Uzma Ahmad; Muqadas Moeen

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THE CLASSIFICATION OF FINITE GROUPS BY USING ITERATION DIGRAPHS

Uzma Ahmad, Muqadas Moeen, Lahore

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Abstract. A digraph is associated with a finite group by utilizing the power map $f\colon G \to G$ defined by $f(x) = x^k$ for all $x \in G$, where k is a fixed natural number. It is denoted by $\gamma_G(n,k)$. In this paper, the generalized quaternion and 2-groups are studied. The height structure is discussed for the generalized quaternion. The necessary and sufficient conditions on a power digraph of a 2-group are determined for a 2-group to be a generalized quaternion group. Further, the classification of two generated 2-groups as abelian or non-abelian in terms of semi-regularity of the power digraphs is completed.

Keywords: 2-group; generalized quaternion group; iteration digraph; cycle; indegree; fixed point; regular digraph

MSC 2010: 05C20, 05C25, 05C50, 20B25, 20D15

1. Introduction

During the last century, the field of graph theory has flourished tremendously. Algebraic graph theory is an important branch which makes an elegant connection of graphs with algebraic structures. Let $f\colon H \to H$ be any map on a finite set H. An iteration digraph is constructed by taking all the elements of H as the vertices of the digraph such that there exists exactly one edge from a vertex x to a vertex y if and only if $f(x) = y$. If, in particular, $f(x) = x^k$ is taken then these iteration digraphs are called power digraphs. The power digraphs modulo n defined on Z_n are extensively studied by Lucheta et al. [9], Wilson [14], Somer and Křížek [11], [12], [13], Ahmad and Husnine [2], [4], [7]. Further, Min Sha [10] explored the properties of power digraphs on finite cyclic groups. Moreover, Ahmad and Husnine [1] extended this work to abelian groups. The obvious motivation is to develop

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this theory for non-abelian groups. The behavior of these digraphs on generalized quaternion is discussed by Ahmad and Moeen [3]. In this paper, the groups under discussion are generalized quaternion groups and finite 2-groups. The power digraph is denoted by $\gamma_G(n,k)$ such that the group G of order n is the set of vertices and the set of edges is $\{(x,y)\colon\ x^k = y,\ \forall\, x,y \in G\}$. For any vertex $a \in G$, $\mathrm{Comp}(a)$ is the component of $\gamma_G(n,k)$ containing a. The minimum distance from a to a cycle vertex of $\mathrm{Comp}(a)$ is called the height of a and is denoted by $\mathrm{height}(a)$. The height of a component is the height of a vertex which possesses maximum height among the other vertices of that component. Also, the height of a digraph is the maximum height of all components of that digraph. For every $a \in G$, let $N(n,k,a)$ denote the number of distinct solutions of the equation $x^k = a$ in G. Then, obviously, $N(n,k,a) = \mathrm{indeg}_n(a)$. If the indegree of every vertex $a \in G$ is 0 or q then $\gamma_G(n,k)$ is semi-regular of degree q. A digraph is said to be *regular* if every vertex acquires the same indegree. A path from a to b is a sequence of directed edges which connect the sequence of vertices from a to b; it is denoted by P^a_b. A component of $\gamma_G(n,k)$ is a sub-digraph which is a maximal connected subgraph of the associated non-directed graph. Also, $\gamma_G(n,k)$ is connected if and only if the associated non-directed graph is connected, i.e., $\gamma_G(n,k)$ consists of only one component. It is well known that 2-groups can be classified as the quaternion free groups and groups containing an isomorphic copy of the quaternion. Thus, to understand the behavior of the power digraph of general 2-groups, it is important to know the structure of the power digraph of generalized quaternion groups. The paper is organized as follows: in Section 2, the formulae for the height of vertices and height for the digraphs of generalized quaternion groups are established. With help of these results, some basic results for generalized quaternion 2-groups are deduced. In Section 3, necessary and sufficient conditions on a power digraph of 2-group are established for a 2-group to be a generalized quaternion group. Further, 2-groups generated by 2 elements are classified with help of the power digraph into abelian and non-abelian groups.

Let Q_{4n} be a quaternion group of order $4n$ with identity element 1. The generators relation form of Q_{4n} can be written as

$$Q_{4n} = \langle a, b\colon\ a^{2n} = 1,\ b^2 = a^n,\ b^{-1}ab = a^{-1}\rangle.$$

The maximal subgroups are $\langle a\rangle$, $\langle a^2, b\rangle$, and $\langle a^2, ab\rangle$, where the first is a cyclic subgroup and the other two are generalized quaternion subgroups. If we take $n = 2^{l-2}$, then the generalized quaternion group of order $4n$ becomes the generalized quaternion 2-group of order $n = 2^l$. Let

$$\delta(k) = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases} \tag{1.1}$$

$$\alpha(k,i) = \begin{cases} 1 & \text{if } k \equiv 2 \pmod 4,\ i = n, \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

$$\beta(k) = \begin{cases} 1 & \text{if } k \equiv 0 \pmod 4, \\ 0 & \text{otherwise,} \end{cases} \tag{1.3}$$

and

$$2n = 2^{\delta(k)}\lambda\mu, \tag{1.4}$$

where $2^{\delta(k)}\lambda$ is the highest factor of $2n$ such that $\gcd(2^{\delta(k)}\lambda, k) = 1$.

