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G-MATRICES, J-ORTHOGONAL MATRICES, AND THEIR SIGN PATTERNS

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This paper is dedicated to the memory of Professor Miroslav Fiedler; it was an honor to work with him. He was an exceptionally kind person, a wonderful friend, a tremendous inspiration, and a great mathematician.

Abstract. A real matrix A is a G-matrix if A is nonsingular and there exist nonsingular diagonal matrices D_1 and D_2 such that $A^{-\mathrm{T}} = D_1AD_2$, where $A^{-\mathrm{T}}$ denotes the transpose of the inverse of A. Denote by $J = \operatorname{diag}(\pm 1)$ a diagonal (signature) matrix, each of whose diagonal entries is $+1$ or -1. A nonsingular real matrix Q is called J-orthogonal if $Q^{\mathrm{T}}JQ = J$. Many connections are established between these matrices. In particular, a matrix A is a G-matrix if and only if A is diagonally (with positive diagonals) equivalent to a column permutation of a J-orthogonal matrix. An investigation into the sign patterns of the J-orthogonal matrices is initiated. It is observed that the sign patterns of the G-matrices are exactly the column permutations of the sign patterns of the J-orthogonal matrices. Some interesting constructions of certain J-orthogonal matrices are exhibited. It is shown that every symmetric staircase sign pattern matrix allows a J-orthogonal matrix. Sign potentially J-orthogonal conditions are also considered. Some examples and open questions are provided.

Keywords: G-matrix; J-orthogonal matrix; Cauchy matrix; sign pattern matrix

MSC 2010: 15A80, 15A15, 15A23

1. Introduction

In [9], a new type of matrix was introduced and studied. A real matrix A is a G-*matrix* if A is nonsingular and there exist nonsingular diagonal matrices D_1 and D_2

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such that

$$A^{-\mathrm{T}} = D_1 A D_2, \tag{1.1}$$

where $A^{-\mathrm{T}}$ denotes the transpose of the inverse of A. Denote by $J = \operatorname{diag}(\pm 1)$ a diagonal (signature) matrix, each of whose diagonal entries is $+1$ or -1. As in [12], a nonsingular real matrix Q is called *J-orthogonal* if

$$Q^{\mathrm{T}} J Q = J, \tag{1.2}$$

or equivalently, if

$$Q^{-\mathrm{T}} = JQJ. \tag{1.3}$$

A *(J_1, J_2)-orthogonal matrix* is defined as a nonsingular real matrix Q such that

$$Q^{\mathrm{T}} J_1 Q = J_2, \tag{1.4}$$

where $J_1 = \operatorname{diag}(\pm 1)$ and $J_2 = \operatorname{diag}(\pm 1)$ are signature matrices having the same inertia [12]. J-orthogonal matrices were studied for example in the context of the group theory [4] or generalized eigenvalue problems [5]. Numerical properties of several orthogonalization techniques with respect to symmetric indefinite bilinear forms have been analyzed recently in [13]. Although J-orthogonality has many numerical connections, this particular paper has more of a combinatorial matrix theory point of view.

In Section 2 we lay the foundation of the paper. We show that a matrix A is a G-matrix if and only if A is diagonally (with positive diagonals) equivalent to a (J_1, J_2)-orthogonal matrix. Hence, as we shall see, a matrix A is a G-matrix if and only if A is diagonally (with positive diagonals) equivalent to a column permutation of a J-orthogonal matrix.

In Section 3 we review sign pattern matrices and recall from [9] some results on the sign patterns of the G-matrices and the sign patterns of the nonsingular Cauchy (generalized Cauchy) matrices. Section 4 is concerned with the connection of the sign patterns of the G-matrices and the sign patterns of the J-orthogonal matrices. In particular, we observe that the sign patterns of the G-matrices are exactly the column permutations of the sign patterns of the J-orthogonal matrices.

In Section 5 we give some interesting constructions of certain J-orthogonal matrices. We also show that every symmetric staircase sign pattern matrix allows a J-orthogonal matrix and we discuss the situation for nonsymmetric staircase sign patterns. Further considerations are made in Section 6, including sign potentially

J-orthogonal conditions. This paper particularly begins an exploration of the sign patterns of the J-orthogonal matrices.

2. G-matrices and J-orthogonal matrices

It was shown in [9] that G-matrices enjoy interesting properties and that many well known special matrices are G-matrices. Two very basic, but useful, properties are the following:

If A is an $n \times n$ G-matrix and D is an $n \times n$ nonsingular diagonal matrix, then both AD and DA are G-matrices, see [9], Theorem 2.4.

If A is an $n \times n$ G-matrix and P is an $n \times n$ permutation matrix, then both AP and PA are G-matrices, see [9], Theorem 2.5.

Obviously, for any nonsingular diagonal matrix A of order at least 2, the matrices D_1 and D_2 are not unique up to scalar multiples. However, it follows from Sylvester's law of inertia that D_1 and D_2 in (1) always have the same inertia, and thus have the same number of positive entries. We now establish several interesting structural properties of G-matrices and characterize the G-matrices A for which the matrices D_1 and D_2 in (1) are unique up to scalar multiples. For the notion of fully indecomposable matrices, we refer the reader to [2].

Theorem 2.1. *Let A be a nonsingular real matrix in block upper triangular form*

$$A = \begin{bmatrix} A_{11} & \dots & A_{1m} \\ & \ddots & \vdots \\ 0 & & A_{mm} \end{bmatrix},$$

where all the diagonal blocks are square. Then A is a G*-matrix if and only if each A_{ii}, $i = 1, \dots, m$, is a* G*-matrix and all the strictly upper triangular blocks A_{ij} are equal to* 0*. Furthermore, if A is a* G*-matrix that has a row (or a column) with no* 0 *entry, then A is fully indecomposable.*

Proof. Assume that A satisfies $A^{-\mathrm{T}} = D_1 A D_2$. Note that $D_1 A D_2$ is block upper triangular and the conformally partitioned $A^{-\mathrm{T}}$ is block lower triangular. It follows that the strictly upper triangular blocks of A are equal to 0. The rest is clear. □

We now characterize those G-matrices A for which the matrices D_1 and D_2 in (1) are unique up to scalar multiples.

