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# COMPUTING THE DETERMINANTAL REPRESENTATIONS OF HYPERBOLIC FORMS

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*Dedicated to the memory of Professor Miroslav Fiedler*

*Abstract.* The numerical range of an $n \times n$ matrix is determined by an $n$ degree hyperbolic ternary form. Helton-Vinnikov confirmed conversely that an $n$ degree hyperbolic ternary form admits a symmetric determinantal representation. We determine the types of Riemann theta functions appearing in the Helton-Vinnikov formula for the real symmetric determinantal representation of hyperbolic forms for the genus $g = 1$. We reformulate the Fiedler-Helton-Vinnikov formulae for the genus $g = 0, 1$, and present an elementary computation of the reformulation. Several examples are provided for computing the real symmetric matrices using the reformulation.

*Keywords*: determinantal representation; hyperbolic form; Riemann theta function; numerical range

*MSC 2010*: 14Q05, 15A60

## 1. Introduction

Let $T$ be an $n \times n$ complex matrix. The numerical range of $T$ is defined as the set

$$W(T) = \{\xi^* T \xi \colon \xi \in \mathbb{C}^n,\ \xi^* \xi = 1\}.$$

The range $W(T)$ is a convex set due to the famous Toeplitz-Hausdorff theorem. Kippenhahn [12] characterized $W(T)$ as the convex hull of the real affine part of the

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dual projective curve of $F_T(x,y,z)=0$, where the real ternary form associated with $T$ is given by

$$F_T(x,y,z)=\det(x\Re(T)+y\Im(T)+zI_n),$$

and $\Re(T)=(T+T^*)/2$, $\Im(T)=(T-T^*)/(2\mathrm{i})$. Obviously, the equation $F_T(x_0,y_0,z)=0$ in $z$ has only real roots for any $(x_0,y_0)\in\mathbb{R}^2$ and $F_T(0,0,1)\neq 0$. The form $F_T(x,y,z)$ possessing this real roots property is called *hyperbolic* with respect to $e=(0,0,1)$. Lax in [13] conjectured that an arbitrary ternary hyperbolic form $F(x,y,z)$ with respect to $e=(e_1,e_2,e_3)\in\mathbb{R}^3$, $e\neq 0$, admits a determinantal representation, i.e.,

$$F(x,y,z)=c\det(xM_1+yM_2+zM_3)$$

for some real symmetric matrices $M_1,M_2,M_3$ with positive definiteness of $e_1M_1+e_2M_2+e_3M_3$, and $c\neq 0$. Independently, Fiedler in [8] made a similar conjecture under a relaxing condition that $M_1$, $M_2$, $M_3$ are Hermitian instead. Fiedler in [7] proved that the Lax conjecture is true provied that $F(x,y,z)=0$ is a rational curve. Recently, Helton and Vinnikov in [10] confirmed that the Lax conjecture is true by using Riemann's theta functions. Based on the confirmation of the Lax conjecture, the authors of this paper in [4] proved that the $c$-numerical range of an $n\times n$ matrix $T$ is reduced to the classical numerical range of an $m\times m$ matrix $A$, such that $W_c(T)=W(A)$ for some $m\leqslant n!$, and Helton and Spitkovsky in [9] proved that any matrix $T$ has a symmetric matrix $S$ satisfying $W(T)=W(S)$.

The construction of real symmetric matrices from the Helton-Vinnikov theorem has attracted attention in studying the numerical range of matrices. One case, for instance, ask, whether the complex symmetric matrix $S$ obtained by the Helton-Vinnikov formula from $F_T(x,y,z)$ is unitarily similar to a given matrix $T$. This question motivated us to compute explicitly the real symmetric matrices of the determinantal representation. In Section 2, we reformulate the formulae in [7], [10] for real symmetric matrices of the determinantal representations of hyperbolic forms with genus $g=0$ or 1. Notice that the entries of the symmetric matrices $M_j$ in the Lax conjecture have to be real. The Riemann theta functions in the Helton-Vinnikov formula may produce imaginary symmetric matrices. We determine the types of Riemann theta functions which lead to real symmetric expressions in the elliptic curve case. In Sections 3 and 4, we present concrete examples of $3\times 3$ and $4\times 4$ matrices, and compute the real symmetric matrices using the reformulation which illustrate the means of the Helton-Vinnikov formula for studying the numerical range of matrices.