Theorem 1.1 ([1])**.** *The power digraph $\gamma_G(n,k)$ is connected if and only if there exists a positive integer m such that $\operatorname{Exp}(G) \mid k^m$.*

Lemma 1.2 ([1])**.** *Let $G \cong C_{n_1} \times C_{n_2} \times \ldots \times C_{n_s}$, where C_{n_i} are cyclic groups of order n_i. Then the indegree of any vertex a of $\gamma_G(n,k)$ is either 0, or*

$$N(n,k,a) = \operatorname{indeg}_n(a) = \prod_{i=1}^{s} \gcd(n_i,k).$$

Theorem 1.3 ([3])**.** *The indegree of any vertex a^ib^j of $\gamma_{Q_{4n}}(2n,k)$ is either 0, or*

$$N(Q_{4n},k,a^ib^j) = \begin{cases} \gcd(2n,k) + 2n\alpha(k,i) & \textit{if } j = 0, \\ 1 & \textit{if } j = 1, \\ \gcd(2n,k) + 2n\beta(k) & \textit{if } i = 0,\ j = 0, \end{cases}$$

where $\alpha(k,i)$ and $\beta(k)$ are defined in (1.2) and (1.3), respectively.

Theorem 1.4 ([3])**.** *The vertex a^i of $\gamma_{Q_{4n}}(4n,k)$ is a cycle vertex if and only if $\operatorname{ord}(a^i) \mid 2^{\delta(k)}\lambda$, where $2^{\delta(k)}\lambda$ is defined in (1.4). Further, the length t of a cycle containing a^i is given by*

$$t = \operatorname{ord}_{d_i} k,$$

where d_i is some divisor of $2^{\delta(k)}\lambda$.

Theorem 1.5 ([3]). *The vertex a^ib is a cycle vertex if and only if k is odd. In particular, there exists no cycle of length $t > 2$ containing the vertex a^ib.*

2. Height structure of $\gamma_{Q_{4n}}(4n, k)$

Consider the factorization of k as

$$k = \varrho_1^{\beta_1}\varrho_2^{\beta_2}\dots\varrho_q^{\beta_q}, \tag{2.1}$$

where $\beta_i > 0$ for all $i \in \{1, \dots, q\}$ and $\varrho_1 < \varrho_2 < \dots < \varrho_q$. Since $\pi(\mu) \subseteq \pi(k)$, where $\pi(x)$ denotes the set of distinct prime divisors of x, we can write

$$\mu = \varrho_1^{\alpha_1}\varrho_2^{\alpha_2}\dots\varrho_q^{\alpha_q}, \tag{2.2}$$

where $\alpha_i \geqslant 0$ for all $i \in \{1, \dots, q\}$. Now if $\operatorname{ord}(x)$ denotes the order of x in Q_{4n}, then $\operatorname{ord}(a^i) \mid 2n$ and it can be written as

$$\operatorname{ord}(a^i) = \lambda_1\varrho^{\gamma_1}\varrho_2^{\gamma_2}\dots\varrho_q^{\gamma_q}, \tag{2.3}$$

where $\gamma_i \geqslant 0$ for all $i \in \{1, \dots, q\}$ and $\lambda_1 \mid 2^{\delta(k)}\lambda$.

Theorem 2.1. *Let n and k be any positive integers defined in* (1.4) *and* (2.1)*, respectively. Then*

$$\operatorname{height}(a^ib^j) = \begin{cases} \max\limits_{1\leqslant i\leqslant q}\left\lceil\dfrac{\nu_{\varrho_i}(2n/\gcd(2n,i))}{\nu_{\varrho_i(k)}}\right\rceil & \text{for } j = 0,\\ (1-\delta(k))\left\lceil\dfrac{2}{\beta_1}\right\rceil & \text{for } j = 1,\end{cases}$$

where $\nu_{\varrho_i}(x)$ denotes the highest power of ϱ_i in x.

Proof. Suppose $h = \operatorname{height}(a^ib^j)$. Then there exists a cycle vertex c in $\gamma_{Q_{4n}}(4n, k)$ such that

$$(a^ib^j)^{k^h} = c.$$

If $j = 0$, then the vertex c must be some power of a, i.e. $c = a^s$, for some $0 \leqslant s \leqslant 2n - 1$. This along with the equations (2.1) and (2.3) implies that

$$\begin{aligned}\operatorname{ord}(c) = \operatorname{ord}((a^i)^{k^h}) &= \frac{\operatorname{ord}(a^i)}{\gcd(k^h, \operatorname{ord}(a^i))},\\ &= \frac{\lambda_1\varrho_1^{\gamma_1}\varrho_2^{\gamma_2}\dots\varrho_q^{\gamma_q}}{\varrho_1^{\min(\gamma_1,\beta_1h)}\varrho_2^{\min(\gamma_2,\beta_2h)}\dots\varrho_s^{\min(\gamma_q,\beta_qh)}}.\end{aligned}$$