Theorem 2.2. *Let A be a fully indecomposable* G*-matrix. Then the diagonal matrices D_1 and D_2 satisfying $A^{-\mathrm{T}} = D_1 A D_2$ are unique up to scalar multiples.*

P r o o f. Replacing A with a matrix permutationally equivalent to A if necessary, without loss of generality, we may assume that all the diagonal entries of A are nonzero. Write $D_1 = \operatorname{diag}(x_1, \ldots, x_n)$ and $D_2 = \operatorname{diag}(y_1, \ldots, y_n)$. It follows that x_i and y_i determine each other, for each $i = 1, \ldots, n$. Since A is fully indecomposable, we know that A is irreducible. Hence, for each $i \neq j$, the presence of a directed path from i to j in the directed graph of A [2] shows that x_i and y_j determine each other. If we assume that the $(1,1)$-entry of D_1 is 1, then all the entries of D_1 and D_2 are uniquely determined. Therefore, D_1 and D_2 are unique up to scalar multiples. $\square$

The following result is then clear.

Theorem 2.3. *Let A be a* G*-matrix such that $A = A_1 \oplus \ldots \oplus A_m$, where A_i is fully indecomposable and is of order n_i, $i = 1, \ldots, m$. Suppose that $\widehat{D}_1$ and $\widehat{D}_2$ are two diagonal matrices satisfying $A^{-\mathrm{T}} = \widehat{D}_1 A \widehat{D}_2$. Then all the D_1 and D_2 satisfying $A^{-\mathrm{T}} = D_1 A D_2$ are given by $D_1 = (c_1 I_{n_1} \oplus \ldots \oplus c_m I_{n_m})\widehat{D}_1$ and $D_2 = (c_1^{-1} I_{n_1} \oplus \ldots \oplus c_m^{-1} I_{n_m})\widehat{D}_2$, where $c_1, \ldots, c_m$ are arbitrary nonzero real numbers.*

As is well known, Cauchy matrices are matrices of the form $C = [c_{ij}]$, where $c_{ij} = 1/(x_i + y_j)$ for some numbers x_i and y_j. We shall restrict ourselves to square, say $n \times n$, Cauchy matrices. Of course, such matrices are defined only if $x_i + y_j \neq 0$ for all pairs of indices i, j, and it is well known that C is nonsingular if and only if all the numbers x_i are mutually distinct and all the numbers y_j are mutually distinct. By Observation 1 in [7], every nonsingular Cauchy matrix is a G-matrix.

For *generalized Cauchy matrices* of order n, additional parameters $u_1, \ldots, u_n$, $v_1, \ldots, v_n$ are considered:

$$\widehat{C} = \Big(\frac{u_i v_j}{x_i + y_j}\Big).$$

Note that then $\widehat{C} = D_1 C D_2$, where $D_1 = \operatorname{diag}(u_i)$, $D_2 = \operatorname{diag}(v_j)$, so that $\widehat{C}$ is a G-matrix.

As mentioned in the introduction, in the recent decades and particularly in numerical mathematics, the class of problems appeared where the scalar products were indefinite, see for example [12], [4], [5] or [13].

Of course, every orthogonal matrix is a J-orthogonal matrix, where J is the identity matrix of the same order as Q. And clearly, from (1.3), every J-orthogonal matrix is a G-matrix. On the other hand, a G-matrix can always be transformed to a J-orthogonal matrix.

Definition 2.4. We say that two real matrices A and B are *positive-diagonally equivalent* if there are diagonal matrices D_1 and D_2 with all diagonal entries positive such that $B = D_1 A D_2$.

Theorem 2.5. *A matrix A is a* G*-matrix if and only if A is positive-diagonally equivalent to a (J_1, J_2)-orthogonal matrix.*

Proof. Let A be a G-matrix, i.e., $A^{-\mathrm{T}} = D_1 A D_2$ for some nonsingular diagonal matrices D_1 and D_2. Consequently, $A^{\mathrm{T}} D_1 A = D_2^{-1}$. Write $D_1 = |D_1|^{1/2} J_1 |D_1|^{1/2}$ and $D_2^{-1} = |D_2|^{-1/2} J_2 |D_2|^{-1/2}$. Thus,

$$(|D_1|^{1/2} A)^{\mathrm{T}} J_1 (|D_1|^{1/2} A) = |D_2|^{-1/2} J_2 |D_2|^{-1/2},$$

which can be written as

$$(|D_1|^{1/2} A |D_2|^{1/2})^{\mathrm{T}} J_1 (|D_1|^{1/2} A |D_2|^{1/2}) = J_2.$$

For $Q = |D_1|^{1/2} A |D_2|^{1/2}$, this is $Q^{\mathrm{T}} J_1 Q = J_2$, so that Q is (J_1, J_2)-orthogonal. Note that due to $A = |D_1|^{-1/2} Q |D_2|^{-1/2}$, A is positive-diagonally equivalent to a (J_1, J_2)-orthogonal matrix.

Conversely, if Q is (J_1, J_2)-orthogonal, then it is a G-matrix and any positive-diagonally equivalent matrix is a G-matrix as well. □

We now have the following.

Theorem 2.6. *A matrix A is a* G*-matrix if and only if A is positive-diagonally equivalent to a column permutation of a J-orthogonal matrix.*

Proof. As mentioned in [12], the matrices J_1 and J_2 in (1.4) have the same inertia, so that $J_2 = P J_1 P^{\mathrm{T}}$ for some permutation matrix P, and hence $(QP)^{\mathrm{T}} J_1 (QP) = J_1$. It follows that the (J_1, J_2)-orthogonal matrices are the column permutations of the J_1-orthogonal matrices. Considering this and Theorem 2.5 we get the statement of our theorem. □

3. Sign pattern matrices

In qualitative and combinatorial matrix theory, we study properties of a matrix based on combinatorial information, such as the sign of entries in the matrix. An $m \times n$ matrix whose entries are from the set $\{+, -, 0\}$ is called a *sign pattern matrix* (or sign pattern). For a real matrix B, $\mathrm{sgn}(B)$ is the sign pattern matrix obtained by replacing each positive, negative, or zero entry of B, respectively, by $+$, $-$, or 0. For a sign pattern matrix A, the *sign pattern class of* A is defined by

$$Q(A) = \{B \colon \mathrm{sgn}(B) = A\}.$$

We denote the set of $n \times n$ sign pattern matrices by Q_n.