## 2. Main theorems

Let $F(x, y, z)$ be an irreducible ternary form of degree $n \geqslant 3$. A point $P_0 = (x_0, y_0, z_0)$ of the complex projective curve

$$\mathcal{V}_{\mathbb{C}}(F) = \{[x, y, z] \in \mathbb{CP}^2 : \ F(x, y, z) = 0\}$$

is called a singular point if

$$\frac{\partial F}{\partial x}(x_0, y_0, z_0) = \frac{\partial F}{\partial y}(x_0, y_0, z_0) = \frac{\partial F}{\partial z}(x_0, y_0, z_0) = 0.$$

We sometimes abbreviate the complex projective curve $\mathcal{V}_{\mathbb{C}}(F)$ as $F(x, y, z) = 0$. For a singular point $P_0 = (x_0, y_0, z_0)$, $z_0 \neq 0$, consider two functions

$$f(X, Y) = F(x_0 + X, y_0 + Y, z_0), \quad f_Y(X, Y) = F_Y(x_0 + X, y_0 + Y, z_0).$$

The Taylor series of these functions define an ideal $(f, f_Y)$ of the ring $\mathbf{C}[[X, Y]]$ of formal power series in $X$, $Y$. We define

$$\delta(P_0) = \frac{1}{2}\Big(\dim\Big(\frac{\mathbf{C}[[X, Y]]}{(g, g_Y)}\Big) - m + s\Big),$$

where $m$ is the multiplicity of $P_0$ and $s$ is the number of irreducible analytic branches of the curve $\mathcal{V}_{\mathbb{C}}(F)$ near $(x_0, y_0, z_0)$. The number $\delta(P_0)$ is always a non-negative integer (cf. [14]). The genus of the curve $F(x, y, z) = 0$ is given by

$$g(F) = \frac{1}{2}(n-1)(n-2) - \sum_{j=1}^{k} \delta(P_j),$$

where $P_1, \ldots, P_k$ are singular points of the curve $F(x, y, z) = 0$. An irreducible curve is called a rational curve or an elliptic curve if its genus is $g = 0$ or $g = 1$, respectively. A rational curve has a rational function parametrization, and an elliptic curve can be parametrized by an elliptic function and its derivative (cf. [17]).

In the formulation of the Helton-Vinnikov theorem, the following two objects play a crucial role:

(i) The Riemann theta functions on a complex torus $\mathbb{C}^g/\Gamma$, where $\Gamma$ is a lattice in $\mathbb{C}^g$.

(ii) The Abel-Jacobi map $\varphi$ of an irreducible algebraic curve with genus $g$ to its corresponding Abel-Jacobi variety $\mathbb{C}^g/\Gamma$.

An accurate numerical computation method of the Riemann theta functions for $g \geqslant 1$ and a program to calculate a basis of $\Gamma$ for an algebraic curve can be found in [5] and [6], respectively. In this paper, we mainly deal with two cases: $g = 0$ and $g = 1$. The first reason is that the general theory of Abel functions and Riemann theta functions for $g \geqslant 2$ is rather complicated. In contrast to this, for $g = 1$, the complex torus $\mathbb{C}^g/\Gamma$ has an abelian fundamental group, and the Riemann functions have a single main variable. Shortly, the case $g = 1$ is more treatable. The second reason is more important from the viewpoint of developing the theory of numerical range. In [3], the authors of this paper proved that any irreducible curve $\mathcal{V}_{\mathbb{C}}(F)$ associated with a weighted shift matrix has genus $g \geqslant 1$, and in [1], they showed that the $j$-invariant of an irreducible elliptic curve associated with a $3 \times 3$ or $4 \times 4$ matrix is real and greater than or equal to 1. There are many tools for computing Riemann theta functions on a Riemann surface with $g = 1$. We used Mathematica (cf. [18]) to implement the numerical computations.

In the rest of this paper, we assume a real ternary form $F(x, y, z)$ of degree $n$ satisfying the following conditions:

(F1) $F(x, y, z)$ is hyperbolic with respect to $e = (0, 0, 1)$ and $F(0, 0, 1) = 1$.
(F2) $F(x, y, z)$ is irreducible.
(F3) The $n$ real intersection points of the complex projective curve $F(x, y, z) = 0$ and the line $x = 0$ are distinct non-singular points $Q_1, \ldots, Q_n$ with coordinates $Q_j = (0, 1, -\beta_j)$, where $\beta_j \neq 0$.