By virtue of Theorem 1.4, the vertex $c = a^s$ is a cycle vertex if and only if $\operatorname{ord}(c) \mid 2^{\delta(K)}\lambda$, i.e. $\gcd(\operatorname{ord} a^i, k) = 1$. Hence, for $c = a^s$ to be a cycle vertex, we must have

$$\beta_i h \geqslant \gamma_i \quad \text{for } i \in \{1, \dots, q\},\ h \geqslant \max_{1 \leqslant i \leqslant q} \left\lceil \frac{\gamma_i}{\beta_i} \right\rceil = \max_{1 \leqslant i \leqslant q} \left\lceil \frac{\nu_{\varrho_i}(\operatorname{ord}\ a^i)}{\nu_{\varrho_i}(k)} \right\rceil.$$

Since we are considering the minimum value,

$$h = \max_{1 \leqslant i \leqslant p} \left\lceil \frac{\gamma_i}{\beta_i} \right\rceil = \max_{1 \leqslant i \leqslant q} \left\lceil \frac{\nu_{\varrho_i}(\operatorname{ord}(a^i))}{\nu_{\varrho_i}(k)} \right\rceil.$$

Now if $j = 1$, then

$$\operatorname{ord}(c) = \operatorname{ord}((a^i b)^{k^h}) = \frac{\operatorname{ord}(a^i b)}{\gcd(k^h, \operatorname{ord}(a^i b))}.$$

Hence,

$$\operatorname{ord}(c) = \frac{4}{\gcd(k^h, 4)}.$$

This implies

$$\operatorname{ord}(c) = \frac{4}{\gcd(\varrho_1^{\beta_1 h} \varrho_2^{\beta_2 h} \dots \varrho_q^{\beta_q h}, 4)}. \tag{2.4}$$

Now if $\varrho_1 = 2$, i.e. k is even, then by Theorem 1.5, $a^i b$ is not a cycle vertex for all i and by Theorem 1.4, a^n is not a cycle vertex either. This along with the fact that $a^i b$ maps onto $1, a^n$ or $a^j b$ implies that $c = 1$. Hence, $\operatorname{ord}(c) = 1$. Therefore, from equation (2.4),

$$\operatorname{ord}(c) = \frac{2^2}{\gcd(2^{\beta_1 h}, 4)} = 1.$$

This implies

$$\beta_1 h \geqslant 2, \quad h \geqslant \left\lceil \frac{2}{\beta_1} \right\rceil.$$

Now if k is odd then by Theorem 1.5, $a^i b$ is a cycle vertex and hence, $\operatorname{height}(a^i b) = 0$. Thus, $\operatorname{height}(a^i b) = (1 - \delta(k))\lceil 2/\beta_1 \rceil$. □

Corollary 2.2. *Let G be a generalized quaternion 2-group of order 2^n and $k = 2$. Then*

$$\operatorname{height}(a^i b^j) = \begin{cases} n-1 & \text{for } j = 0 \text{ and } i \text{ odd}, \\ n - 1 - \min(n-1, m) & \text{for } j = 0 \text{ and } i = 2^m s,\ (s, 2) = 1, \\ 2 & \text{for } j = 1. \end{cases}$$

Theorem 2.3. *Let k be a positive integer defined in* (2.1). *Then the height of* $\gamma_{Q_{4n}}(4n,k)$ *is*

$$\max\Big\{(1-\delta(k))\Big\lceil\frac{2}{\beta_1}\Big\rceil, \max_{0\leqslant i\leqslant q}\Big\lceil\frac{\nu_{\varrho_i}(2n)}{\nu_{\varrho_i(k)}}\Big\rceil\Big\},$$

where $\delta(k)$ is defined in (1.1) *and $\nu_{\varrho_i}(x)$ denotes the highest power of ϱ_i in x.*

Proof. Since $Q_{4n}=A\cup B$ and $A\cap B=\varphi$, it is easy to see that

$$\text{height}(\gamma_{Q_n}(2n,k))=\max\{\text{height}(\gamma_A(2n,k)),\text{height}(\gamma_B(2n,k))\}. \tag{2.5}$$

Due to Theorem 2.1 for any vertex a^j

$$\text{height}(a^j)=\max_{1\leqslant i\leqslant q}\Big\lceil\frac{\nu_{\varrho_i}(2n/\gcd(2n,j))}{\nu_{\varrho_i(k)}}\Big\rceil.$$

Since $\text{ord}(a^j)\mid 2n$, we have

$$\Big\lceil\frac{\nu_{\varrho_i}(\text{ord}(a^j))}{\nu_{\varrho_i}(k)}\Big\rceil\leqslant\Big\lceil\frac{\nu_{\varrho_i}(2n)}{\nu_{\varrho_i}(k)}\Big\rceil,\quad 1\leqslant i\leqslant q,$$
$$\max_{1\leqslant i\leqslant q}\Big\lceil\frac{\nu_{\varrho_i}(\text{ord}(a^j))}{\nu_{\varrho_i}(k)}\Big\rceil\leqslant\max_{1\leqslant i\leqslant q}\Big\lceil\frac{\nu_{\varrho_i}(2n)}{\eta_{\varphi_i}(k)}\Big\rceil,$$
$$\text{height}(a^j)\leqslant\max_{1\leqslant i\leqslant q}\Big\lceil\frac{\nu_{\varrho_i}(2n)}{\nu_{\varrho_i}(k)}\Big\rceil,\quad 0\leqslant j\leqslant 2n-1.$$