A sign pattern matrix P is called a *permutation sign pattern* (*generalized permutation sign pattern*) if exactly one entry in each row and column is equal to + (+ or −) and all the other entries are 0. A *permutation similarity* of the $n \times n$ sign pattern A has the form $P^{\mathrm{T}}AP$, where P is an $n \times n$ permutation matrix. A *signature pattern* is a diagonal sign pattern matrix, each of whose diagonal entries is + or −. A sign pattern B is *signature equivalent* to the sign pattern A provided $B = S_1AS_2$, where S_1 and S_2 are signature patterns. A *signature similarity* of the $n \times n$ sign pattern A has the form SAS, where S is an $n \times n$ signature pattern.

Suppose P is a property referring to a real matrix. A sign pattern A is said to *require* P if every matrix in $Q(A)$ has property P; A is said to *allow* P if some real matrix in $Q(A)$ has property P. The reader is referred to [3] or [11] for more information on sign pattern matrices.

As in [9], we let $\mathcal{G}_n$ denote the class of all $n \times n$ sign pattern matrices A that allow a G-matrix, that is, there exists a nonsingular matrix $B \in Q(A)$ such that $B^{-\mathrm{T}} = D_1BD_2$ for some nonsingular diagonal matrices D_1 and D_2. The following assertion is Theorem 3.1 of [9]: The class $\mathcal{G}_n$ is closed under

(i) multiplication (on either side) by a permutation pattern, and
(ii) multiplication (on either side) by a signature pattern.

The use of these operations in $\mathcal{G}_n$ then produces "equivalent" sign patterns.

Also as in [9], next let $\mathcal{C}_n$ ($\mathcal{GC}_n$) be the class of all sign patterns of the $n \times n$ nonsingular Cauchy (generalized Cauchy) matrices. It should be clear that $\mathcal{C}_n$ ($\mathcal{GC}_n$) is closed under operation (i) (operations (i) and (ii)) above. The classes $\mathcal{C}_n$ and $\mathcal{GC}_n$ are two particular sub-classes of $\mathcal{G}_n$.

The class $\mathcal{C}_n$ is the same as the class of $n \times n$ sign patterns permutation equivalent to a sign pattern of the form

$$\begin{pmatrix} + & + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + & - \\ + & + & + & + & + & + & + & + & - \\ + & + & + & + & + & + & + & - & - \\ + & + & + & + & + & + & - & - & - \\ + & + & + & - & - & - & - & - & - \\ + & + & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - \end{pmatrix},$$

where the part above (below) the staircase is all + (−) [9], Theorem 3.2. In this form, whenever there is a minus, then to the right and below there are also minuses. Note that this form includes the all + and all − patterns.

4. G-matrix/J-orthogonal matrix sign patterns

Of course, when $J = I_n$, a J-orthogonal matrix is an orthogonal matrix. An old question raised by M. Fiedler in 1964, [8], is the following: what are the sign patterns which allow an orthogonal matrix? Since that time, much research has been done on these sign patterns, [11]. Letting $\mathcal{PO}_n$ denote the class of $n \times n$ sign patterns that allow an orthogonal matrix, we give the connection with G-matrices.

Proposition 4.1. *An $n \times n$ sign pattern A allows a* G*-matrix with associated diagonal matrices having positive diagonal entries if and only if $A \in \mathcal{PO}_n$.*

Proof. Let A be an $n \times n$ sign pattern. Suppose there exist a nonsingular matrix $B \in Q(A)$ and nonsingular diagonal matrices D_1, D_2 with $+$ diagonal entries such that $B^{-\mathrm{T}} = D_1 B D_2$. Let $E_1 = {D_1}^{1/2}, E_2 = {D_2}^{1/2}$. Then

$$(E_1 B E_2)^{-1} = (E_1 B E_2)^{\mathrm{T}}.$$

So, $E_1 B E_2$ is an orthogonal matrix in $Q(A)$. Conversely, if C is an orthogonal matrix in $Q(A)$, then $C^{-\mathrm{T}} = C = I_n C I_n$. □

Remark 4.2. In [6] the class $\mathcal{T}_n$ of all $n \times n$ sign patterns A for which there exists a nonsingular matrix $B \in Q(A)$ where $B^{-1} \in Q(A^{\mathrm{T}})$ was studied. There it was asked if the class $\mathcal{T}_n$ is the same as the subclass $\mathcal{PO}_n$. This question is still unanswered.

A more general question than characterizing $\mathcal{PO}_n$ is the following: what are the sign patterns which allow a J-orthogonal matrix? Specifically, it is of interest to find sign patterns which allow a J-orthogonal matrix, but do not allow an orthogonal matrix. We shall let $\mathcal{J}_n$ denote the class of all sign patterns of the $n \times n$ J-orthogonal matrices, that is, the class of $n \times n$ sign patterns that allow a J-orthogonal matrix.

From Theorem 2.6 we immediately have the following connection with G-matrices.

Theorem 4.3. *The sign patterns of the $n \times n$* G*-matrices are exactly the column permutations of the sign patterns in $\mathcal{J}_n$.*

Now, the all $+$ (also, all $-$) $n \times n$ sign pattern is the sign pattern of a nonsingular Cauchy matrix, which is a G-matrix. Thus:

Theorem 4.4. *The all $+$ (also, all $-$) $n \times n$ sign pattern allows a J-orthogonal matrix (but of course not an orthogonal matrix, unless $n = 1$).*

Remark 4.5. In general, every sign pattern in $\mathcal{C}_n$ ($\mathcal{GC}_n$) is the sign pattern of a nonsingular Cauchy (generalized Cauchy) matrix, which is a G-matrix. So, every such sign pattern is a column permutation of a sign pattern in $\mathcal{J}_n$. This implicitly provides many sign patterns that allow a J-orthogonal matrix, but not an orthogonal matrix.

