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GENERAL PROPORTIONAL MEAN RESIDUAL LIFE MODEL

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Abstract. By considering a covariate random variable in the ordinary proportional mean residual life (PMRL) model, we introduce and study a general model, taking more situations into account with respect to the ordinary PMRL model. We investigate how stochastic structures of the proposed model are affected by the stochastic properties of the baseline and the mixing variables in the model. Several characterizations and preservation properties of the new model under different stochastic orders and aging classes are provided. In addition, to illustrate different properties of the model, some examples are presented.

Keywords: stochastic order; preservation property; decreasing failure rate (DFR); increasing mean residual life (IMRL)

MSC 2010: 60E05, 60E15, 62N05

1. Introduction

The mean residual life (MRL) and the hazard rate (HR) functions are commonly used to characterize the lifetime of a system. The HR function gives the instantaneous failure rate at any point of time, whereas the MRL function summarizes the entire residual life; and this is why the MRL function is found to be more relevant than the HR function. Let X be the lifetime of a system having survival function (sf) denoted $\overline{F}_X$. Then the MRL of the system at age t is defined as the expectation of $X_t = [X - t \mid X > t]$, the remaining lifetime of the system after t, that is

$$m_X(t) = \frac{\int_t^\infty \overline{F}_X(x)\,\mathrm{d}x}{\overline{F}_X(t)}, \quad t \in \mathbb{R}^+.$$

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For an exhaustive monograph on the MRL function, we refer the readers to Lai and Xie [10]. As a useful model to describe failure time data, Zahedi [19] proposed the proportional mean residual life (PMRL) model in which the MRL function of an individual is expressed as the product of an individual-specific quantity and a baseline MRL function, written as

$$m_T(t) = cm_X(t) \quad \forall\, t \geqslant 0, \tag{1.1}$$

where m_T and m_X are MRL functions of two non-negative random variables (rv's) T and X, respectively, with finite expectations. The PMRL model can be interpreted in several ways when the rv represents the lifetime of a system or a unit, which yields several applications of the model. For example, let a series system be formed with k components, one of which has lifetime distribution F_X and the other $k-1$ components have i.i.d. life distributions, which is the equilibrium distribution corresponding to F_X. An equilibrium distribution is obtained as a limiting distribution of a renewal process. Then the MRL function of the system so formed and the MRL function corresponding to F_X are proportional with constant of proportionality $c = 1/k$. Other interpretations, properties and applications of the PMRL model can be found in Magulury and Zhang [11], Gupta and Kirmani [6], Nanda et al. [13] and Nanda et al. [14]. Recently, Nanda et al. [15] replaced c in Eq. (1.1) by some non-negative function of t, say $c(t)$, and extended the corresponding model to the dynamic proportional mean residual life (DPMRL) model written as

$$m_T(t) = c(t)m_X(t), \quad t \geqslant 0. \tag{1.2}$$

For interpretation, properties and applications of the DPMRL model we refer the readers to Nanda et al. [15]. The purpose of this paper is to introduce, study and analyze the general proportional mean residual life (GPMRL) model. By considering a covariate rv in the ordinary PMRL model given in (1.1), our model takes more situations into consideration. Because of the fact that the covariate variable may be unobservable in some situations, mixture of distributions that follow GPMRL model, is also considered. In the context of the mixture GPMRL model, a number of characterizations and preservation properties of some dependence structures, stochastic orders and aging notions are established. To detect how the variation of the baseline variable and the variation of the mixing variable with respect to some stochastic orders each affects the model, we seek for conditions under which some ordering relations between overall variables hold true. The rest of the paper is organized as follows. In Section 2, we present some definitions and basic properties which will be used throughout the paper. In Section 3, some basic representations of the GPMRL model are given. Preservation properties of the model with respect to some dependence structures and some stochastic orders are studied in Section 4. In that section, we obtain some implications that indicate the preservation properties of some aging notions under the formation of the mixture GPMRL model. In Section 5, in view of the new model, we establish some useful stochastic order relations. Finally, in Section 6, we conclude the paper with some remarks related to the current research. Throughout the paper, we use *increasing* and *decreasing* in place of non-decreasing and non-increasing, respectively. In addition, all the integrals and the expectations are assumed to exist wherever they appear.

2. Preliminaries

This section presents some definitions and basic properties which will be used across the paper. First, we bring definitions of some stochastic orders and aging notions. For the stochastic orders we refer to Shaked and Shanthikumar [18] and for the aging notions we refer to Lai and Xie [10] and Barlow and Proschan [1]. In the sequel, X and Y are two non-negative rv's with distribution functions F and G, sf's $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y = 1 - F_Y$, probability density functions (pdf's) f_X and f_Y, MRL functions m_X and m_Y and HR functions $h_X(t) = f_X(t)/\overline{F}_X(t)$ and $h_Y(t) = f_Y(t)/\overline{F}_Y(t)$, respectively. We also denote by X_e the equilibrium rv associated with X having sf $\overline{F}_{X_e}(x) = \int_x^\infty \overline{F}_X(u)\,\mathrm{d}u/E(X)$ in which $E(X) = \int_0^\infty \overline{F}_X(u)\,\mathrm{d}u < \infty$.

Definition 2.1. The rv X is said to be smaller than the rv Y in the:

(i) Hazard rate order (denoted as $X \leqslant_{\mathrm{HR}} Y$) if $h_X(t) \geqslant h_Y(t)$ for all $t \geqslant 0$;
(ii) Usual stochastic order (denoted as $X \leqslant_{\mathrm{ST}} Y$) if $\overline{F}_X(t) \leqslant \overline{F}_Y(t)$ for all $t \geqslant 0$;
(iii) Mean residual life (denoted as $X \leqslant_{\mathrm{MRL}} Y$) if $m_X(t) \leqslant m_Y(t)$ for all $t \geqslant 0$.