Since $\langle a\rangle$ is a cyclic subgroup of Q_{4n} such that $\text{ord}(a)=2n$, the existence of such vertex shows that

$$\text{height}(\gamma_A(2n,k))=\max_{1\leqslant i\leqslant q}\Big\lceil\frac{\nu_{\varrho_i}(2n)}{\nu_{\varrho_i}(k)}\Big\rceil. \tag{2.6}$$

Theorem 2.1 yields

$$\text{height}(\gamma_B(2n,k))=(1-\delta(k))\Big\lceil\frac{2}{\beta_1}\Big\rceil. \tag{2.7}$$

(2.5), (2.6), and (2.7) complete the proof. □

Corollary 2.4. *Let G be a generalized quaternion 2-group of order 2^n. Then*

$$\text{height}(\gamma_{Q_{2^n}}(2^n,2))=n-1.$$

Lemma 2.5. *Let G be a generalized quaternion 2-group of order 2^n and $k=2$. Then $N_{Q_{2^n}}(2^n,a^i,2)=0$, i.e., (a^i is an end vertex) if and only if i is an odd integer.*

P r o o f. Suppose i is an odd integer but $N_{Q_{2^n}}(2^n, a^i, 2) \neq 0$. This implies that there must exist a vertex x such that

$$(x)^2 = a^i.$$

Since $(a^j b)^2 = a^{2^{n-2}}$ for all $0 \leqslant j \leqslant 2^{n-2}$ and i is odd, x must be equal to a^r for some $0 \leqslant r \leqslant 2^{n-2}$. Therefore,

$$\begin{aligned}(a^r)^2 &= a^i,\\ a^{2r-i} &= 1,\\ 2^{n-1} &\mid 2r - i.\end{aligned}$$

This contradicts the assumption that i is an odd integer.

Conversely, assume $N(2^{n-1}, a^i, 2) = 0$ but i is an even integer. Thus, i can be written as $i = 2r$ for some integer $0 < r < i$. This implies that a^r is a vertex of $\gamma_{Q_{4n}}(4n, k)$ such that $(a^r)^2 = a^i$. This contradicts the assumption that $N(2^{n-1}, a^i, 2) = 0$. □

Corollary 2.6. *Let G be a generalized quaternion 2-group of order 2^n and $k = 2$. Then*

$$N_{Q_{2^n}}(2^n, a^i, 2) = \begin{cases} (1 - \delta(i))2 & \text{if } i \neq 2^{n-2},\\ 2 + 2^{n-1} & \text{otherwise,} \end{cases}$$

where $\delta(i)$ is defined in (1.1).

P r o o f. It is evident from Theorem 1.3 and Lemma 2.5. □

Lemma 2.7. *Let G be a generalized quaternion 2-group of order 2^n and $k = 2$. Then all vertices a^i, where i is odd and $1 \leqslant i < 2^{n-2}$, are at of the same height.*

P r o o f. It is clear from Lemma 2.5 and Corollaries 2.2, 2.6. □

Corollary 2.8. *Let G be a generalized quaternion 2-group and $k = 2$. Then the vertex a^i is of the maximum height if and only if i is odd, where $1 \leqslant i < 2^{n-2}$.*

P r o o f. It is obvious from Lemma 2.5 and Lemma 2.7. □

3. Classification of 2-groups

Theorem 3.1. *Let G be a finite 2-group. Then $G \cong Q_{2^n}$ if and only if $\gamma_G(2^n, 2)$ is isomorphic to the digraph S given in Figure 1 such that the sub-digraph H containing 2^{n-1} vertices is a binary digraph rooted at the fixed point and having height $n-1$ with all end vertices at the same height.*

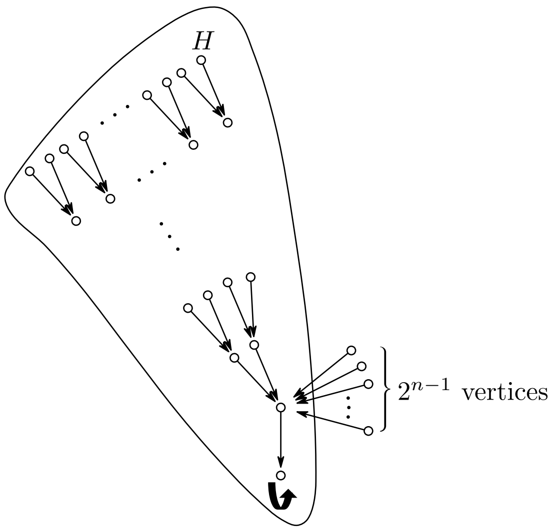

Figure 1. The digraph S.