According to the determinantal representation theorem [7], [10], there exist real symmetric matrices $B$ and $C$ of dimension $n$ such that

$$F(x, y, z) = \det(zI_n + yB + xC), \tag{2.1}$$

where $B = \operatorname{diag}(\beta_1, \ldots, \beta_n)$, and the diagonal entries $c_{jj}$ of the real symmetric matrix $C$ are given by

$$c_{jj} = \beta_j \frac{F_x(0, 1, -\beta_j)}{F_y(0, 1, -\beta_j)}. \tag{2.2}$$

The crucial problem is the construction of the off-diagonal entries of $C$. If $g = 0, 1$, we denote by $Q'_j$ the point on the parameter space (the real line for $g = 0$, the complex torus for $g = 1$) corresponding to $Q_j$. In the expression (2.1), if we replace $C$ by

$$\widetilde{C} = \operatorname{diag}(\eta_1, \eta_2, \ldots, \eta_n)\, C \operatorname{diag}(\eta_1, \eta_2, \ldots, \eta_n),$$

$(\eta_1, \eta_2, \ldots, \eta_n = \pm 1)$, we have another determinantal representation

$$F(x, y, z) = \det(zI_n + yB + x\widetilde{C}).$$

The choice of the sign pattern of the off-diagonal entries of $C$ is determined for the case $g=0$, and is open for $g=1$. We reformulate Fiedler formula ([7], Theorem 1), for the determinantal representation if $g=0$.

**Theorem 2.1.** *Let $F(t,x,y)$ be a ternary form of degree $n$ satisfying conditions* (F1)–(F3). *Assume the genus of the complex projective curve $F(x,y,z)=0$ is 0. Then the off-diagonal entries of $C$ in the determinantal representation* (2.1) *are given by*

$$c_{jk}=\varepsilon\,\frac{\beta_k-\beta_j}{Q'_k-Q'_j}\,\frac{1}{\sqrt{\Big(\mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q'_j)\Big)\Big(\mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q'_k)\Big)}},\tag{2.3}$$

*where*

$$x=R_1(s)=\frac{u(s)}{w(s)},\quad y=R_2(s)=\frac{v(s)}{w(s)}$$

*are real rational functions parametrizing the affine part $F(x,y,1)=0$, and $\varepsilon\in\{+1,-1\}$ satisfies $\varepsilon\,u'(Q'_j)v(Q'_j)>0$ for all $j$.*

Proof. It is shown in [7], Theorem 1, that we can choose $\varepsilon\in\{+1,-1\}$ such that $\varepsilon\,u'(Q'_j)v(Q'_j)>0$ for all $j$. Further, we compute that

$$\frac{1}{\mathrm{d}\big(\frac{R_1}{R_2}\big)(Q'_j)}=\frac{1}{\mathrm{d}\big(\frac{u}{v}\big)(Q'_j)}=\frac{v(Q'_j)^2}{u'(Q'_j)v(Q'_j)-u(Q'_j)v'(Q'_j)}=\frac{v(Q'_j)}{u'(Q'_j)},$$

and hence the formula (2.3) essentially coincides with the formula obtained in [7], Theorem 1. □

The formulation in Theorem 2.1 is just a slight modification of Fiedler formula. This reformulation is be consistent with the formula pattern in Theorem 2.4 for the case $g=1$.

The Helton-Vinnikov Formula in [10], Theorem 2.2 (see also [15], Theorem 6), for a hyperbolic form with genus $g$ reads as follows:

**Theorem 2.2.** *Let $F(x,y,z)$ be a ternary form of degree $n$ satisfying conditions* (F1)–(F3). *Assume the genus of the complex projective curve $F(x,y,z)=0$ is $g\geqslant 1$. Then the off-diagonal entries of $C$ in the determinantal representation* (2.1) *are given by*

$$c_{jk}=\frac{\beta_k-\beta_j}{\theta[\delta](0)}\,\frac{\theta[\delta](\varphi(Q_k)-\varphi(Q_j),S)}{E(\varphi(Q_k),\varphi(Q_j))}\,\frac{1}{\sqrt{\mathrm{d}\big(\frac{x}{y}\big)(Q'_j)}\sqrt{\mathrm{d}\big(\frac{x}{y}\big)(Q'_k)}},$$

*where $\theta[\delta](\cdot,\cdot)$ is a Riemann theta function with an even characteristic $\delta$, $E(\cdot,\cdot)$ is the prime form on the Jacobi-variety given as a constant multiple of a Riemann theta function $\theta[\varepsilon](\cdot,\cdot)$ with an odd characteristic $\varepsilon$, the two Riemann theta functions are defined for $(z,S)\in\mathbb{C}^g\times\mathcal{H}_g$, the matrix $S$ is determined by the curve $\mathcal{V}_{\mathbb{C}}(F)$, $\varphi$ is the Abel-Jacobi map from $\mathcal{V}_{\mathbb{C}}(F)$ into the Jacobian variety, and $Q'_j$ is the point on the Riemann surface corresponding to $Q_j$. Symbol $\mathcal{H}_g$ denotes the set of the $g\times g$ Riemann matrices, i.e., symmetric matrices whose imaginary parts are positive definite.*

The Helton-Vinnikov formula in Theorem 2.2 involves computing the Riemann theta functions and Abel-Jacobi maps. The Riemann theta functions are explicit, but the non-explicitness arises because of the complexity in computation when the genus satisfies $g\geqslant 2$. For instance, we have a quartic curve with integral coefficients and $g=2$ for which the computation of the Riemann matrix $S$ is not possible by the usual software. We restrict our attention to the case $g=1$, and reformulate Theorem 2.2 using Riemann theta functions with a single main variable, and the Weierstrass canonical forms of non-singular cubic curves.