Definition 2.2. The non-negative rv X is said to have:

(i) Increasing [Decreasing] Failure Rate (IFR [DFR]) property if h_X is an increasing [a decreasing] function or equivalently if $\overline{F}$ is a log-concave [log-convex] function;
(ii) Decreasing [Increasing] Mean Residual Life (DMRL [IMRL]) property if m_X is a decreasing [an increasing] function or equivalently if $\int_x^\infty \overline{F}_X(u)\,\mathrm{d}u$ is log-concave [log-convex] for $x \geqslant 0$;
(iii) New Better [Worse] than Used in Expectation (NBUE [NWUE]) if $m_X(0) \geqslant (\leqslant) m_X(t)$ for all $t > 0$;
(iv) Decreasing [Increasing] Mean Residual Life in Harmonic Average (DMRLHA [IMRLHA]) if

$$\left[\frac{1}{x}\int_0^x \frac{\mathrm{d}t}{m_X(t)}\right]^{-1} \text{ is decreasing [increasing] in } x \geqslant 0.$$

Definition 2.3 (Karlin [7]). A non-negative function $\beta(x, y)$ is said to be Totally Positive (Reverse Regular) of order 2, denoted as TP_2 (RR_2), in $(x, y) \in \chi \times \gamma$ if

$$\begin{vmatrix} \beta(x_1, y_1) & \beta(x_1, y_2) \\ \beta(x_2, y_1) & \beta(x_2, y_2) \end{vmatrix} \geqslant (\leqslant)\ 0$$

for all $x_1 \leqslant x_2 \in \chi$ and $y_1 \leqslant y_2 \in \gamma$, in which χ and γ are two real subsets of the real line $\mathbb{R}$.

3. The model and its representations

Consider Y represents the changes in the conditions that has a multiplicative effect on X through the relationship

$$m^*(x \mid y) = a(y)m_X(x) \tag{3.1}$$

for some positive function $a(y)$, where X has finite expectation. Let X^* denote the random variable with its conditional MRL given that $Y = y$ satisfies (3.1). The formal definition of the model is as follows. Assume that Y is an rv with support $S_Y = (l_Y, u_Y)$, where $l_Y = \inf\{y \in \mathbb{R} \colon F_Y(y) > 0\}$ and $l_Y = \sup\{y \in \mathbb{R} \colon F_Y(y) < 1\}$.

Definition 3.1. Suppose that $X(\theta)$ is a non-negative rv with MRL $m(x; \theta)$, where $\theta \in \mathbb{R}$ is a parameter. The rv's X and $X(\theta)$ are said to have GPMRL model whenever there exists a non-negative function $a(\cdot)$ such that $m(x; \theta) = a(\theta)m_X(x)$ for all $x \geqslant 0$.

Special cases of the GPMRL model have found some practical applications in the literature (see, e.g., Oakes and Dasu [17], Chen and Cheng [3], and Mansourvar et al. [12]) including some inferential properties based on real data sets. As mentioned in Nanda et al. [14], there is a one-to-one correspondence between the conditional MRL function $m^*(\cdot \mid y)$ and the conditional survival function $\overline{F}^*(\cdot \mid y)$ of X^*, given $Y = y$, as below

$$\overline{F}^*(x \mid y) = \frac{m^*(0 \mid y)}{m^*(x \mid y)} \exp\Bigl(-\int_0^x \frac{\mathrm{d}u}{m^*(u \mid y)}\Bigr).$$

Now, since

$$\int_0^x \frac{\mathrm{d}u}{m_X(u)} = \ln\Bigl(\frac{m_X(0)}{\int_x^\infty \overline{F}_X(u)\,\mathrm{d}u}\Bigr),$$

it follows from (3.1) that

$$\begin{aligned}
\overline{F}^*(x \mid y) &= \frac{a(y)m_X(0)}{a(y)m_X(x)} \exp\biggl(-\frac{1}{a(y)}\int_0^x \frac{\mathrm{d}u}{m_X(u)}\biggr) \\
&= \frac{E(X)\overline{F}_X(x)}{\int_x^\infty \overline{F}_X(u)\,\mathrm{d}u}\biggl(\frac{\int_x^\infty \overline{F}_X(u)\,\mathrm{d}u}{E(X)}\biggr)^{1/a(y)} \\
&= \overline{F}_X(x)\biggl(\frac{\int_x^\infty \overline{F}_X(u)\,\mathrm{d}u}{E(X)}\biggr)^{1/a(y)-1}.
\end{aligned}$$

By the formula of the sf of an equilibrium rv, the sf of X^*, given that $Y = y$, can be written as

$$\overline{F}^*(x \mid y) = \overline{F}_X(x)\overline{F}_{X_e}^{1/a(y)-1}(x) = R(x)\overline{F}_{X_e}^{1/a(y)}(x), \tag{3.2}$$

where $R(x) = \overline{F}_X(x)/\overline{F}_{X_e}(x)$. In the sequel, the rv's X, Y and X^* will be referred to as the baseline, the mixing and the overall variables, respectively. It is assumed that X^* has the sf $\overline{F}^*$, the HR h^* and the MRL m^*. From (3.2), the unconditional sf of X^* is

$$\overline{F}^*(x) = R(x)\int_{-\infty}^{\infty} \overline{F}_{X_e}^{1/a(y)}(x) f_Y(y)\,\mathrm{d}y. \tag{3.3}$$

Taking $x = 0$ in (3.1) gives

$$a(y) = \frac{E(X^* \mid Y = y)}{E(X)}, \tag{3.4}$$

which is the regression curve of X^* on Y divided by the mean of X. An equivalent representation of (3.3) based on the MRL function instead of the sf is obtained as follows. The conditional density of $(Y \mid X^* > x)$ is given by

$$\begin{aligned}
g(y \mid X^* > x) &= \frac{\overline{F}^*(x \mid y) f_Y(y)}{\overline{F}^*(x)}, \quad \overline{F}^*(x) > 0, \\
&= \frac{\overline{F}_{X_e}^{1/a(y)}(x) f_Y(y)}{\int_{-\infty}^{\infty} \overline{F}_{X_e}^{1/a(y)}(x) f_Y(y)\,\mathrm{d}y} \qquad \forall\, x, y \geqslant 0.
\end{aligned} \tag{3.5}$$

From (3.1)–(3.5) one can get

$$\begin{aligned}
m^*(x) &= \frac{\int_x^\infty \overline{F}^*(u)\,\mathrm{d}u}{\overline{F}^*(x)} \\
&= \int_{-\infty}^{\infty} \frac{\int_x^\infty \overline{F}^*(u \mid y)\,\mathrm{d}u}{\overline{F}^*(x \mid y)} \frac{\overline{F}^*(x \mid y) f_Y(y)}{\overline{F}^*(x)}\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} m^*(x \mid y) g(y \mid X^* > x)\,\mathrm{d}y \\
&= m_X(x) E[a(Y) \mid X^* > x].
\end{aligned} \tag{3.6}$$

The identity given in (3.6) reveals that by taking $c(x) = E[a(Y) \mid X^* > x]$ in (1.2), X^* and X satisfy the DPMRL model which was introduced by Nanda et al. [15].