Proof. Suppose $G \cong Q_{2^n}$. We label the digraph S as in Figure 2. Let $f\colon V(S) \to V(\gamma_{Q_{2^n}}(2^n, 2))$ be a digraph mapping. Since $k = 2$, Theorem 1.1 implies that $\gamma_{Q_{2^n}}(2^n, 2)$ is a connected digraph. Since 1 (identity of Q_{2^n}) is always a fixed point of $\gamma_{Q_{2^n}}(2^n, 2)$, it consists of only one component containing the fixed point 1. Therefore, we can take $f(x) = 1$. Futher, Q_{2^n} has only one involution, i.e., only one nontrivial element $a^{2^{n-2}}$ such that $(a^{2^{n-2}})^2 = 1$. This implies that there is an edge between $a^{2^{n-2}}$ and 1 in $\gamma_{Q_{2^n}}(2^n, 2)$. Hence, we can take $f(y) = a^{2^{n-2}}$. Now it is easy to see that $a^i b$ goes to $a^{2^{n-2}}$ for all $1 \leqslant i \leqslant 2^{n-1}$. Thus, we define $f(x_i) = a^i b$. From Corollary 2.6 we have $N(a^i, 2, 2^{n-1}) = 0$ when i is odd while for even i, a^i has indegree 2 except $a^{2^{n-2}}$ whose indegree is $2 + 2^{n-1}$ so that all the vertices $a^i b$ for $0 \leqslant i \leqslant 2^{n-1}$ are adjacent to $a^{2^{n-2}}$. From Lemma 2.7, all the end vertices are of the same height. Also from the proof of Theorem 2.3, height$(\langle a \rangle) = n-1$. Thus, $\gamma_{\langle a \rangle}(2^{n-1}, 2)$ is a binary digraph of height $n-1$ such that all the end vertices are of the same height. Therefore, $H \cong \gamma_{\langle a \rangle}(2^{n-1}, 2)$ and for $h_i \in V(H)$, we can define $f(h_i) = a^j$ for some j, where $1 \leqslant i \leqslant 2^{n-3}$ and $1 \leqslant j \leqslant 2^{n-3}$. Now it is easy to see that f is a graph isomorphism.

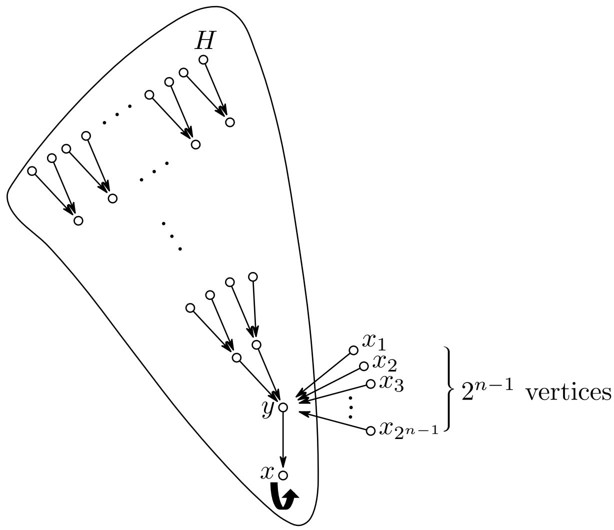

Figure 2. The labeled digraph S.

Conversely, suppose $\gamma_G(2^n,2)$ is isomorphic to the digraph S. The digraph S shows that there is only one nontrivial element x of G such that $x^2=1$. Since G is a 2-group and has only one element of order 2 (i.e., only one involution) G must be cyclic or a generalized quaternion 2-group. Suppose that G is a cyclic group. Then, from Lemma 1.2, for any vertex b having nonzero indegree in $\gamma_G(2^n,2)$,

$$N(2^n,b,2)=\gcd(2^n,2)=2.$$

But the digraph S shows that there is a vertex having degree $2^{n-1}+2$ which is greater than 2. Thus, G is a generalized quaternion 2-group, i.e. $G\cong Q_{2^n}$. $\square$

Corollary 3.2. *Let G be a generalized quaternion 2-group, $k=2$ and Q a maximal subgroup of G. Then $Q=\langle a\rangle$ if and only if*

$$\gamma_Q(|Q|,2)=\bigcup_{\mathrm{height}(x)=n-1} P_1^x.$$

Proof. It is obvious from the proof of Theorem 3.1 that the sub-digraph H in Figure 1 is isomorphic to $\gamma_{\langle a\rangle}(2^{n-1},2)$ and $H=\bigcup\limits_{\mathrm{height}(x)=n-1} P_1^x$. $\square$

Corollary 3.3. *Let G be a generalized quaternion 2-group, $k=2$ and let Q_1,Q_2 be two generalized quaternion subgroups of G. Then*

$$\gamma_{Q_1\cup Q_2}(|Q_1\cup Q_2|,2)=\bigcup_{h\leqslant l} P_1^h,$$

where $l=\max\{n-2,2\}$.