Let $F(t,x,y)$ be a ternary form satisfying conditions (F1)–(F3) with genus $g=1$, i.e., $\mathcal{V}_{\mathbb{C}}(F)$ is an elliptic curve. Then there is a real birational transformation $\Phi$ for which $\Phi(\mathcal{V}_{\mathbb{C}}(F))$ is a non-singular cubic curve of the Weierstrass standard form

$$Y^2Z=4X^3-g_2X^2Z-g_3Z^3$$

for some real constants $g_2$, $g_3$ such that $g_2^3-27g_3^2>0$. The complex affine algebraic curve $Y^2=4X^3-g_2X-g_3$ is parametrized as

$$X=\mathcal{P}(s\colon\ g_2,g_3),\quad Y=\mathcal{P}'(s\colon\ g_2,g_3),$$

where $\mathcal{P}(s\colon\ g_2,g_3)$ and $\mathcal{P}'(s\colon\ g_2,g_3)$ are the Weierstrass $P$-functions and its derivative with parameters $g_2,g_3$ satisfying the differential equation

$$\Big(\frac{\mathrm{d}\mathcal{P}'}{\mathrm{d}s}\Big)^2=4\mathcal{P}^3(s\colon\ g_2,g_3)-g_2\mathcal{P}(s\colon\ g_2,g_3)-g_3.$$

The meromorphic function $\mathcal{P}(s\colon\ g_2,g_3)$ on the Gaussian plane $\mathbb{C}$ has two linearly independent half-periods $\omega_1$ and $\omega_2$ in the sense that

$$\mathcal{P}(s+2\omega_1\colon\ g_2,g_3)=\mathcal{P}(s\colon\ g_2,g_3)\quad\text{and}\quad\mathcal{P}(s+2\omega_2\colon\ g_2,g_3)=\mathcal{P}(s\colon\ g_2,g_3),$$

where $\omega_1$ is a positive real number and $w_2$ is a purely imaginary number with $\Im(\omega_2)>0$ (cf. [1]). The $\tau$-invariant of the curve $\mathcal{V}_{\mathbb{C}}(F)$ is defined by $\tau=\omega_2/\omega_1$.

The real affine part $F(x, y, 1) = 0$ of the curve $\mathcal{V}_{\mathbb{C}}(F)$ is then parametrized as

$$\begin{aligned}(2.4)\ \ \{(x,y,1) = (\mathrm{R}_1(\mathcal{P}(u),\mathcal{P}'(u)),\mathrm{R}_2(\mathcal{P},\mathcal{P}'(u)),1)\colon\ &\Im(u)=0,\ 0<\Re(u)<2\omega_1 \text{ or}\\ &\Im(u)=\Im(\omega_2),\ 0\leqslant\Re(u)\leqslant 2\omega_1\}\end{aligned}$$

by real rational functions $R_1, R_2$ of $\mathcal{P}$ and $\mathcal{P}'$ over the torus $\mathbb{T}$. This parametrization $s \mapsto (x, y, 1)$ is the inverse of the Abel-Jacobi map $\varphi\colon\ \mathcal{V}_{\mathbb{C}}(F) \to \mathrm{Jac}(X)$.

Denote by $\mathcal{H}$ the upper half-plane $\mathcal{H} = \{z \in \mathbb{C}\colon\ \Im(z) > 0\}$. The *Riemann theta function* is the holomorphic function on $\mathbb{C} \times \mathcal{H}$ defined by the exponential series

$$\theta(u,\tau) = \sum_{m\in\mathbb{Z}} \exp(\pi\mathrm{i}(m^2\tau + 2mu)),$$

which is quasi-periodic with respect to the lattice $\mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$:

$$\theta(u + m + \tau n, \tau) = \exp(\pi\mathrm{i}(-2nu - n^2\tau))\,\theta(u,\tau)$$

for all integers $m, n$. We consider four Riemann theta functions $\theta[\varepsilon](u)$ with characteristics $\varepsilon$ defined as

$$\theta[\varepsilon](u,\tau) = \exp(\pi\mathrm{i}(a^2\tau + 2au + 2ab))\theta(u + \tau a + b, \tau)$$

for $\varepsilon = a + \tau b$ with

$$(a,b) = (0,0), \Big(\frac{1}{2},0\Big), \Big(0,\frac{1}{2}\Big), \Big(\frac{1}{2},\frac{1}{2}\Big).$$