4. Basic properties of the model

4.1. Dependence structures. In the model of (3.3), we will show the existence of some dependence structures between the overall variable X^* and the mixing variable Y. Let the vector (X^*, Y) have joint sf $\overline{F}^*(\cdot,\cdot)$ and joint pdf $f^*(\cdot,\cdot)$. According to Nelsen [16], we have the following dependence properties.

Definition 4.1.

(i) The rv's X^* and Y have positive [negative] likelihood ratio dependence structure (PLRD(X^*, Y)) [NLRD(X^*, Y)] if $f^*(x,y)$ is TP$_2$ [RR$_2$] in $(x,y) \in \{(x,y) \in \mathbb{R}^2\colon\ f^*(x,y) > 0\}$.

(ii) The rv X^* is stochastically increasing [decreasing] in Y (SI$(X^* \mid Y)$) [SD$(X^* \mid Y)$] if $P(X^* > x \mid Y = y)$ is non-decreasing [non-increasing] in y for all $x \in \mathbb{R}$.

(iii) The rv's X^* and Y are right corner set increasing [decreasing] (RCSI(X^*, Y)) [RCSD(X^*, Y)] if $\overline{F}^*(x,y)$ is TP$_2$ [RR$_2$] in $(x,y) \in \{(x,y) \in \mathbb{R}^2\colon\ \overline{F}^*(x,y) > 0\}$.

Before stating the first result we need to state the following lemma which is essentially due to Karlin [7].

Lemma 4.1. *Let $\varphi(x,s)$ be an* TP$_2$(RR$_2$) *function in $(x,s) \in A \times B$ and let $\psi(s,y)$ be* TP$_2$ *in $(s,y) \in B \times C$, where A, B and C are three arbitrary subsets of $\mathbb{R}$. Then*

$$\varrho(x,y) = \int_{s \in B} \varphi(x,s)\psi(s,y)\,\mathrm{d}s$$

is TP$_2$(RR$_2$) *in $(x,y) \in A \times C$.*

Theorem 4.1.

(i) *X^* is stochastically increasing (decreasing) in Y if and only if $a(y)$ is increasing (decreasing) in y.*

(ii) *Let $a(y)$ be increasing (decreasing) in y. Then X^* and Y are right corner set increasing (decreasing).*

P r o o f. (i) First assume that a is an increasing (a decreasing) function. Based on (3.2) for any $y_1 \leqslant y_2 \in S_Y$,

$$\begin{aligned}\overline{F}^*(x \mid y_2) - \overline{F}^*(x \mid y_1) &= R(x)[\overline{F}_{X_e}^{1/a(y_2)}(x) - \overline{F}_{X_e}^{1/a(y_1)}(x)] \\ &\geqslant (\leqslant)\ 0 \quad \forall\, x \geqslant 0,\end{aligned}$$

which means $\mathrm{SI}(X^* \mid Y)$ $(\mathrm{SD}(X^* \mid Y))$. Conversely, let $\mathrm{SI}(X^* \mid Y)$ $(\mathrm{SD}(X^* \mid Y))$ hold. Equivalently,

$$(X^* \mid Y = y_1) \leqslant_{\mathrm{ST}} (\geqslant_{\mathrm{ST}}) (X^* \mid Y = y_2) \quad \forall\, y_1 \leqslant y_2 \in S_Y.$$

Since the ST order implies the expectation order,

$$E(X^* \mid Y = y_1) \leqslant (\geqslant) E(X^* \mid Y = y_2) \quad \forall\, y_1 \leqslant y_2 \in S_Y.$$

By the above inequality it follows form (3.4) that $a(y)$ is increasing (decreasing) in $y \in S_Y$.

(ii) It suffices to show that the joint sf of X^* and Y is TP_2 in $(x,y) \in \mathbb{R}^+ \times S_Y$. One has

$$\begin{aligned} \overline{F}^*(x,y) &= \int_y^\infty \int_x^\infty f^*(t,s)\,\mathrm{d}t\,\mathrm{d}s \\ &= \int_y^\infty \overline{F}_X(x)\overline{F}_{X_e}^{1/a(s)-1}(x) f_Y(s)\,\mathrm{d}s \\ &= \int_{-\infty}^\infty \varphi(x,s)\psi(s,y)\,\mathrm{d}s, \end{aligned} \tag{4.1}$$

where $\varphi(x,s) = \overline{F}_X(x)\overline{F}_{X_e}^{1/a(s)-1}(x)$ and $\psi(s,y) = f_Y(s)I[s > y]$, in which $x \geqslant 0$ and $s \in S_Y$. Easily, $\varphi(x,s)$ is $\mathrm{TP}_2(\mathrm{RR}_2)$ in $(x,s) \in \mathbb{R}_+ \times S_Y$ when $a(s)$ is increasing (decreasing) in $s \in S_Y$. In addition, $\psi(s,y)$ is TP_2 in $(s,y) \in S_Y \times S_Y$. Now, by applying Lemma 4.1 to (4.1) the proof is obtained at once. □

Theorem 4.2. *Let X have a convex (concave)* MRL *function and let $a(y)$ be increasing (decreasing) in y. Then X^* and Y are* PLRD (NLRD).