Finally in this section, we digress to the (J_1, J_2)-orthogonal matrices and utilize Theorem 2.5.

Theorem 4.6. *The sign patterns of the* G*-matrices are the same as the sign patterns of the* (J_1, J_2)*-orthogonal matrices.*

In particular, we have the following.

Corollary 4.7. *If* $A \in \mathcal{GC}_n$ *(in particular, if* $A \in \mathcal{C}_n$*), then* A *allows a* (J_1, J_2)*-orthogonal matrix.*

From [9] we know that every 2×2 $(+,-)$ sign pattern is a matrix in $\mathcal{GC}_2$ and that every 3×3 $(+,-)$ sign pattern is a matrix in $\mathcal{GC}_3$. Hence:

Corollary 4.8. *For* $n \leqslant 3$*, every* $n \times n$ $(+,-)$ *sign pattern allows a* (J_1, J_2)*-orthogonal matrix.*

5. Construction of certain J-orthogonal matrices

It follows from Theorem 4.4 that there exists a 2×2 matrix with all $+$ sign pattern that is a J-orthogonal matrix. It is also clear that the all $+$ sign pattern does not allow an orthogonal matrix with respect to the standard inner product, where $J = I_2$. For example, the symmetric matrix $Q_2 = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$ is J-orthogonal with respect to the matrix $J_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ due to

$$Q_2^{\mathrm{T}} J_2 Q_2 = Q_2 J_2 Q_2 = J_2,$$

but there is no 2×2 matrix Q_2 with all positive entries that satisfies $Q_2^{\mathrm{T}} Q_2 = I_2$. We arrive at the following result.

Theorem 5.1. *If we take the* 2×2 *sign pattern matrix* $A_2 = \begin{pmatrix} + & + \\ + & + \end{pmatrix}$ *and for each* $n = 1, 2, \ldots$ *define recursively the* $2^{n+1} \times 2^{n+1}$ *sign pattern matrix*

$$A_{2^{n+1}} = \begin{pmatrix} A_{2^n} & -A_{2^n} \\ -A_{2^n} & A_{2^n} \end{pmatrix} \tag{5.1}$$

then each sign pattern matrix A_{2^n} *allows a* J*-orthogonal matrix and does not allow an orthogonal matrix.*

P r o o f. As was already pointed out the statement is true for $n = 1$. Inductively, if there exists a $2^n \times 2^n$ matrix Q_{2^n} such that $Q_{2^n}^{\mathrm{T}} J_{2^n} Q_{2^n} = J_{2^n}$ and we define

$$Q_{2^{n+1}} = \begin{pmatrix} \sqrt{2}Q_{2^n} & -Q_{2^n} \\ -Q_{2^n} & \sqrt{2}Q_{2^n} \end{pmatrix}$$

then $Q_{2^{n+1}}^{\mathrm{T}} J_{2^{n+1}} Q_{2^{n+1}} = J_{2^{n+1}}$, i.e. the matrix $Q_{2^{n+1}}$ is J-orthogonal with respect to the matrix $J_{2^{n+1}}$ given as

$$J_{2^{n+1}} = \begin{pmatrix} J_{2^n} & 0 \\ 0 & -J_{2^n} \end{pmatrix}.$$

It is also clear from the definition that $\operatorname{sgn}(Q_{2^{n+1}}) = A_{2^{n+1}}$. Moreover, the sign pattern $A_{2^{n+1}}$ does not allow orthogonality since its first two rows or columns are equal. $\square$

Remark 5.2. Note that $Q_2 = \begin{pmatrix} \sqrt{2} & -1 \\ -1 & \sqrt{2} \end{pmatrix}$ is also J-orthogonal with respect to the matrix $J_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We could alternatively take the matrix A_2 as $A_2 = \begin{pmatrix} + & - \\ - & + \end{pmatrix}$ as the starting point in Theorem 5.1. Then, the sign pattern A_2 also allows a J-orthogonal matrix, but does not allow an orthogonal matrix. The sign pattern matrices $A_{2^{n+1}}$ can still be defined as in (5.1). The proof of Theorem 5.1 works in the same way and we generate a different sequence of sign patterns that allow J-orthogonality but not orthogonality.

We now return to the staircase patterns.

Theorem 5.3. *Each symmetric staircase sign pattern matrix allows a J-orthogonal matrix.*

P r o o f. Let us recall that each symmetric staircase sign pattern matrix A corresponds to the symmetric Cauchy matrix $C = [c_{ij}]$ with $c_{ij} = 1/(x_i + x_j)$, where the numbers x_i are ordered so that $x_1 > x_2 > \ldots > x_n$. It follows then from [10] that if we define the diagonal matrix D as $D = \operatorname{diag}(d_i)$ with

$$d_i = 2x_i \prod_{k \neq i} \frac{x_i + x_k}{x_i - x_k}, \tag{5.2}$$

then indeed $C^{-\mathrm{T}} = DCD$. If we write $D = |D| \operatorname{diag}(\operatorname{sign}(d_i))$, then the matrix $Q = |D|^{1/2} C |D|^{1/2}$ is J-orthogonal with respect to the matrix $J = \operatorname{diag}(\operatorname{sign}(d_i))$ satisfying $Q^{\mathrm{T}} J Q = QJQ = J$. It is clear from the construction that the sign pattern of Q coincides with the sign pattern A as $\operatorname{sgn}(Q) = \operatorname{sgn}(C) = A$. $\square$

Remark 5.4. The i-th diagonal entry of J defined in Theorem 5.3 is actually equal to the sign of d_i from (5.2). If we denote by m_i the number of negative signs in the i-th row of the staircase sign pattern matrix corresponding to the numbers $x_1 > x_2 > \ldots > x_n$, then $m_n \geqslant \ldots \geqslant m_2 \geqslant m_1 \geqslant 0$. It is clear that $x_i + x_k < 0$ for $k = n - m_i + 1, \ldots, n$ and $x_i - x_k < 0$ for $k = 1, \ldots, i-1$. Taking into account all negative terms in (5.2) we get that the sign of d_i is equal to $(-1)^{m_i+i-1}$ for $i = 1, \ldots, n$.