Using the parameter $q = \exp(\mathrm{i}\pi\tau)$, we have

$$\theta(u,[q]) = \theta(u,\tau) = \sum_{m\in\mathbb{Z}} q^{m^2}\exp(2m\pi\mathrm{i}u).$$

The four Riemann theta functions are also denoted as

$$\begin{aligned}
\theta_1(u,[q]) &= -\theta\Big[\frac{1}{2},\frac{1}{2}\Big](u,\tau) = 2q^{1/4}\sum_{m=0}^{\infty} q^{(m+1)m}\sin((2m+1)\pi u),\\
\theta_2(u,[q]) &= \theta\Big[\frac{1}{2},0\Big](u,\tau) = 2q^{1/4}\sum_{m=0}^{\infty} q^{(m+1)m}\cos((2m+1)\pi u),\\
\theta_3(u,[q]) &= \theta[0,0](u,\tau) = \theta(u,\tau) = 1 + 2\sum_{m=1}^{\infty} q^{m^2}\cos(2m\pi u),\\
\theta_4(u,[q]) &= \theta\Big[0,\frac{1}{2}\Big](u,\tau) = 1 + 2\sum_{m=1}^{\infty} (-1)^m q^{m^2}\cos(2m\pi u).
\end{aligned}$$

For references on the Weierstrass $P$-functions and Riemann theta functions, one may see, for instance, [11], [17].

**Theorem 2.3.** *The four Riemann theta functions $\theta_\delta$, $\delta = 1, 2, 3, 4$, are quasi-periodic, and the elliptic functions $\theta_\delta/\theta_1$, $\delta = 2, 3, 4$ have respective double periods* $1$, $2\tau$ $(\delta = 2)$, $2$, $2\tau$ $(\delta = 3)$, $2$, $\tau$ $(\delta = 4)$. *Moreover, the function $\theta_4/\theta_1$ takes on real values on the real part of the Jacobian variety, and the functions $\theta_2/\theta_1$, $\theta_3/\theta_1$ take on real values or purely imaginary values depending on the two connected components of the real part of the Jacobi variety.*

P r o o f. Direct computations show that

$$\begin{aligned}
\theta_1(u+1,[q]) &= -\theta_1(u,[q]), & \theta_1(u+\tau,[q]) &= -q^{-1}\exp(-2\pi \mathrm{i} u)\theta_1(u,[q]),\\
\theta_2(u+1,[q]) &= -\theta_2(u,[q]), & \theta_2(u+\tau,[q]) &= q^{-1}\exp(-2\pi \mathrm{i} u)\theta_2(u,[q]),\\
\theta_3(u+1,[q]) &= \theta_3(u,[q]), & \theta_3(u+\tau,[q]) &= q^{-1}\exp(-2\pi \mathrm{i} u)\theta_3(u,[q]),\\
\theta_4(u+1,[q]) &= \theta_4(u,[q]), & \theta_4(u+\tau,[q]) &= -q^{-1}\exp(-2\pi \mathrm{i} u)\theta_4(u,[q]).
\end{aligned}$$

Thus the functions $\theta_\delta/\theta_1$, $\delta = 2, 3, 4$, are elliptic functions with double periods $1$, $2\tau$ $(\delta = 2)$, $2$, $2\tau$ $(\delta = 3)$, $2$, $\tau$ $(\delta = 4)$.

Suppose that $\tau$ is a purely imaginary number. Then the four functions $\theta_\delta(u,[q])$ take on real values on the real line. On the line $\Im(z) = \Im(\tau)/2$, we have

$$\frac{\theta_4}{\theta_1}\Big(u+\frac{\tau}{2}\Big) = \frac{\theta_1}{\theta_4}(u),$$

and

$$\frac{\theta_2}{\theta_1}\Big(u+\frac{\tau}{2}\Big) = -\mathrm{i}\frac{\theta_3}{\theta_4}(u), \quad \frac{\theta_3}{\theta_1}\Big(u+\frac{\tau}{2}\Big) = -\mathrm{i}\frac{\theta_2}{\theta_4}(u)$$

for any $u \in \mathbb{R}$. Hence, $\theta_2/\theta_1$, $\theta_3/\theta_1$ take on either real or purely imaginary values on the real part of the Jacobi variety. $\square$

Using the notation of Theorem 2.3, we reformulate the Helton-Vinnikov Formula in [10], Theorem 2.2, (cf. [15], Theorem 6) for $g = 1$, and determine the types of Riemann theta functions which lead to real symmetric determinantal representations.