P r o o f. From (3.2) the joint pdf of X^* and Y is obtained, by total probability formula, as

$$f^*(x,y) = f_Y(y)\overline{F}_{X_e}^{1/a(y)-1}(x)\overline{F}_X(x)\Big[\Big(\frac{1}{a(y)} - 1\Big)h_{X_e}(x) + h_X(x)\Big], \ x \geqslant 0,$$

where

$$h_X(x) = \frac{f_X(x)}{\overline{F}_X(x)} \quad \text{and} \quad h_{X_e}(x) = \frac{\overline{F}_X(x)}{\int_x^\infty \overline{F}_X(u)\,\mathrm{d}u}$$

are, respectively, the baseline HR and the HR of X_e. As in the proof of Theorem 4.1, when $a(y)$ is increasing (decreasing) in $y \in S_Y$, then $\overline{F}_{X_e}^{1/a(y)-1}(x)$ is TP_2 (RR_2) in

$(x,y) \in B = \{(x,y) \in \mathbb{R}_+ \times S_Y\colon\ f^*(x,y) > 0\}$. Because the product of two TP_2 (RR_2) functions is itself a TP_2 (RR_2) function, we only need to show that

$$\eta(x,y) = \Big[\Big(\frac{1}{a(y)} - 1\Big) h_{X_e}(x) + h_X(x)\Big]$$

is TP_2 (RR_2) in $(x,y) \in B$. We prove that $\eta(x,y_2)/\eta(x,y_1)$ is increasing (decreasing) in x for all $y_1 \leqslant y_2 \in S_Y$ and $x \geqslant 0$. Take $\gamma(x) = h_X(x)/h_{X_e}(x)$ and $\gamma'(x) = \mathrm{d}\gamma(x)/\mathrm{d}x$. Then we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[\frac{\eta(x,y_2)}{\eta(x,y_1)}\Big] = \frac{[1/a(y_1) - 1/a(y_2)]\gamma'(x)}{[1/a(y_1) - \gamma(x)]^2},$$

which is non-negative (non-positive) if $\gamma'(x) \geqslant 0$ for all $x \geqslant 0$. Because

$$\gamma(x) = h_X(x) m_X(x) = 1 + m'_X(x),$$

the convexity (concavity) of $m_X(x)$ is equivalent to saying that $\gamma(x)$ is increasing (decreasing) and hence the proof is completed. $\square$

4.2. Preservation properties with respect to some stochastic orders. Based on the hazard rate function, an alternative representation of (3.3) can be obtained. By (3.3), for all $x \geqslant 0$ with $\overline{F}^*(x) > 0$,

$$\begin{aligned}
h^*(x) &= -\frac{\mathrm{d}}{\mathrm{d}x}\ln[\overline{F}^*(x)] \\
&= -\frac{\mathrm{d}}{\mathrm{d}x}\ln(R(x)) - \frac{\mathrm{d}}{\mathrm{d}x}\ln\Big[\int_{-\infty}^{\infty} \overline{F}_{X_e}^{1/a(y)}(x) f_Y(y)\,\mathrm{d}y\Big] \\
&= h_X(x) - \frac{\mathrm{d}}{\mathrm{d}x}\ln(\overline{F}_{X_e}(x))\Big[1 - \frac{\int_{-\infty}^{\infty} a(y)^{-1}\overline{F}_{X_e}^{1/a(y)}(x) f_Y(y)\,\mathrm{d}y}{\int_{-\infty}^{\infty} \overline{F}_{X_e}^{1/a(y)}(x) f_Y(y)\,\mathrm{d}y}\Big] \\
&= h_X(x) + \frac{1}{m_X(x)}\Big[E\Big(\frac{1}{a(Y)} \mid X^* > x\Big) - 1\Big].
\end{aligned} \tag{4.2}$$

In what follows, we discuss some preservation properties of the GPMRL model given in (3.3) with respect to some stochastic orders and aging classes of life distributions. Before stating the next result we need the following lemma which is known as covariance inequality in the literature.

Lemma 4.2 (Behboodian [2]). *Let f and g be two real functions.*

(i) *If f is increasing (decreasing) and g is increasing (decreasing), then*

$$\operatorname{Cov}(f(X), g(X)) \geqslant 0.$$

(ii) *If f is increasing (decreasing) and g is decreasing (increasing), then*

$$\operatorname{Cov}(f(X), g(X)) \leqslant 0.$$

For the sake of comparison of HR's of X and X^* we can say using (4.2) that $X \leqslant_{\mathrm{HR}} (\geqslant_{\mathrm{HR}})\, X^*$ if and only if $E[1/a(Y) \mid X^* > x] \leqslant (\geqslant)\, 1$ for all $x \geqslant 0$. To refine this result we consider the following characterization

Theorem 4.3. *Let X and X^* be the baseline and the overall rv's. Then*

$$X \leqslant_{\mathrm{HR}} X^* \Leftrightarrow E(1/a(Y)) \leqslant 1.$$

P r o o f. Set $\eta(x) = E[1/a(Y) \mid X^* > x]$ and observe that $\eta(0) = E[1/a(Y)]$. We first prove the "if" part of the theorem. Suppose that $E[1/a(Y)] \leqslant 1$. In view of (3.5)

$$\begin{aligned} \eta(x) &= \int_{-\infty}^{\infty} \frac{1}{a(y)} g(y \mid X^* > x)\,\mathrm{d}y \\ &= \frac{\int_{-\infty}^{\infty} a(y)^{-1} \overline{F}_{X_e}^{1/a(y)}(x) f_Y(y)\,\mathrm{d}y}{\int_{-\infty}^{\infty} \overline{F}_{X_e}^{1/a(y)}(x) f_Y(y)\,\mathrm{d}y} \\ &= E\Big[\frac{\overline{F}_{X_e}^{1/a(Y)}(x)}{a(Y)}\Big] \Big/ E[\overline{F}_{X_e}^{1/a(Y)}(x)] \quad \forall\, x \geqslant 0. \end{aligned} \tag{4.3}$$