P r o o f. Since $Q_1 = \langle a^2, b\rangle$ and $Q_2 = \langle a^2, ab\rangle$,

$$Q_1 \cup Q_2 = \langle a^2\rangle \cup B.$$

The subgroup $\langle a^2\rangle$ must contain elements of the form a^i such that i is even. This implies that $\gamma_{Q_1\cup Q_2}(|Q_1 \cup Q_2|, 2)$ contains all the vertices of G except a^j, where j is odd. From Lemma 2.5, all a^i, where i is odd, are of the maximum height which is $n-1$. Now the vertices from B are of height 2. Let $l = \max\{n-2, 2\}$. Therefore, $\gamma_{Q_1\cup Q_2}(|Q_1 \cup Q_2|, 2)$ is the union of all paths from the vertices of height at most l to identity 1. Hence,

$$\gamma_{Q_1\cup Q_2}(|Q_1 \cup Q_2|, 2) = \bigcup_{h\leqslant l} P_1^h.$$

□

Lemma 3.4. *Let G be a semi-dihedral 2-group. Then $\gamma_G(2^n, 2)$ is not a semi-regular digraph.*

P r o o f. Suppose G is a semi-dihedral 2-group. The group representation is given as

$$SD_{2^n} = \langle a, b\colon\ a^{2^{n-1}} = b^2 = 1,\ bab = a^{2^{n-2}-1}\rangle.$$

Now consider the indegree of $a^{2^{n-2}}$, i.e., the number of solutions of equation $x^2 = a^{2^{n-2}}$. It is easy to see that for $k = 2$,

- ▷ a^i goes to some a^j for all $1 \leqslant i \leqslant 2^{n-1}$,
- ▷ $a^i b$ goes to 1 if i is even otherwise it maps to $a^{2^{n-2}}$. This shows that either $x = a^i$ or $x = a^j b$, where j is an odd number between 1 and 2^{n-1}. The solutions of the form a^i belong to the cyclic subgroup $\langle a\rangle = \{1, a, a^2, \ldots, a^{2^{n-2}}\}$. Thus, the number of solutions of the form a^i, by Theorem 1.2, is $\gcd(2^{n-1}, 2)$. Hence,

$$N(SD_{2^n}, a^{2^{n-2}}, 2) = \gcd(2^{n-1}, 2) + 2^{n-2}.$$

If all vertices a^i are adjacent to either $a^{2^{n-2}}$ or 1 then the orders of all these vertices are 4 or 2, respectively. This contradicts the fact that $\langle a\rangle$ is a finite cyclic subgroup of order 2^{n-1} and corresponding to each divisor m of 2^{n-1}, there exists an element of order m. Therefore, there must exist vertices a^j and a^r such that $(a^j)^2 = a^r$ and $a^r \neq 1$ or $a^{2^{n-2}}$. This shows that $N(SD_{2^n}, a^r, 2) \neq 0$. Also, as $a^r \neq a^{2^{n-2}}$ or 1, all the solutions of the equation $x^2 = a^r$ are of the form a^i and hence lie in $\langle a\rangle$. By using Theorem 1.2, $N(SD_{2^n}, a^r, 2) = \gcd(2n, k)$. This implies that

$$N(SD_{2^n}, a^{2^{n-2}}, 2) \neq N(SD_{2^n}, a^r, 2).$$

Hence, $\gamma_G(2^n, 2)$ is not a semi-regular digraph. □

Lemma 3.5. *Let G be a generalized dihedral 2-group. Then $\gamma_G(2^n, 2)$ is not a semi-regular digraph.*

P r o o f. The proof is similar to that of Lemma 3.4. □

Theorem 3.6. *Let G be a finite two generated 2-group of order 2^n. Then $G \cong C_{n_1} \times C_{n_2} \times \ldots \times C_{n_k}$ if and only if $\gamma_G(2^n, 2)$ is a connected semi-regular digraph of degree 2^k.*

P r o o f. Suppose $\gamma_G(2^n, 2)$ is a connected semi-regular digraph of degree 2^k but $G \cong C_{n_1} \times C_{n_2} \times \ldots \times C_{n_k}$ for any integers $n_1, n_2, \ldots, n_k$, i.e., G is a non-abelian group. Also $\gamma_G(2^n, 2)$, by Theorem 1.1, consists of only one component containing the fixed point 1 (identity of G). Now two cases arise:

Case 1: Let G be a non-abelian non-metacyclic 2-group. Then it can be further classified into two categories: in the first all proper subgroups of G are metacyclic and in the other G has at least one proper non-metacyclic subgroup. In either case, by using Theorem 3.2 of [5] and the main theorem of [6], G must be generated by three generators, which contradicts our assumption.

Case 2: If G is meta cyclic then there are three possibilities of involutions:

▷ Exactly one involution in G.

▷ More than three involutions in G.

▷ Exactly three involutions in G.