**Theorem 2.4.** *Let $F(t, x, y)$ be a ternary form of degree $n$ satisfying conditions* (F1)–(F3). *Assume the genus of the complex projective curve $F(x, y, z) = 0$ is 1, and $x = R_1(\mathcal{P}(u), \mathcal{P}'(u))$, $y = R_2(\mathcal{P}, \mathcal{P}'(u))$ parametrize the elliptic curve $\mathcal{V}_{\mathbb{C}}(F)$ in* (2.4). *Let $Q_j' = \varphi(Q_j)$ be the point of the torus $\mathbb{T}$ corresponding to the point*

*$Q_j \in \mathcal{V}_{\mathbb{C}}(F)$. For $\delta = 2, 3$, the matrix $C$ in the determinantal representation* (2.1) *is real symmetric, and its off-diagonal entries are given by*

$$(2.5) \qquad c_{jk} = \frac{(\beta_k - \beta_j)\theta_1'(0)}{2\omega_1\,\theta_\delta(0)} \frac{\theta_\delta((Q_k' - Q_j')/2\omega_1)}{\theta_1((Q_k' - Q_j')/2\omega_1)} \frac{1}{\sqrt{\mathrm{d}\big(\frac{R_1}{R_2}\big)(Q_j')}\sqrt{\mathrm{d}\big(\frac{R_1}{R_2}\big)(Q_k')}}.$$

P r o o f. We use a non-normalized Jacobi variety $\mathbb{C}/(2\mathbb{Z}\omega_1 + 2\mathbb{Z}\omega_2)$ in place of the normalized Jacobi variety $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ for $\tau = \omega_2/\omega_1$. According to this frame, we easily use the Weierstrass $P$-function to express the inverse of the Abel-Jacobi map $\varphi$. By this parameter change, a new factor $1/(2\omega_1)$ appears in the formulation (2.5). For $g = 1$, the Riemann theta function with an odd characteristic is uniquely given by $\theta_1(\cdot)$. As the prime form $E(\cdot,\cdot)$, it generates the term $\theta_1((Q_k'-Q_j')(2\omega_1)^{-1})/\theta_1'(0)$ in (2.5). The Riemann theta function $\theta[\delta](\cdot)$ with an even characteristic appearing in (2.5), is given by $\theta_2, \theta_3$ or $\theta_4$.

We claim that $\delta = 4$ produces an imaginary $C$ in (2.1). For a real parameter $\theta$, we consider the equation $F(-\cos\theta, -\sin\theta, z) = 0$ in $z$. By the hyperbolicity of $F(x, y, z)$, this equation has $n$ real roots $z_j(\theta)$ counting multiplicities. In particular, for $\theta = -\pi/2$, the $n$ distinct real roots are $Q_j = (0, 1, -\beta_j)$, $j = 1, 2, \ldots, n$. By the Helton-Vinnikov theorem and Rellich's theorem, the roots $z_j(\theta)$ of the equation depend analytically on $\theta$. Every real point of the curve $\mathcal{V}_{\mathbb{C}}(F)$ is joined to some $Q_j$. By a birational transformation, each point $Q_j$ is mapped to $Q_j'$ on a non-singular cubic curve, and the curve $z_j(\theta)$ is mapped to the real part of the cubic curve consisting of a pseudo line and an oval. The image of $z_j(\theta)$ covers the real part of the cubic curve except for a finite number of points. There are $j \neq k$ for which $Q_j'$ lies on the pseudo line and $Q_k'$ lies on the oval. We use the same symbol $Q_j'$ for the point on the non-normalized torus $\mathbb{C}/(2\omega_1\mathbb{C} + 2\omega_2\mathbb{Z})$ corresponding to the point $Q_j'$ on the cubic curve. Then $\Im(Q_j') = 0$ and $\Im(Q_k') = \Im(\omega_2)$, and thus $\sqrt{\mathrm{d}(x/y)(Q_j')}\sqrt{\mathrm{d}(x/y)(Q_k')}$ is purely imaginary. Hence, the entries $c_{jk}$ in (2.5) are real if and only if the ratio $\theta_\delta((Q_k' - Q_j')(2\omega_1)^{-1})/\theta_1((Q_k' - Q_j')(2\omega_1)^{-1})$ of a purely imaginary value. This happens only for $\delta = 2, 3$, by Theorem 2.3, since $\Im(Q_k' - Q_j') = \Im(\omega_2)$. The case $\delta = 4$ results in complex entries of $C$. $\square$

**Remarks.** 1. Applying the formulae mentioned in the proof of Theorem 2.3, we find that the function $\theta_\delta/\theta_1$ on the normalized torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ is defined up to multiplicative constants $\pm 1$.

2. For $g = 1$, the advantage of the formulation (2.5) is that the torus $\mathbb{C}^g/\Gamma$ is a one-dimensional analytic manifold which is realized as a complex projective curve $\mathcal{V}_{\mathbb{C}}(G)$ for some ternary form $G$ using a birational transformation.