Note that $\overline{F}_{X_e}^{w}(x)$ is decreasing in w for all $x \geqslant 0$. Take $W = 1/a(Y)$. Then Lemma 4.2 (ii) provides that

$$\operatorname{Cov}[W, \overline{F}_{X_e}^{W}(x)] = E[W\overline{F}_{X_e}^{W}(x)] - E(W)E[\overline{F}_{X_e}^{W}(x)] \leqslant 0 \quad \forall\, x \geqslant 0.$$

Therefore, from (4.3), $\eta(x) \leqslant E(1/a(Y))$ for all $x \geqslant 0$. Because of this and using (4.2), we get

$$\begin{aligned} h_X(x) - h^*(x) &= \frac{1}{m_X(x)}[1 - \eta(x))] \\ &\geqslant \frac{1}{m_X(x)}\Big[1 - E\Big(\frac{1}{a(Y)}\Big)\Big] \geqslant 0 \quad \forall\, x \geqslant 0, \end{aligned}$$

which means that $X \leqslant_{\mathrm{HR}} X^*$. We now prove the "only if" right part. We know that if $X \leqslant_{\mathrm{HR}} X^*$, then $\eta(x) \leqslant 1$ for all $x \geqslant 0$. Hence, $\eta(0) = E[1/a(Y)] \leqslant 1$. $\square$

The following example indicates that "$E(1/a(Y)) \leqslant 1$" is a necessary condition in Theorem 4.3.

Example 4.1. Let X have exponential distribution with survival function $\overline{F}_X(x) = \mathrm{e}^{-5x}$, $x \geqslant 0$, and let Y have Gamma distribution with pdf $f_Y(y) = y\mathrm{e}^{-y}$, $y \geqslant 0$. Now, if $a(y) = 1/y$ is chosen, then $E(1/a(Y)) = 2$ and $\overline{F}^*(x) = (5x+1)^{-2}$ for $x \geqslant 0$. Also, we observe that $\overline{F}^*(x)/\overline{F}_X(x) = \mathrm{e}^{5x}(5x+1)^{-2}$, which is decreasing for $x \leqslant 1/5$ and it is increasing for $x \geqslant 1/5$. Hence $X \ngeqslant_{\mathrm{HR}} X^*$. The necessary and sufficient condition of Theorem 4.3 cannot, therefore, be dropped.

To establish the MRL order between X and X^* using (3.6) we have $X \leqslant_{\mathrm{MRL}}$ ($\geqslant_{\mathrm{MRL}}$) X^* if and only if $E[a(Y) \mid X^* > x] \geqslant (\leqslant)\ 1$ for all $x \geqslant 0$. The following result is also useful.

Theorem 4.4. *Let X and X^* be the baseline and the overall rv's. Then*

$$X \leqslant_{\mathrm{MRL}} X^* \Leftrightarrow E(X) \leqslant E(X^*).$$

Proof. First, we prove that $X \leqslant_{\mathrm{MRL}} X^*$ if and only if $E[a(Y)] \geqslant 1$. Let $E[a(Y)] \geqslant 1$ and set $c(x) = E[a(Y) \mid X^* > x]$ for $x \geqslant 0$. We have

$$\begin{aligned} c(x) &= \int_{-\infty}^{\infty} a(y) g(y \mid X^* > x)\,\mathrm{d}y \\ &= \frac{\int_{-\infty}^{\infty} a(y)\overline{F}_{X_e}^{1/a(y)}(x) f_Y(y)\,\mathrm{d}y}{\int_{-\infty}^{\infty} \overline{F}_{X_e}^{1/a(y)}(x) f_Y(y)\,\mathrm{d}y} \\ &= \frac{E[a(Y)\overline{F}_{X_e}^{1/a(Y)}(x)]}{E[\overline{F}_{X_e}^{1/a(Y)}(x)]} \quad \forall\, x \geqslant 0. \end{aligned} \tag{4.4}$$

It can be seen that $\overline{F}_{X_e}^{1/v}(x)$ is increasing in $v = a(y)$ for all $x \geqslant 0$. Thus, if we take $V = a(Y)$, then by Lemma 4.2 (i),

$$\operatorname{Cov}(V, \overline{F}_{X_e}^{1/V}(x)) = E[V\overline{F}_{X_e}^{1/V}(x) - E(V)E(\overline{F}_{X_e}^{1/V}(x))] \geqslant 0 \quad \forall\, x \geqslant 0.$$

Now, from (4.4) we get $c(x) \geqslant E[a(Y)]$ for all $x \geqslant 0$. By (3.6),

$$\begin{aligned} m^*(x) - m_X(x) &= m_X(x)(c(x) - 1) \\ &\geqslant m_X(x)[E(a(Y)) - 1] \\ &\geqslant 0 \quad \forall\, x \geqslant 0, \end{aligned}$$

which means that $X \leqslant_{\mathrm{MRL}} X^*$. To prove the reversed implication, note that $X \leqslant_{\mathrm{MRL}} X^*$ implies $c(x) \geqslant 1$ for all $x \geqslant 0$. Thus $c(0) = E(a(Y)) \geqslant 1$. Observe from (3.4) that

$$E[a(Y)] = \frac{E[E(X^* \mid Y)]}{E(X)} = \frac{E(X^*)}{E(X)}.$$

Hence, $E[a(Y)] \geqslant 1$ if and only if $E(X) \leqslant E(X^*)$, which completes the proof. $\square$

In the context of Theorem 4.4, Example 4.2 below indicates that the condition $E(X^*) < E(X)$ cannot be a sufficient condition to conclude $X^* \leqslant_{\mathrm{MRL}} X$. Therefore, the set up of Theorem4.4 does not remain true in the reversed direction.

