}.\end{aligned}$$

From (3.45), (3.46) and Lemma 2.6 we deduce that if δ is small enough, then the corresponding solution (u,τ) exists globally and is also unique. We note that we only obtain a partial answer to the uniqueness of solutions. Meanwhile, if the initial value is in the ball $B_{\varepsilon_1}(0)$, then the solution of system (1.5) is unique in the ball $B_{2\varepsilon_1}(0)$. We need to get rid of this restrictive condition. Let (u_1,τ_1), (u_2,τ_2) be two solutions to system (1.5) and $\delta u = u_1 - u_2$, $\delta\tau = \tau_1 - \tau_2$. From the second step in Subsection 3.1, if we choose $T_1 > 0$ sufficiently small for all $t \in [0,T_1]$ we have

$$\|\delta u\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{n/p-1}} + \|\delta\tau\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{n/p}} = 0.$$

Repeating the above procedure on time intervals $[0,T_1]$, $[T_1,2T_1]$, $[2T_1,3T_1],\dots$ enables us to deduce

$$\|\delta u\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{n/p-1}} + \|\delta\tau\|_{\widetilde{L}_t^\infty \dot{B}_{p,1}^{n/p}} = 0,$$

which implies $\delta u = 0$, $\delta\tau = 0$. □

4. Blow-up

Proof of Theorem 1.2. Applying Lemma 2.3 to the first equation and the second equation in system (1.5), respectively, we have

$$(4.1)\qquad \|u\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{n/p-1}} \lesssim \|u_0\|_{\dot{B}_{p,1}^{n/p-1}} + \|u\cdot\nabla u + \nabla\cdot(\nabla\tau\odot\nabla\tau)\|_{\widetilde{L}_T^{\varrho_1} \dot{B}_{p,1}^{n/p-1+2/\varrho_1-2}},$$

$$(4.2)\qquad \|\tau\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{n/p}} \lesssim \|\tau_0\|_{\dot{B}_{p,1}^{n/p}} + \|u\cdot\nabla\tau + |\nabla\tau|^2\tau + |\nabla\tau|^2\bar{d}_0\|_{\widetilde{L}_T^{\varrho_2} \dot{B}_{p,1}^{n/p+2/\varrho_2-2}}.$$

Using Lemma 2.7, we deduce

$$(4.3)\qquad \|u\cdot\nabla u\|_{\widetilde{L}_T^{\varrho_1} \dot{B}_{p,1}^{n/p-1+2/\varrho_1-2}} \lesssim \|u\|_{\widetilde{L}_T^{\varrho_1} \dot{B}_{\infty,\infty}^{2/\varrho_1-1}} \|u\|_{\widetilde{L}_T^\infty \dot{B}_{p,1}^{n/p-1}}.$$

For the term

$$\|\nabla\cdot(\nabla\tau\odot\nabla\tau)\|_{\widetilde{L}^{\varrho_1}_T\dot{B}^{n/p-1+2/\varrho_1-2}_{p,1}}\lesssim\|\nabla\tau\odot\nabla\tau\|_{\widetilde{L}^{\varrho_1}_T\dot{B}^{n/p+2/\varrho_1-2}_{p,1}} \tag{4.4}$$

we have

$$\begin{aligned}\|\mathcal{T}_{\nabla\tau}\nabla\tau\|_{\widetilde{L}^{\varrho_1}_T\dot{B}^{n/p+2/\varrho_1-2}_{p,1}}&\lesssim\|\nabla\tau\|_{\widetilde{L}^{\varrho_1}_T\dot{B}^{2/\varrho_1-1}_{\infty,\infty}}\|\nabla\tau\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p-1}_{p,1}}\\&\lesssim\|\tau\|_{\widetilde{L}^{\varrho_1}_T\dot{B}^{2/\varrho_1}_{\infty,\infty}}\|\tau\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p}_{p,1}},\\ \|\mathcal{R}(\nabla\tau,\nabla\tau)\|_{\widetilde{L}^{\varrho_1}_T\dot{B}^{n/p+2/\varrho_1-2}_{p,1}}&\lesssim\|\tau\|_{\widetilde{L}^{\varrho_1}_T\dot{B}^{2/\varrho_1}_{\infty,\infty}}\|\tau\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p}_{p,1}},\end{aligned}$$

which combined with (4.4) gives

$$\|\nabla\cdot(\nabla\tau\odot\nabla\tau)\|_{\widetilde{L}^{\varrho_1}_T\dot{B}^{n/p-1+2/\varrho_1-2}_{p,1}}\lesssim\|\tau\|_{\widetilde{L}^{\varrho_1}_T\dot{B}^{n/p+2/\varrho_1}_{p,\infty}}\|\tau\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p}_{p,1}}. \tag{4.5}$$