Remark 5.5. The all $+$ $n \times n$ sign pattern corresponds to the situation with $m_i = 0$ for $i = 1, \ldots, n$. Consequently, the signs in J alternate according to $(-1)^{i-1}$ for $i = 1, \ldots, n$. The all $-$ $n \times n$ sign pattern corresponds to the situation with $m_i = n$ for $i = 1, \ldots, n$. Consequently, the signs in J alternate according to $(-1)^{n+i-1}$ for $i = 1, \ldots, n$.

Example 5.6. The symmetric sign patterns

$$\begin{pmatrix} + & + & + & + \\ + & - & - & - \\ + & - & - & - \\ + & - & - & - \end{pmatrix}, \begin{pmatrix} + & + & + & + \\ + & + & + & + \\ + & + & - & - \\ + & + & - & - \end{pmatrix}, \begin{pmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & - \end{pmatrix},$$
$$\begin{pmatrix} + & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix}, \begin{pmatrix} + & + & - & - \\ + & + & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix}, \begin{pmatrix} + & + & + & - \\ + & + & + & - \\ + & + & + & - \\ - & - & - & - \end{pmatrix}$$

allow J-orthogonal matrices but do not allow orthogonal matrices.

Remark 5.7. One can consider also nonsymmetric staircase sign patterns. Let us recall that each nonsymmetric staircase sign pattern matrix A corresponds to the (nonsymmetric) Cauchy matrix $C = [c_{ij}]$ with $c_{ij} = 1/(x_i + y_j)$, where the numbers x_i and y_j are ordered so that $x_1 > x_2 > \ldots > x_n > 0$ and $y_1 > y_2 > \ldots > y_n$. It follows then from [10] that if we define the diagonal matrices D_1 and D_2 as $D_1 = \operatorname{diag}(u_i)$ and $D_2 = \operatorname{diag}(v_j)$ with

$$u_i = (x_i + y_i) \prod_{k \neq i} \frac{x_i + y_k}{x_i - x_k}, \quad v_j = (x_j + y_j) \prod_{k \neq j} \frac{y_j + x_k}{y_j - y_k}, \tag{5.3}$$

then indeed $C^{-\mathrm{T}} = D_1 C D_2$. If we write $D_1 = |D_1| J_1$ and $D_2 = |D_2| J_2$ where $J_1 = \operatorname{diag}(\operatorname{sign}(u_i))$ and $J_2 = \operatorname{diag}(\operatorname{sign}(v_j))$ there exists a permutation matrix P such that it provides the transformation $J_2 = P J_1 P^{\mathrm{T}}$. Then the matrix $Q = |D_1|^{1/2} C |D_2|^{1/2} P$ is J-orthogonal with respect to J_1 satisfying $Q^{\mathrm{T}} J_1 Q = J_1$. The sign pattern of Q is equal to a column permutation of the sign pattern A as $\operatorname{sgn}(Q) = \operatorname{sgn}(CP) = AP$.

It is easy to see from the construction that the i-th diagonal entry of J_1 is actually equal to the sign of u_i from (5.3). If we denote by m_i the number of negative signs in the i-th row of the staircase sign pattern matrix corresponding to the numbers $x_1 > x_2 > \ldots > x_n > 0$ and $y_1 > y_2 > \ldots > y_n$, then $m_n \geqslant \ldots \geqslant m_2 \geqslant m_1 \geqslant 0$. It is clear that $x_i + y_k < 0$ for $k = n - m_i + 1, \ldots, n$ and $x_i - x_k < 0$ for $k = 1, \ldots, i-1$. Taking into account all negative terms in (5.3) we get that the sign of u_i is equal to $(-1)^{m_i+i-1}$ for $i = 1, \ldots, n$. Similarly the j-th diagonal entry of J_2 is equal to the sign of v_j from (5.3). If we denote by n_j the number of negative signs in the j-th columns of the staircase sign pattern matrix corresponding to the numbers $x_1 > x_2 > \ldots > x_n > 0$ and $y_1 > y_2 > \ldots > y_n$, then $n_n \geqslant \ldots \geqslant n_2 \geqslant n_1 \geqslant 0$. It is clear that $y_j + x_k < 0$ for $k = n - n_j + 1, \ldots, n$ and $y_j - y_k < 0$ for $k = 1, \ldots, j-1$. Taking into account all negative terms in (5.3) we get that the sign of u_j is equal to $(-1)^{n_j+j-1}$ for $j = 1, \ldots, n$.

Example 5.8. Note that there exist also nonsymmetric staircase sign patterns such as

$$\begin{pmatrix} + & + & - & - \\ + & + & - & - \\ + & + & - & - \\ + & + & - & - \end{pmatrix}, \begin{pmatrix} + & + & + & + \\ + & + & + & + \\ - & - & - & - \\ - & - & - & - \end{pmatrix}$$

that allow J-orthogonal matrices but do not allow orthogonal matrices. In such cases we have $(m_i - n_i) \bmod 2 = 0$ and this leads to $J_2 = J_1$ in Remark 5.7 with the permutation matrix P equal to the identity $P = I_4$.

Example 5.9. Note that the nonsymmetric staircase sign patterns

$$\begin{pmatrix} + & + & + & + \\ + & + & + & - \\ + & + & + & - \\ + & + & + & - \end{pmatrix}, \begin{pmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & - & - & - \end{pmatrix},$$
$$\begin{pmatrix} + & - & - & - \\ + & - & - & - \\ + & - & - & - \\ - & - & - & - \end{pmatrix}, \begin{pmatrix} + & + & + & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix}$$

also allow J-orthogonal matrices but do not allow orthogonal matrices. The situation is more complicated for these four sign patterns as $P = [e_1, e_3, e_2, e_4] \neq I_4$ but we still have $\mathrm{sgn}(Q) = \mathrm{sgn}(CP) = AP = A$.