3. Plaumann et al. mentioned in [15], page 270, that they do not know why sometimes the off-diagonal entries of $C$ are wrong by a constant factor when applying

the Helton-Vinnikov formula. The authors of this paper are not able to fix their problem, but the formulation in Theorem 2.4 for $g = 1$ has no such trouble.

4. It is also shown in [15], Theorem 7, that for a smooth curve $\mathcal{V}_{\mathbb{C}}(f)$ there are $2^g$ real positive definite representations. We prove in Theorem 2.4 that for an elliptic curve ($g = 1$) with singular points (non-smooth), there are $2 = 2^g$ real positive definite representations.

In Theorems 2.1 and 2.4, we use a parametrization of an irreducible projective algebraic curve $F(x, y, z) = 0$. An irreducible curve $F(x, y, z) = 0$ is transformed into an algebraic curve $G(x, y, z) = 0$ for which every singular point $(x_0, y_0, z_0) \neq (0, 0, 0)$ of $G(x, y, z) = 0$ has pairwise distinct tangents by successive Cremona transformations (cf. [16], Theorem 7.4). Such a birational transformation preserves the genus of the curve. We assume that $G(x, y, z)$ is an irreducible homogeneous polynomial of degree $n$, the curve $G(x, y, z) = 0$ has ordinary multiple points of multiplicities $m_1, \ldots, m_k$ and has no singular points other than the ordinary ones. Then the genus $g$ of $G(x, y, z) = 0$ is given by

$$g = \frac{1}{2}(n-1)(n-2) - \frac{1}{2}\sum_{j=1}^{k} m_j(m_j - 1) \tag{2.6}$$

(cf. [16]). The number $g$ can be evaluated by the function 'genus' of algcurves package in Maple. For $g = 0$, the method of constructing a parametrization of the curve $G(x, y, z) = 0$ as $x = u(s)$, $y = v(s)$, $z = w(s)$ of degree at most $n$ is given in [16], pages 67–68. For $g = 1$, the curve $G(x, y, z) = 0$ is transformed into the Weierstrass canonical form

$$-y^2 z + 4x^3 - g_2 x z^2 - g_3 = 0$$

with $g_2^3 - 27g_3^2 \neq 0$. The affine curve $G(x, y, 1) = 0$ is then expressed as

$$x = R_1(\mathcal{P}, \mathcal{P}'), \quad y = R_2(\mathcal{P}, \mathcal{P}')$$

by some rational functions $R_1$, $R_2$ of two variables (cf. [16], page 72, [17], pages 489–493). The Riemann theta functions $\theta_\delta(u, [q])$, $\delta = 1, 2, 3, 4$, can be numerically computed using Mathematica function 'EllipticTheta $[\delta, \pi u, q]$' (cf. [18]).

## 3. Computing rational curves

We explain the formula in Theorem 2.1 by practical computation on an algebraic curve with genus $g = 0$. Consider a typical roulette curve defined by a trigonometric polynomial

$$\varphi(\theta) = \exp(2\mathrm{i}\theta) + \frac{4}{5}\exp(-\mathrm{i}\theta).$$

The determinantal representation of this curve has been studied in [2]. We apply Theorem 2.1 to find the real symmetric matrices $B$ and $C$. By using a parameter $s = \tan(\theta)/2$, this roulette curve is parametrized as

$$x = \Re(\varphi(\theta)) = \frac{u(s)}{w(s)}, \quad y = \Im(\varphi(\theta)) = \frac{v(s)}{w(s)},$$

where

$$\begin{aligned} u(s) &= \frac{1}{5}(s^2+6s+3)(s^2-6s+3),\\ v(s) &= -\frac{4}{5}(7s^2-3)s,\\ w(s) &= (s^2+1)^2, \end{aligned}$$

and the roulette curve as an affine curve $F(x, y, 1) = 0$ is parametrized as

$$\begin{aligned} L_1(x,s) &= -(s^2+1)^2 x + \frac{1}{5}(s^2+6s+3)(s^2-6s+3) = 0,\\ L_2(y,s) &= -(s^2+1)^2 y - \frac{4}{5}(7s^2-3)s = 0. \end{aligned}$$

By taking the resultant of $L_1(x, s)$ and $L_2(y, s)$ with respect to $s$, we obtain the equation $F(x, y, 1) = 0$ of the roulette curve which, in homogeneous form, is expressed as

$$F(x,y,z) = \frac{15{,}625}{729}(x^2+y^2)^2 - \frac{20{,}000}{729}(x^3z - 3xy^2z) - \frac{550}{27}(x^2+y^2)z^2 + z^4.$$