It follows from some calculations that

$$\|u\cdot\nabla\tau\|_{\widetilde{L}^{\varrho_2}_T\dot{B}^{n/p+2/\varrho_2-2}_{p,1}}\lesssim\|u\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p-1}_{p,1}}\|\tau\|_{\widetilde{L}^{\varrho_2}_T\dot{B}^{2/\varrho_2}_{\infty,\infty}}. \tag{4.6}$$

Similarly to (4.5), we deduce

$$\||\nabla\tau|^2\tau\|_{\widetilde{L}^{\varrho_2}_T\dot{B}^{n/p+2/\varrho_2-2}_{p,1}}\lesssim\|\nabla\tau\tau\|_{\widetilde{L}^{\varrho_2}_T\dot{B}^{2/\varrho_2-1}_{\infty,\infty}}\|\tau\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p}_{p,1}}. \tag{4.7}$$

Applying Lemma 2.5 and the embedding properties in Besov space, we have

$$\|\nabla\tau\tau\|_{\widetilde{L}^{\varrho_2}_T\dot{B}^{2/\varrho_2-1}_{\infty,\infty}}\lesssim\|\tau\tau\|_{\widetilde{L}^{\varrho_2}_T\dot{B}^{2/\varrho_2}_{\infty,\infty}}\lesssim\|\tau\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p}_{p,1}}\|\tau\|_{\widetilde{L}^{\varrho_2}_T\dot{B}^{2/\varrho_2}_{\infty,\infty}}, \tag{4.8}$$

which together with (4.7) gives

$$\||\nabla\tau|^2\tau\|_{\widetilde{L}^{\varrho_2}_T\dot{B}^{n/p+(2/\varrho_2)-2}_{p,1}}\lesssim\|\tau\|_{\widetilde{L}^{\varrho_2}_T\dot{B}^{2/\varrho_2}_{\infty,\infty}}\|\tau\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p}_{p,1}}\|\tau\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p}_{p,1}}. \tag{4.9}$$

Similarly to (4.7), we obtain

$$\||\nabla\tau|^2\bar{d}_0\|_{\widetilde{L}^{\varrho_2}_T\dot{B}^{n/p+2/\varrho_2-2}_{p,1}}\lesssim\|\tau\|_{\widetilde{L}^{\varrho_2}_T\dot{B}^{2/\varrho_2}_{\infty,\infty}}\|\tau\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p}_{p,1}}. \tag{4.10}$$

Thus, we have

$$\begin{aligned}\|u\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p-1}_{p,1}}\lesssim{}&\|u_0\|_{\dot{B}^{n/p-1}_{p,1}}+\|u\|_{\widetilde{L}^{\varrho_1}_T\dot{B}^{(2/\varrho_1)-1}_{\infty,\infty}}\|u\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p-1}_{p,1}}\\&+\|\tau\|_{\widetilde{L}^{\varrho_1}_T\dot{B}^{n/p+2/\varrho_1}_{p,\infty}}\|\tau\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p}_{p,1}},\end{aligned} \tag{4.11}$$

$$\begin{aligned}\|\tau\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p}_{p,1}}\lesssim{}&\|\tau_0\|_{\dot{B}^{n/p}_{p,1}}+\|u\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p-1}_{p,1}}\|\tau\|_{\widetilde{L}^{\varrho_2}_T\dot{B}^{2/\varrho_2}_{\infty,\infty}}\\&+\|\tau\|_{\widetilde{L}^{\varrho_2}_T\dot{B}^{2/\varrho_2}_{\infty,\infty}}\|\tau\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p}_{p,1}}\|\tau\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p}_{p,1}}+\|\tau\|_{\widetilde{L}^{\varrho_2}_T\dot{B}^{2/\varrho_2}_{\infty,\infty}}\|\tau\|_{\widetilde{L}^{\infty}_T\dot{B}^{n/p}_{p,1}}.\end{aligned} \tag{4.12}$$