6. Sign potentially J-orthogonal conditions

First, we develop some conditions for J-orthogonal matrices which extend the sign potentially orthogonal (SPO) conditions. As in [6], we use the symbol # to denote an *ambiguous* quantity, namely, $\# = (+) + (-)$. We define a *generalized sign pattern matrix* $A = (a_{ij})$ as a $(+, -, 0, \#)$ matrix, and the sign pattern class of such an $n \times n$ matrix is given by

$$Q(A) = \{B = (b_{ij}) \in M_n(\mathbb{R})\colon\ a_{ij} = \# \text{ or } a_{ij} = \operatorname{sgn}(b_{ij})\}.$$

Note that every sign pattern matrix is also a generalized sign pattern matrix. We denote the set of $n \times n$ generalized sign pattern matrices by $\overline{Q}_n$. We say two patterns $A, A' \in \overline{Q}_n$ are *compatible* if, for all $i, j \in \{1, 2, \ldots, n\}$, either $a_{ij} = a'_{ij}$, or one of a_{ij} and a'_{ij} is #. Equivalently, A and A' are compatible if and only if $Q(A) \cap Q(A') \neq \emptyset$. We write $A \overset{c}{\leftrightarrow} A'$ when A and A' are compatible. For example,

$$\begin{pmatrix} \# & 0 \\ + & - \end{pmatrix} \overset{c}{\leftrightarrow} \begin{pmatrix} - & \# \\ + & \# \end{pmatrix}.$$

Let A be an $n \times n$ sign pattern matrix. If $A \in \mathcal{J}_n$, then there exists $B \in Q(A)$ such that

$$\begin{aligned} B^{\mathrm{T}} J B &= J, \\ (B^{\mathrm{T}} J)(BJ) &= I, \\ (BJ)(B^{\mathrm{T}} J) &= I, \\ BJB^{\mathrm{T}} &= J. \end{aligned}$$

With a slight abuse of notation, we will identify J with $\operatorname{sgn}(J)$. Thus the *sign potentially J-orthogonal* (SPJO) conditions are that

$$A^{\mathrm{T}} J A \overset{c}{\leftrightarrow} J$$

and

$$AJA^{\mathrm{T}} \overset{c}{\leftrightarrow} J$$

for some $(+, -)$ signature pattern J.

These are necessary conditions for $A \in \mathcal{J}_n$. If these conditions do not hold, then $A \notin \mathcal{J}_n$. When $J = I$, we get the normal SPO conditions for orthogonal matrices, see for example [6]. The SPJO conditions are not sufficient for an $n \times n$ sign pattern matrix to allow J-orthogonality.

Example 6.1. Let

$$A = \begin{pmatrix} + & + & 0 & 0 \\ + & + & 0 & 0 \\ + & + & + & + \\ + & + & + & + \end{pmatrix}.$$

It is easily checked that A satisfies the SPJO conditions with $J = \operatorname{diag}(+,-,+,-)$. Other signature patterns also work, such as $J = \operatorname{diag}(+,+,+,-)$. However, suppose that A allows a J-orthogonal matrix. Since every J-orthogonal matrix is a G-matrix, A then allows a G-matrix. But then by Theorem 2.1, A would have to be block-diagonal, which is a contradiction. Thus, $A \notin \mathcal{J}_4$.

For sign vectors $c, x \in \{+,-,0\}^n$, we have that $c^{\mathrm{T}}x \overset{c}{\leftrightarrow} 0$ if at least one of the following holds:

(1) for each i, we have $c_i = 0$ or $x_i = 0$, or
(2) there are indices i, j with $c_i = x_i \neq 0$ and $c_j = -x_j \neq 0$.

For a set of sign vectors $S \subseteq \{+,-,0\}^n$, the *orthogonal complement* of S is

$$S^{\perp} = \{c \in \{+,-,0\}^n \colon\ c^{\mathrm{T}}x \overset{c}{\leftrightarrow} 0 \text{ for all } x \in S\}.$$

Specifically, if $c, x \in \{+,-\}^n$, we have only the second condition.

Theorem 6.2. *If A is an $n \times n$ $(+,-)$ sign pattern matrix and $n \geqslant 6$, then A satisfies the SPJO conditions.*

Proof. Let $A = (a_{ij})$ be an $n \times n$ $(+,-)$ sign pattern matrix. We need to show that there exists a $(+,-)$ signature pattern J such that

$$A^{\mathrm{T}}JA \overset{c}{\leftrightarrow} J \tag{6.1}$$

and

$$AJA^{\mathrm{T}} \overset{c}{\leftrightarrow} J. \tag{6.2}$$

Observe that $A^{\mathrm{T}}JA$ and AJA^{T} are symmetric generalized sign pattern matrices. So, we need only to find a J which fulfils the upper-triangular part of the compatible conditions.

Let $J = \operatorname{diag}(\omega_1, \ldots, \omega_n)$. Note that (6.1) and (6.2) may be restated as

$$\sum_{k=1}^{n} \omega_k a_{ki} a_{kj} \overset{c}{\leftrightarrow} \delta_{ij}\omega_j \quad \forall\, i, j \tag{6.3}$$

and

$$\sum_{k=1}^{n}\omega_k a_{ik}a_{jk} \overset{c}{\leftrightarrow} \delta_{ij}\omega_j \quad \forall\, i,j. \tag{6.4}$$

Then, for $i = j$, (6.3) and (6.4) automatically hold for any J. For the $i < j$ positions, (6.3) and (6.4) each yield $n(n-1)/2$ linear expressions in J. Letting $v = (\omega_1, \ldots, \omega_n)^{\mathrm{T}}$, we have

$$C_1 v \overset{c}{\leftrightarrow} 0$$

and

$$C_2 v \overset{c}{\leftrightarrow} 0$$

to solve simultaneously, where C_1 and C_2 are $n(n-1)/2 \times n$ $(+,-)$ sign patterns. Let S be the set of rows of C_1 together with the set of rows of C_2. Let S' be $S \cup (-S)$. To find a possible J, we choose a $(+,-)$ n-vector v such that $v \notin S'$. For $n \geqslant 6$, $2n(n-1) < 2^n$, so that such a choice of v is always possible. Then for any $c \in S'$, v will be different from c in at least one component and different from $-c$ in at least one component. Hence, $c^{\mathrm{T}} v \overset{c}{\leftrightarrow} 0$, i.e., $v \in (S')^{\perp}$. Letting $J = \operatorname{diag}(v)$, we have a signature pattern that fulfils (6.1) and (6.2). □

If we allow zero entries, then Theorem 6.2 may fail. For example, an $n \times n$ sign pattern A with a zero column does not satisfy $A^{\mathrm{T}} J A \overset{c}{\leftrightarrow} J$ and an $n \times n$ sign pattern A with a zero row does not satisfy $A J A^{\mathrm{T}} \overset{c}{\leftrightarrow} J$, for any signature pattern J.