Example 4.2. Let Y be such that $P(Y = 1/2) = 1/2$, $P(Y = 1/3) = 1/4$, and $P(Y = 2) = 1/4$. Take $a(y) = y$ for $y = 1/2, 1/3, 2$. We see that $E[a(Y)] = 10/12 < 1$. Because $E[a(Y)] = E(X^*)/E(X)$, it follows that $E(X^*) < E(X)$. Suppose that X has exponential distribution with sf $\overline{F}_X(x) = \mathrm{e}^{-x}$ for $x \geqslant 0$. After some calculation, $\overline{F}^*(x) = 0.5\mathrm{e}^{-2x} + 0.25(\mathrm{e}^{-3x} + \mathrm{e}^{-x/2})$ for $x \geqslant 0$. Note that the MRL function of X is $m_X(x) = 1$ for any $x \geqslant 0$, and the MRL function of X^* is $m^*(x) = (3\mathrm{e}^{-2x} + \mathrm{e}^{-3x} + 6\mathrm{e}^{-x/2})/(6\mathrm{e}^{-2x} + 3\mathrm{e}^{-3x} + 3\mathrm{e}^{-x/2})$ for any $x \geqslant 0$. One can easily see that $m_X(x) - m^*(x)$ is not always non-negative, that is $X \ngeqslant_{\mathrm{MRL}} X^*$.

Consider a situation where data are coming from the population with sf $\overline{F}^*$. Then Theorems 4.3 and 4.4 state that if the baseline distribution is mistakenly used in place of the distribution of the overall variable, then the HR and the MRL functions of the overall variable are underestimated under some conditions. On the other hand, it is well-known that the HR order implies the MRL order but in general the converse is not true (cf. Nanda et al. [14] and Shaked and Shanthikumar [18]). However, in the context of the GPMRL model, the converse is true as we demonstrate below.

Theorem 4.5. *Let X and X^* be the baseline and the overall rv's. If $X^* \leqslant_{\mathrm{MRL}} X$, then $X^* \leqslant_{\mathrm{HR}} X$, and so $X^* \leqslant_{\mathrm{ST}} X$.*

Proof. Suppose that $X^* \leqslant_{\mathrm{MRL}} X$. Then $E[a(Y) \mid X^* > x] \leqslant 1$ for all $x \geqslant 0$. Because of Jensen's inequality,

$$E\Big[\frac{1}{a(Y)} \mid X^* > x\Big] \geqslant \frac{1}{E[a(Y) \mid X^* > x]} \geqslant 1 \quad \forall\, x \geqslant 0,$$

which means that $X^* \leqslant_{\mathrm{HR}} X$. Since the HR order implies the usual stochastic order, we also have $X^* \leqslant_{\mathrm{ST}} X$. $\square$

4.3. Preservation properties with respect to several aging notions. Here we discuss the preservation properties of some aging classes of life distributions under the transformation $X \to X^*$ and also under the reversed transformation $X^* \to X$.

Theorem 4.6.

(i) *Let* $E(1/a(Y) \mid X^* > x)$ *be decreasing in* x *such that* $E(1/a(Y) \mid X^* > x) \geqslant 1$ *for all* $x \geqslant 0$. *If* X *is* DFR, *then* X^* *is* DFR.

(ii) *Let* $E[a(Y) \mid X^* > x]$ *be increasing for* $x \geqslant 0$. *If* X *is* IMRL (IMRLHA), *then* X^* *is* IMRL (IMRLHA). *In addition, if* X^* *is* DMRL (DMRLHA), *then* X *is* DMRL (DMRLHA).

(iii) *If* X *is* NWUE, *then* X^* *is* NWUE. *Also, if* X^* *is* NBUE, *then* X *is* NBUE.

Proof. (i) Set $l(x) = E(1/a(Y) \mid X^* > x) - 1$ for $x \geqslant 0$. By assumption, $l(x)$ is non-negative and non-increasing. Since X is DFR, h_X is decreasing and since DFR $\subset$ IMRL, $1/m_X(x)$ is also decreasing. Now, from (4.2), $h^*(x) = h_X(x) + l(x)/m_X(x)$, which leaves h^* decreasing. That is, X^* is DFR.

(ii) For the cases of IMRL and DMRL, using (3.6), the proof is straightforward. Take $c(x) = E[a(Y) \mid X^* > x]$ for $x \geqslant 0$, which is increasing by assumption. Suppose that X is IMRLHA, then $\int_0^x [1/m_X(t) - 1/m_X(x)]\,\mathrm{d}t \geqslant 0$ for all $x \geqslant 0$. Because of (3.6), since $1/c(t) \geqslant 1/c(x)$ for all $t \leqslant x$, one can see that

$$\begin{aligned}\int_0^x \Big[\frac{1}{m^*(t)} - \frac{1}{m^*(x)}\Big]\,\mathrm{d}t &= \int_0^x \Big[\frac{1}{c(t)m_X(t)} - \frac{1}{c(x)m_X(x)}\Big]\,\mathrm{d}t \\ &\geqslant \frac{1}{c(x)} \int_0^x \Big[\frac{1}{m_X(t)} - \frac{1}{m_X(x)}\Big]\,\mathrm{d}t \geqslant 0 \quad \forall\, x \geqslant 0,\end{aligned}$$

which means that X^* is IMRLHA. Now, assume that X^* is DMRLHA, which gives $\int_0^x (1/m^*(x) - 1/m^*(t))\,\mathrm{d}t \geqslant 0$ for all $x \geqslant 0$. We need to show that $\int_0^x [1/m_X(x) - 1/m_X(t)]\,\mathrm{d}t \geqslant 0$ for all $x \geqslant 0$. By (3.6) and because $-c(t) \geqslant -c(x)$ for all $t \leqslant x$, one derives

$$\begin{aligned}\int_0^x \Big[\frac{1}{m_X(x)} - \frac{1}{m_X(t)}\Big]\,\mathrm{d}t &= \int_0^x \Big[\frac{c(x)}{m^*(x)} - \frac{c(t)}{m^*(t)}\Big]\,\mathrm{d}t \\ &\geqslant c(x) \int_0^x \Big[\frac{1}{m^*(x)} - \frac{1}{m^*(t)}\Big]\,\mathrm{d}t \geqslant 0 \quad \forall\, x \geqslant 0.\end{aligned}$$