From Theorem 2.1 of [8], if G has one involution then G must be cyclic or generalized quaternion. The assumption that G is a non abelian group having a semi-regular digraph along with the fact that the digraph $\gamma_{Q_{2^n}}(2^n, 2)$ is not semi-regular due to Theorem 3.1, imply that G cannot be isomorphic to a cyclic or generalized quaternion. Hence, the case of exactly one involution is not possible. In the situation of more than three involutions, again from Theorem 2.1 of [8], G is equivalent to a dihedral or semi-dihedral group. This is not possible as the digraphs of a dihedral and semi-dihedral group are not semi-regular for $k = 2$ due to Lemmas 3.4 and 3.5. Hence, we may assume that G has exactly three involutions such that $S = \{x_1, x_2, x_3\}$ is the set of nontrivial involutions. Since the center of G is nontrivial, by the Cauchy theorem, the center of G must contain at least one involution, say 'x_1'. Since the power digraph is semi-regular and $a^2 = 1$ has four solutions (three involutions and identity 1), $\gamma_G(|G|, 2)$ is semi-regular of degree 4. Now as G is non abelian, therefore, cardinality of the set of vertices must be greater than 4. Hence, there is at least one involution $x \in S$ such that $N(x, 2^r, 2) \neq 0$. Let a_1 be one of the vertices adjacent to x, then it is very easy to see that the set of vertices adjacent to x is $P = \{a_1, a_1^{-1}, a_1x_1, a_1^{-1}x_1\}$, where the order of each element of P is 4. Now two cases arise: either $[a_1, x_2] = 1$ or $[a_1, x_2] \neq 1$. If $[a_1, x_2] = 1$, then

$$a_1 x_2 a_1^{-1} x_2^{-1} = 1,$$
$$a_1 x_2 = x_2 a_1,$$
$$(a_1 x_2)^2 = a_1^2 x_2^2 = a_1^2 = x.$$

Similarly, $a_1 x_2^{-1}$ is also adjacent to x. It is easy to see that $a_1 x_2$ and $a_1 x_2^{-1}$ are different from the members of P. This shows that $\operatorname{indeg}(x) \geqslant 6$, which contradicts the fact that $\gamma_G(|G|, 2)$ is semi-regular of degree 4. Hence, we may assume that $[a_1, x_2] \neq 1$. Now consider $(x_2)^{a_1}$ which is an involution, therefore, $(x_2)^{a_1} = a_1^{-1} x_2 a_1$ can be x_1, x_2 or x_3. Since $[a_1, x_2] \neq 1$ and x_1 is a central element, $a_1^{-1} x_2 a_1 \neq x_1$ or x_2. Hence, $a_1^{-1} x_2 a_1 = x_3$, where $x_3 = x_1 x_2$. Therefore,

$$a_1^{-1} x_2 a_1 = x_1 x_2,$$
$$a_1^3 x_2 a_1 = x_1 x_2,$$
$$a_1 x x_2 a_1 = x_1 x_2.$$

Since x is an involution, there are the following three possibilities:

If $x = x_2$, then

$$a_1 x_2^2 a_1 = x_1 x_2,$$
$$a_1^2 = x_1 x_2,$$
$$x = x_1 x_2,$$
$$x_2 = x_1 x_2.$$

This contradicts the assumption that x_1 is a nontrivial involution.

If $x = x_1$, then as x_1 is a central element, we can write

$$a_1 x_1 x_2 a_1 = x_1 x_2,$$
$$a_1 x_2 a_1 = x_2,$$
$$a_1 x_2 a_1 x_2^{-1} = 1,$$
$$a_1 x_2 a_1 x_2 = 1.$$

This shows that $a_1 x_2 = 1, x_1, x_2, x_3$. In all cases, we get either $a_1 = 1$ or a_1 is an involution, which contradicts the choice of a_1 (nontrivial element of order 4).

Now, if $x = x_3 = x_1 x_2$, then as x_1 is a central element and x_2 is an involution, we get

$$a_1 x_1 x_2^2 a_1 = x_1 x_2,$$
$$a_1 x_1 a_1 = x_1 x_2,$$
$$a_1^2 = x_2,$$
$$x_1 x_2 = x = x_2.$$

This implies that $x_1 = 1$, which is not possible as x_1 is a nontrivial involution. All the cases lead to a contradiction. Hence, G is an abelian 2-group.

Conversely, suppose G can be written as

$$G \cong C_{n_1} \times C_{n_2} \times \ldots \times C_{n_k},$$

where C_{n_i} is a cyclic group for all $1 \leqslant i \leqslant k$, i.e. G is abelian. Then by Lemma 1.2, the indegree of any vertex a of G is either 0 or

$$N(G, 2, a) = \prod_{i=1}^{k} \gcd(n_i, 2) = \prod_{i=1}^{k} 2 = 2^k.$$

This shows that $\gamma_G(|G|, 2)$ is a semi-regular power digraph. Now, since the order of each element of G is a power of 2 for all $a \in G$, $a^{2^m} = 1$ for some positive integer m. This implies that every vertex a has a path from a to 1 which is a fixed point. This further shows that $a \in \mathrm{Comp}(1)$ for all $a \in G$. Hence, $\gamma_G(|G|, 2) = \mathrm{Comp}(1)$. Thus, $\gamma_G(|G|, 2)$ is a connected power digraph. This completes the proof. $\square$

Example. Let G be a non-abelian group of order 2^5 such that

$$G = \langle a, b\colon\ a^{16} = b^2 = 1,\ aba = b\rangle.$$

From Figure 3 we can see that the power digraph $\gamma_G(2^5, 2)$ is not semi-regular. Whereas, if we take $G = C_8 \times C_4$, then the power digraph of $\gamma_G(2^5, 2)$ is semi-regular of degree 4. This is shown in Figure 4.