The following is straightforward.

Lemma 6.3. *The class $\mathcal{J}_n$ is closed under the following operations:*

i) *negation;*
ii) *transposition;*
iii) *permutation similarity;*
iv) *multiplication (on either side) by a signature pattern;*
v) *signature equivalence.*

The use of these operations yields "equivalent" sign patterns. We now investigate the question of whether the $(+,-)$ $n \times n$ sign patterns always allow a J-orthogonal matrix.

Remark 6.4. It was observed in [6] that for $n \leqslant 4$, the SPO patterns are the same as the sign patterns in $\mathcal{PO}_n$, and that this is also the case for $(+,-)$ sign patterns of order 5, see [1] and [14]. So, regarding the above question with $n \leqslant 5$, we need only to consider non-SPO patterns.

By what we have previously done, all the $(+,-)$ sign patterns of orders 1 or 2 allow a J-orthogonal matrix. By Theorem 5.3, every symmetric staircase pattern allows a J-orthogonal matrix. By Remark 6.4, for $n \leqslant 5$, every $n \times n$ $(+,-)$ SPO sign pattern allows orthogonality. If A is a 3×3 $(+,-)$ sign pattern, by signature multiplications, A is equivalent to a sign pattern of the form

$$\begin{pmatrix} + & + & + \\ + & \multicolumn{2}{c}{\multirow{2}{*}{A_1}} \\ + & & \end{pmatrix}$$

where A_1 is a 2×2 $(+,-)$ sign pattern. By analyzing the 16 choices for A_1, it can be seen that A is equivalent to at least one of the following: a symmetric staircase pattern; a SPO pattern; the pattern

$$\widehat{A} = \begin{pmatrix} + & + & + \\ + & + & + \\ + & - & - \end{pmatrix}.$$

By Remark 5.7 it follows that this nonsymmetric staircase pattern allows a J-orthogonal matrix since the counts $m_1 = m_2 = 0$, $m_3 = 2$ and $n_1 = 0$, $n_2 = n_3 = 1$ lead to $P = [e_1, e_3, e_2]$ that satisfies $\widehat{A} = \widehat{A}P$, similarly to Example 5.9. So, $A \in \mathcal{J}_3$. We arrive at the following result.

Proposition 6.5. *If A is a 3×3 $(+,-)$ sign pattern, then $A \in \mathcal{J}_3$.*

This result improves Corollary 4.8. In fact, given a 3×3 $(+,-)$ sign pattern A, we can easily enough construct $B \in Q(A)$ that is J-orthogonal.

Example 6.6. If

$$A = \begin{pmatrix} + & - & - \\ + & + & - \\ + & - & - \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & -1 \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}, \quad J = \operatorname{diag}(1, 1, -1),$$

then $B \in Q(A)$ and $B^{\mathrm{T}} J B = J$.

Given a 4×4 $(+,-)$ sign pattern we can proceed similarly.

Example 6.7. If

$$A = \begin{pmatrix} + & - & + & + \\ - & + & + & - \\ + & - & - & + \\ - & - & + & - \end{pmatrix}, \quad B = \begin{pmatrix} \frac{14}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & \frac{1}{2\sqrt{6}} & \frac{17}{2\sqrt{6}} \\ -\frac{11}{\sqrt{15}} & \frac{4}{\sqrt{15}} & \frac{1}{\sqrt{6}} & -\frac{7}{\sqrt{6}} \\ \frac{14}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & -\frac{3}{\sqrt{6}} & \frac{9}{\sqrt{6}} \\ -\frac{16}{\sqrt{15}} & -\frac{1}{\sqrt{15}} & \frac{3}{2\sqrt{6}} & -\frac{21}{2\sqrt{6}} \end{pmatrix},$$

$J = \operatorname{diag}(1, -1, 1, -1)$, then $B \in Q(A)$ and $B^{\mathrm{T}} J B = J$.

The same can be done for 5×5 $(+,-)$ sign patterns.

Open Question 6.8. Is every $n \times n$ $(+,-)$ sign pattern in $\mathcal{J}_n$?

Now we return to the SPJO conditions. If A is a 4×4 $(+,-)$ sign pattern, by signature multiplications, A is equivalent to a sign pattern of the form

$$\begin{pmatrix} + & + & + & + \\ + & & & \\ + & & A_1 & \\ + & & & \end{pmatrix}$$

where A_1 is a 3×3 $(+,-)$ sign pattern. We denote the columns and rows of A_1 as follows:

$$A_1 = (\, c_1 \quad c_2 \quad c_3 \,) = \begin{pmatrix} r_1^{\mathrm{T}} \\ r_2^{\mathrm{T}} \\ r_3^{\mathrm{T}} \end{pmatrix}.$$

The SPJO conditions (6.1) and (6.2) for the matrix A have then the form of linear expressions in diagonal elements of J so that $Cv \overset{c}{\leftrightarrow} 0$ and $J = \operatorname{diag}(v)$, where

$$C = \begin{pmatrix} + & c_1^{\mathrm{T}} \\ + & c_2^{\mathrm{T}} \\ + & c_3^{\mathrm{T}} \\ + & c_1^{\mathrm{T}} \circ c_2^{\mathrm{T}} \\ + & c_1^{\mathrm{T}} \circ c_3^{\mathrm{T}} \\ + & c_2^{\mathrm{T}} \circ c_3^{\mathrm{T}} \\ + & r_1^{\mathrm{T}} \\ + & r_2^{\mathrm{T}} \\ + & r_3^{\mathrm{T}} \\ + & r_1^{\mathrm{T}} \circ r_2^{\mathrm{T}} \\ + & r_1^{\mathrm{T}} \circ r_3^{\mathrm{T}} \\ + & r_2^{\mathrm{T}} \circ r_3^{\mathrm{T}} \end{pmatrix}. \tag{6.5}$$

By observing (6.5), it can be seen that any permutation of the rows or columns of A_1 leads to the same SPJO conditions for A.