(iii) As in the proof of Theorem 4.4, $E[a(Y)\overline{F}_{X_e}^{1/a(Y)}(x)] \geqslant E[a(Y)]E[\overline{F}_{X_e}^{1/a(Y)}(x)]$ for all $x \geqslant 0$. Hence,

$$\begin{aligned}\frac{m^*(x)}{m^*(0)} &= \frac{m_X(x)}{m_X(0)} \frac{E[a(Y) \mid X^* > x]}{E[a(Y)]} \\ &= \frac{m_X(x)}{m_X(0)} \frac{E[a(Y)\overline{F}_{X_e}^{1/a(Y)}(x)]}{E[\{a(Y)\}E\{\overline{F}_{X_e}^{1/a(Y)}(x)\}]} \\ &\geqslant \frac{m_X(x)}{m_X(0)} \quad \forall\, x \geqslant 0.\end{aligned}$$

The result now follows by a simple discussion. $\square$

R e m a r k 4.1. The stated sufficient condition in Theorem 4.6 (i) holds true if $a(y)$ is monotone increasing (or monotone decreasing) in $y \in S_Y$. We use Theorem 4.1 (ii) to show it. Suppose that $a(y)$ is increasing (decreasing) in $y \in S_Y$. Then Theorem 4.1 (ii) says that $\overline{F}^*(x,y)$ is TP_2 (RR_2) in $(x,y) \in \mathbb{R}_+ \times S_Y$, which can be simply translated to

$$(Y \mid X^* > x_1) \leqslant_{\mathrm{HR}} (\geqslant_{\mathrm{HR}}) \, (Y \mid X^* > x_2) \quad \forall \, x_1 \leqslant x_2,$$

and hence,

$$(Y \mid X^* > x_1) \leqslant_{\mathrm{ST}} (\geqslant_{\mathrm{ST}}) \, (Y \mid X^* > x_2) \quad \forall \, x_1 \leqslant x_2.$$

Because $1/a(y)$ is decreasing (increasing) in $y \in S_Y$,

$$E[1/a(Y) \mid X^* > x_1] \geqslant E[1/a(Y) \mid X^* > x_2] \quad \forall \, x_1 \leqslant x_2,$$

which means that $E(1/a(Y) \mid X^* > x)$ is decreasing in $x \geqslant 0$. Similarly, to make Theorem 4.6 (ii) applicable, we see from Theorem 4.1 (ii) that if $a(y)$ is increasing (decreasing) in $y \in S_Y$, then $E[a(Y) \mid X^* > x]$ is increasing in $x \geqslant 0$.

5. Mean residual life comparisons

In this section, in order to demonstrate how the variation of the baseline variable and the variation of the mixing variable each has an effect on the model, we make a stochastic comparisons of the MRL functions between the two overall variables arisen from the model. Assume that X_i is the baseline variable in the model that has sf $\overline{F}_{X_i}$, and assume that X_i^* is the associated overall variable with sf

$$\overline{F}_i^*(x) = R_i(x) E[\overline{F}_{X_{ie}}^{1/a(Y)}(x)], \tag{5.1}$$

where $R_i(x) = \overline{F}_{X_i}(x)/\overline{F}_{X_{ie}}(x)$ for all $x \geqslant 0$, and $\overline{F}_{X_{ie}}$ is the sf of the equilibrium rv associated with X_i, $i = 1, 2$. Denote by m_{X_i} the MRL function of the rv X_i, $i = 1, 2$. In the following result, under some appropriate assumptions, we show that the MRL order between X_1 and X_2 is translated to the MRL order between X_1^* and X_2^*.

Theorem 5.1. *Let $a(y)$ be increasing (decreasing) in $y \geqslant 0$ and let*

$$(Y \mid X_1^* > x) \leqslant_{\mathrm{ST}} (\geqslant_{\mathrm{ST}}) \, (Y \mid X_2^* > x)$$

for all $x \geqslant 0$. Then

$$X_1 \leqslant_{\mathrm{MRL}} X_2 \Rightarrow X_1^* \leqslant_{\mathrm{MRL}} X_2^*.$$

P r o o f. First, denote by m_i^* the MRL function of X_i^* for $i=1,2$. As in (3.6), we have $m_i^*(x) = m_{X_i}(x)E[a(Y) \mid X_i^* > x]$ for $i=1,2$. Thus,

$$\begin{aligned} (5.2) \qquad m_2^*(x) - m_1^*(x) &= m_{X_2}(x)E[a(Y) \mid X_2^* > x] - m_{X_1}(x)E[a(Y) \mid X_1^* > x] \\ &\geqslant m_{X_1}(x)(E[a(Y) \mid X_2^* > x] - E[a(Y) \mid X_1^* > x]) \\ &= m_{X_1}(x)\int_{-\infty}^{\infty} a(y)[g(y \mid X_2^* > x) - g(y \mid X_1^* > x)]\,\mathrm{d}y \\ &\quad \forall\, x \geqslant 0, \end{aligned}$$

where the inequality is due to $X_1 \leqslant_{\rm MRL} X_2$. Since $(Y \mid X_1^* > x) \leqslant_{\rm ST} (\geqslant_{\rm ST})$ $(Y \mid X_2^* > x)$ for all $x \geqslant 0$, we have for all $x \geqslant 0$ that

$$\int_{\nu}^{\infty}\Big(\int_{-\infty}^{\nu}\Big)[g(y \mid X_2^* > x) - g(y \mid X_1^* > x)]\,\mathrm{d}y \geqslant 0 \quad \forall\, \nu \geqslant 0.$$

Now, since $a(y)$ is increasing (decreasing), by an application of Lemma 7.1 of Barlow and Proschan [9], the non-negativity of the integral given in (5.2) is guaranteed, which completes the proof. $\square$

Next, we consider the influence of variation of the mixing variable on the model. Let Y_i be a mixing rv with pdf f_{Y_i} for $i=1,2$. The resulted overall variable X_i^*, $i=1,2$ has sf

$$\overline{F}_i^*(x) = R(x)E[\overline{F}_{X_e}^{1/a(Y_i)}(x)],$$

where $R(x) = \overline{F}_X(x)/\overline{F}_{X_e}(x)$ for all $x \geqslant 0$. Let $g_i(y \mid X_i^* > x)$ denote the pdf of $(Y_i \mid X_i^* > x)$ which is given by

$$g_i(y \mid X_i^* > x) = \frac{\overline{F}_{X_e}^{1/a(y)}(x)f_{Y_i}(y)}{\int_{-\infty}^{\infty}\overline{F}_{X_e}^{1/a(y)}(x)f_{Y_i}(y)\,\mathrm{d}y}.$$

Below, we provide some conditions to make the MRL order between X_1^* and X_2^*.