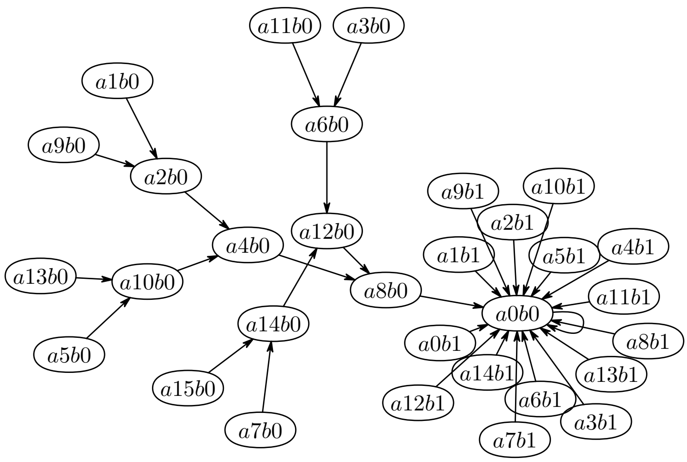

Figure 3. The power digraph of a non-abelian group of order 32.

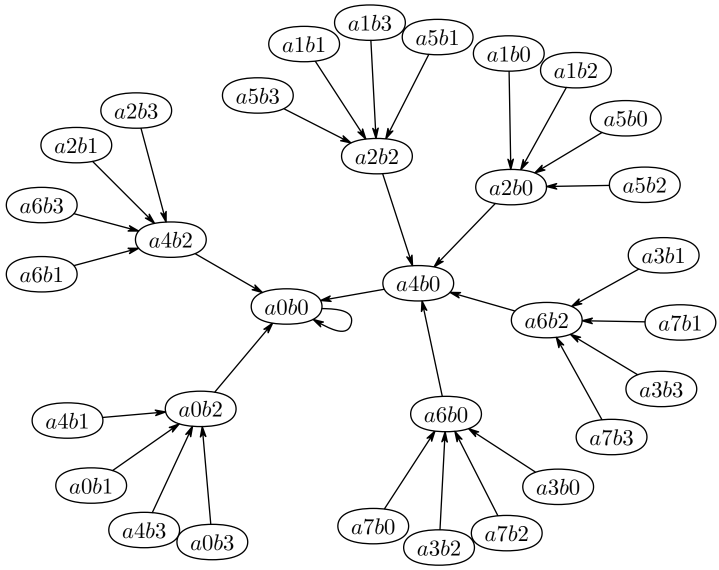

Figure 4. The power digraph of an abelian group of order 32.

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References

[1] *U. Ahmad, S. Husnine*: The power digraphs of finite groups. To appear in Util. Math.
[2] *U. Ahmad, S. Husnine*: Characterization of power digraphs modulo n. Commentat. Math. Univ. Carol. *52* (2011), 359–367.
[3] *U. Ahmad, M. Moeen*: The digraphs arising by the power maps of generalized quaternion groups. To appear in J. Algebra Appl. DOI:10.1142/S0219498817501791.
[4] *U. Ahmad, H. Syed*: On the heights of power digraphs modulo n. Czech. Math. J. *62* (2012), 541–556.
[5] *N. Blackburn*: Generalizations of certain elementary theorems on p-groups. Proc. Lond. Math. Soc. (3) *11* (1961), 1–22.
[6] *V. Ćepulić, O. S. Pyliavska*: A class of nonabelian nonmetacyclic finite 2-groups. Glas. Mat., Ser. (3) *41* (2006), 65–70.
[7] *S. M. Husnine, U. Ahmad, L. Somer*: On symmetries of power digraphs. Util. Math. *85* (2011), 257–271.
[8] *Z. Janko*: A classification of finite 2-groups with exactly three involutions. J. Algebra *291* (2005), 505–533.
[9] *C. Lucheta, E. Miller, C. Reiter*: Digraphs from powers modulo p. Fibonacci Q. *34* (1996), 226–239.
[10] *M. Sha*: Digraphs from endomorphisms of finite cyclic groups. J. Comb. Math. Comb. Comput. *83* (2012), 105–120.
[11] *L. Somer, M. Křížek*: On a connection of number theory with graph theory. Czech. Math. J. *54* (2004), 465–485.

[12] *L. Somer, M. Křížek*: On semiregular digraphs of the congruence $x^k \equiv y \pmod n$. Commentat. Math. Univ. Carol. *48* (2007), 41–58.
[13] *L. Somer, M. Křížek*: On symmetric digraphs of the congruence $x^k \equiv y \pmod n$. Discrete Math. *309* (2009), 1999–2009.
[14] *B. Wilson*: Power digraphs modulo n. Fibonacci Q. *36* (1998), 229–239.

Authors' address: Uzma Ahmed, Muqadas Moeen, University of the Punjab, New Campus, Lahore, 54000, Punjab, Pakistan, e-mail: uzma.math@pu.edu.pk, muqadasomi@hotmail.com.