From Theorem 1.1 one deduces $(u,\tau)\in C([0,T];\dot B^{n/p-1}_{p,1})\times C([0,T];\dot B^{n/p}_{p,1})$. In order to prove Theorem 1.2, it suffices to prove that if

$$\|u\|_{\widetilde L^{\varrho_1}_{T^*}\dot B^{2/\varrho_1-1}_{\infty,\infty}}+\|\tau\|_{\widetilde L^{\varrho_2}_{T^*}\dot B^{2/\varrho_2}_{\infty,\infty}}+\|\tau\|_{\widetilde L^{\varrho_1}_{T^*}\dot B^{n/p+2/\varrho_1}_{p,\infty}}=\varepsilon_3<\infty, \tag{4.13}$$

then $T^*>T$. Now for all $t\in[0,T]$ we take $(u(t,x),\tau(t,x))$ as a new initial value of system (1.5) and split $u=u_h+u_l$ so that

$$\hat u(t,\xi)=\hat u 1_{|\xi|>2^{N_1}}(t,\xi)+\hat u 1_{|\xi|<2^{N_1}}(t,\xi)=\hat u_h+\hat u_l.$$

Similarly, we split $\tau=\tau_h+\tau_l$. From Lemma 2.4 we deduce that there exists a sufficiently large constant $N_1\in N$ such that

$$\|(u_h,\tau_h)\|_{\dot B^{n/p-1}_{p,1}\times\dot B^{n/p}_{p,1}}\lesssim\frac{\varepsilon}{2}. \tag{4.14}$$

We denote $I=[t,t+T_\varepsilon]$ and $M=\|(u(t),\tau(t))\|_{\dot B^{n/p-1}_{p,1}\times\dot B^{n/p}_{p,1}}$. Choosing $T_0>t$ and a suitable constant ϱ such that $T_0-t\leqslant[\varepsilon(C_2 2^{1+2N_1/\varrho}M)^{-1}]^\varrho=T_\varepsilon$, we deduce

$$\|(\mathrm{e}^{t\Delta}u_l,\mathrm{e}^{t\Delta}\tau_l)\|_{\widetilde L^{\varrho_1}_I\dot B^{n/p-1+2/\varrho_1-2}_{p,1}\times\widetilde L^{\varrho_2}_I\dot B^{n/p+2/\varrho_2-2}_{p,1}}\lesssim\frac{\varepsilon}{2}. \tag{4.15}$$

From (4.11), (4.12) and (4.14), one gets

$$\begin{aligned}\|u\|_{\widetilde L^\infty_I\dot B^{n/p-1}_{p,1}}+\|\tau\|_{\widetilde L^\infty_I\dot B^{n/p}_{p,1}}\lesssim{}&\|u(t)\|_{\dot B^{n/p-1}_{p,1}}+\|\tau(t)\|_{\dot B^{n/p}_{p,1}}\\&+\theta[\|u\|_{\widetilde L^\infty_I\dot B^{n/p-1}_{p,1}}+\|\tau\|_{\widetilde L^\infty_I\dot B^{n/p}_{p,1}}],\end{aligned} \tag{4.16}$$

where $\theta\in(0,1)$ providing ε_3 small enough in (4.13). Then we have

$$\|u\|_{\widetilde L^\infty_I\dot B^{n/p-1}_{p,1}}+\|\tau\|_{\widetilde L^\infty_I\dot B^{n/p}_{p,1}}\lesssim\|u(t)\|_{\dot B^{n/p-1}_{p,1}}+\|\tau(t)\|_{\dot B^{n/p}_{p,1}}. \tag{4.17}$$

Therefore, from (4.17) we find a constant T_ε depending only on ε and M such that system (1.5) has a solution on the time interval $[t,t+T_\varepsilon]$ for all $t\in[0,T]$. From the uniqueness we deduce that all solutions obtained in this way coincide in their existence interval. Thus, the solution can be extended to the time interval $[0,T+T_\varepsilon]$. This implies $T^*>T$, which completes the proof of Theorem 1.2. $\square$

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