Theorem 5.2. *Let $a(y)$ be increasing (decreasing) in $y \geqslant 0$ and let*

$$(Y_1 \mid X_1^* > x) \leqslant_{\rm ST} (\geqslant_{\rm ST}) (Y_2 \mid X_2^* > x)$$

for all $x \geqslant 0$. Then $X_1^* \leqslant_{\rm MRL} X_2^*$.

P r o o f. From (3.6), $m_i^*(x) = m_X(x)E[a(Y_i) \mid X_i^* > x]$ for $i=1,2$. Therefore,

$$\begin{aligned} (5.3) \qquad m_2^*(x) - m_1^*(x) &= m_X(x)[E(a(Y_2) \mid X_2^* > x) - E(a(Y_1) \mid X_1^* > x)] \\ &= m_X(x)\int_{-\infty}^{\infty} a(y)[g_2(y \mid X_2^* > x) - g_1(y \mid X_1^* > x)]\,\mathrm{d}y \\ &\quad \forall\, x \geqslant 0. \end{aligned}$$

Now, appealing to the assumptions, as in the proof of Theorem 5.1, Lemma 7.1 of Barlow and Proschan [9] can be applied to (5.3), which ends the proof. □

6. Conclusion

Due to the importance of modeling failure time data in reliability and survival analysis, there is a crucial need to find an appropriate model to fit the data in various practical situations (cf. Chen et al. [4], Zhao and Elsayed [20], Nanda et al. [15], Gupta [5], Kayid and Izadkhah [8] and Kayid et al. [9]). In this investigation, a model called general proportional mean residual life (GPMRL) was studied. This model is an extension of the ordinary proportional mean residual life model initiated firstly by Zahedi [19]. Formally, let $m_X(\cdot)$ be the MRL function in the baseline population and consider a covariate rv Y that has an effect on the population dividing it to some subpopulations. By considering the conditional MRL function arisen in each subpopulations (denoted by $m^*(\cdot \mid y)$) as the product of the baseline MRL m_X and a general function $a(\cdot)$ of Y, the GPMRL model was established. A number of alternative representations for the GPMRL model based on other reliability measures were first given. Then, to determine how the overall variable in the model is affected by the covariate variable, some dependence properties between these two variables are investigated. After that, preservation properties of the model with respect to the HR and the MRL orders are provided. In addition, preservation properties of some aging classes of life distributions under the formation of the model are demonstrated. Finally, in order to make the MRL order between two GPMRL models with either different baseline variables or different mixing variables, two results were obtained.

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References

[1] *R. E. Barlow, F. Proschan*: Statistical Theory of Reliability and Life Testing. International Series in Decision Processes. Holt, Rinehart and Winston, New York, 1975.
[2] *J. Behboodian*: Covariance inequality and its applications. Int. J. Math. Educ. Sci. Technol. *25* (1994), 643–647.
[3] *Y. Q. Chen, S. Cheng*: Semiparametric regression analysis of mean residual life with censored survival data. Biometrika *92* (2005), 19–29.
[4] *Y. Q. Chen, N. P. Jewell, X. Lei, S. C. Cheng*: Semiparametric estimation of proportional mean residual life model in presence of censoring. Biometrics *61* (2005), 170–178.
[5] *R. C. Gupta*: Mean residual life function for additive and multiplicative hazard rate models. Probab. Eng. Inf. Sci. *30* (2016), 281–297.
[6] *R. C. Gupta, S. N. U. A. Kirmani*: On the proportional mean residual life model and its implications. Statistics *32* (1998), 175–187.

[7] *S. Karlin*: Total Positivity. Vol. I. Stanford University Press, Stanford, 1968.
[8] *M. Kayid, S. Izadkhah*: A new extended mixture model of residual lifetime distributions. Oper. Res. Lett. *43* (2015), 183–188.
[9] *M. Kayid, S. Izadkhah, D. ALmufarrej*: Random effect additive mean residual life model. IEEE Trans. Reliab. *65* (2016), 860–866.
[10] *C.-D. Lai, M. Xie*: Stochastic Ageing and Dependence for Reliability. Springer, New York, 2006.
[11] *G. Maguluri, C.-H. Zhang*: Estimation in the mean residual life regression model. J. R. Stat. Soc., Ser. B *56* (1994), 477–489.
[12] *Z. Mansourvar, T. Martinussen, T. H. Scheike*: Semiparametric regression for restricted mean residual life under right censoring. J. Appl. Stat. *42* (2015), 2597–2613.
[13] *A. K. Nanda, S. Bhattacharjee, S. S. Alam*: Properties of proportional mean residual life model. Stat. Probab. Lett. *76* (2006), 880–890.
[14] *A. K. Nanda, S. Bhattacharjee, N. Balakrishnan*: Mean residual life function, associated orderings and properties. IEEE Trans. Reliab. *59* (2010), 55–65.
[15] *A. K. Nanda, S. Das, N. Balakrishnan*: On dynamic proportional mean residual life model. Probab. Eng. Inf. Sci. *27* (2013), 553–588.
[16] *R. B. Nelsen*: An Introduction to Copulas. Springer Series in Statistics, Springer, New York, 2006.
[17] *D. Oakes, T. Dasu*: Inference for the proportional mean residual life model. Crossing Boundaries: Statistical Essays in Honor of J. Hall. IMS Lecture Notes Monogr. Ser. 43, Inst. Math. Statist., Beachwood, 2003, pp. 105–116.
[18] *M. Shaked, J. G. Shanthikumar*: Stochastic Orders. Springer Series in Statistics, Springer, New York, 2007.
[19] *H. Zahedi*: Proportional mean remaining life model. J. Stat. Plann. Inference *29* (1991), 221–228.
[20] *W. Zhao, E. A. Elsayed*: Modelling accelerated life testing based on mean residual life. Int. J. Syst. Sci. *36* (2005), 689–696.

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