Assume that columns c_1, c_2, c_3 are mutually different and assume the same for the vectors r_1, r_2, r_3. Then it is clear that none of the vectors $c_1^{\mathrm{T}} \circ c_2^{\mathrm{T}}$, $c_1^{\mathrm{T}} \circ c_3^{\mathrm{T}}$, $c_2^{\mathrm{T}} \circ c_3^{\mathrm{T}}$, $r_1^{\mathrm{T}} \circ r_2^{\mathrm{T}}$, $r_1^{\mathrm{T}} \circ r_3^{\mathrm{T}}$ and $r_2^{\mathrm{T}} \circ r_3^{\mathrm{T}}$ is equal to $(+ \quad + \quad +)^{\mathrm{T}}$. Assuming that none of c_1, c_2, c_3, r_1, r_2, r_3 is equal to $(+ \quad + \quad +)^{\mathrm{T}}$, we have $Cv \overset{c}{\leftrightarrow} 0$ with $J = \operatorname{diag}(v)$ and $v = (+ \quad + \quad + \quad +)^{\mathrm{T}}$, so that A satisfies the SPO conditions. If at least one of vectors c_1, c_2, c_3, r_1, r_2, r_3 is equal to $(+ \quad + \quad +)^{\mathrm{T}}$, the matrix A has two identical columns $(+ \quad + \quad + \quad +)^{\mathrm{T}}$ or rows $(+ \quad + \quad + \quad +)$ (and thus it does not satisfy the

SPO conditions). Assume without loss of generality that $c_1 = (+\ \ +\ \ +)^{\mathrm{T}}$. Then $c_1 \circ c_2 = c_2$ and $c_1 \circ c_3 = c_3$. In addition, either $r_1 = c_1$ and thus also $r_1 \circ r_2 = r_2$ and $r_1 \circ r_3 = r_3$, or none of r_1, r_2, r_3 is equal to c_1, but then either $r_1 = r_2$ or $r_1 \circ r_2 = r_3$. All these cases lead to at most 7 different conditions in (6.5) so that there exists a vector v satisfying $Cv \overset{c}{\leftrightarrow} 0$.

It remains to treat the cases of at least two identical columns or rows in the submatrix A_1. The case $c_1 = c_2 = c_3$ leads to three vectors r_1, r_2, r_3 that are equal to $(+\ \ +\ \ +)^{\mathrm{T}}$ or to $(-\ \ -\ \ -)^{\mathrm{T}}$. Here, $c_1 \circ c_2 = c_1 \circ c_3 = c_2 \circ c_3 = (+\ \ +\ \ +)^{\mathrm{T}}$. Therefore, it is easy to find v such that $(+\ \ +\ \ +\ \ +)\, v \overset{c}{\leftrightarrow} 0$, $(+\ \ -\ \ -\ \ -)\, v \overset{c}{\leftrightarrow} 0$ and $(+\ \ c_1^{\mathrm{T}})\, v \overset{c}{\leftrightarrow} 0$. For the next case, assume without loss of generality that $c_1 = c_2 \neq c_3$, so that $c_1 \circ c_2 = (+\ \ +\ \ +)^{\mathrm{T}}$ and $c_2 \circ c_3 = c_1 \circ c_3$. Then it is not difficult to show that at least one of the vectors r_1, r_2, r_3 must be equal to $(+\ \ +\ \ +)^{\mathrm{T}}$ or $(-\ \ -\ \ -)^{\mathrm{T}}$, or, all the three vectors r_1, r_2, r_3 are the same, or two are the same and the third is negative of those two (in which cases our desired result easily holds). Then, without loss of generality, $r_1 = (+\ \ +\ \ +)^{\mathrm{T}}$ so that $r_1 \circ r_2 = r_2$ and $r_1 \circ r_3 = r_3$, or $r_1 = (-\ \ -\ \ -)^{\mathrm{T}}$ so that $r_1 \circ r_2 = -r_2$ and $r_1 \circ r_3 = -r_3$. All these cases also lead to at most 7 different conditions in (6.5) so that there exists a vector v satisfying $Cv \overset{c}{\leftrightarrow} 0$.

We have proved the following.

Proposition 6.9. *If A is a 4×4 $(+, -)$ sign pattern, then A satisfies the SPJO conditions.*

The case for $n = 5$ can be handled in a generally similar way. We omit the proof.

Proposition 6.10. *If A is a 5×5 $(+, -)$ sign pattern, then A satisfies the SPJO conditions.*

In view of Proposition 6.9, Proposition 6.10, and Theorem 6.2, we have all the cases covered (the cases $n = 1$ and $n = 2$ are trivial).

Theorem 6.11. *For all $n \geqslant 1$, each $n \times n$ $(+, -)$ sign pattern A satisfies the SPJO conditions.*

We finish with some more open questions.

Open Question 6.12. Let A be an $n \times n$ $(+, -)$ sign pattern and A_1 a principal submatrix of A. Are there relations between signature patterns J satisfying the SPJO conditions for A and the signature patterns J_1 satisfying the SPJO conditions for A_1?

Open Question 6.13. Let A be an $n \times n$ $(+, -)$ sign pattern that satisfies the SPJO conditions. Are there some sufficient conditions on submatrices of A to ensure that $A \in \mathcal{J}_n$?

7. Concluding remarks

In this paper we have established connections between G-matrices and J-orthogonal matrices, and we have begun an exploration of the sign patterns of the J-orthogonal matrices. This opens an interesting new topic for further research and there are many questions still to be resolved. We will continue this investigation in a follow-up paper.

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