Solving the equation $F(0, 1, -\beta_j) = 0$, we find that the matrix $B$ is given by

$$B = \operatorname{diag}\left(\frac{5}{9}(-3+2\sqrt{6}), -\frac{5}{9}(3+2\sqrt{6}), \frac{5}{9}(3+2\sqrt{6}), -\frac{5}{9}(-3+2\sqrt{6})\right).$$

The corresponding points $Q'_j = s$ of the real line are characterized as

$$(s^2+6s+3)(s^2-6s+3) = 0, \quad -\frac{v(s)}{w(s)} = -\frac{1}{\beta_j}.$$

It follows that

$$Q_1' = 3+\sqrt{6}, \quad Q_2' = 3-\sqrt{6}, \quad Q_3' = -3+\sqrt{6}, \quad Q_4' = -3-\sqrt{6}.$$

We conclude by (2.3) that the matrix $C$ and its entries are

$$C = \begin{bmatrix} -c_{22} & c_{12} & c_{13} & c_{14} \\ c_{12} & c_{22} & c_{23} & c_{13} \\ c_{13} & c_{23} & c_{22} & c_{12} \\ c_{14} & c_{13} & c_{12} & -c_{22} \end{bmatrix},$$

where

$$c_{22} = \frac{25\sqrt{2}}{9\sqrt{3}}, \quad c_{12} = -\frac{5\sqrt{10}}{9\sqrt{3}}, \quad c_{13} = \frac{5\sqrt{5}}{9\sqrt{6}},$$
$$c_{14} = -\frac{5}{9\sqrt{6}}\sqrt{73+28\sqrt{6}}, \quad c_{23} = \frac{5}{9\sqrt{6}}\sqrt{73-28\sqrt{6}}.$$

## 4. Computing elliptic curves

In the paper [1], the so-called $j$-invariant of an irreducible elliptic curve associated with the following $4\times 4$ matrix is explicitly formulated. The $4\times 4$ cyclic weighted shift matrix is

$$S = \begin{bmatrix} 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \\ a_1 & 0 & 0 & 0 \end{bmatrix},$$

where $a_1 = \sqrt{2}k(1-s^2)/(1+s^2)$, $a_2 = \sqrt{2}k(2s)/(1+s^2)$ for $0<k$, $0<s<\sqrt{2}$. Then

$$(s^2+1)^4F_S(x,y,z) = (s^2+1)^4z^4 - 2k^2(s^2+1)^4(x^2+y^2)z^2 + 16k^4s^2(s^2-1)^2(x^2+y^2)^2.$$

This form is hyperbolic with respect to $(0,0,1)$, and has two ordinary double points at $(0,1,0)$ and $(1,0,0)$. Accordingly, by the genus formula (2.6), $g(F_S)=1$.

The curve $F_S(x,y,z)=0$ intersects the line $x=0$ at four distinct points $(0,1,-\beta_j)$ with

$$\beta_1 = \sqrt{2}k\frac{1-s^2}{1+s^2}, \quad \beta_2 = -\beta_1, \quad \beta_3 = \sqrt{2}k\frac{2s}{1+s^2}, \quad \beta_4 = -\beta_3.$$

Then the diagonal matrix is $B = \mathrm{diag}(\beta_1,\beta_2,\beta_3,\beta_4)$. The quartic form $F_S(x,y,z)$ has a rather simple symmetric determinantal representation

$$F(x,y,z) = \det(zI_4 + yB + xA_1),$$

where

$$A_1 = \begin{bmatrix} 0 & 0 & \varepsilon a_{13} & a_{14} \\ 0 & 0 & \varepsilon a_{14} & a_{13} \\ \varepsilon a_{13} & a_{14} & 0 & 0 \\ \varepsilon a_{14} & a_{13} & 0 & 0 \end{bmatrix}$$

with

$$a_{13} = \frac{k(1+2s-s^2)}{\sqrt{2}(1+s^2)}, \quad a_{14} = \frac{k(1-2s-s^2)}{\sqrt{2}(1+s^2)}, \quad \varepsilon = \pm 1.$$

For $k = 1/\sqrt{2}$ and $s = 1/5$, we have $a_{13} = \varepsilon 17/26$, $a_{14} = 7/26$.

Another symmetric determinantal representation is given by

$$F_S(x,y,z) = \det(zI_4 + yB + xA_2),$$

where

$$A_2 = \begin{bmatrix} 0 & \varepsilon a_{12} & \eta a_{13} & 0 \\ \varepsilon a_{12} & 0 & 0 & -\eta a_{13} \\ \eta a_{13} & 0 & 0 & \varepsilon a_{34} \\ 0 & -\eta a_{13} & \varepsilon a_{34} & 0 \end{bmatrix},$$

with

$$a_{12} = \frac{2\sqrt{2}ks(1-2s-s^2)}{(1+2s-s^2)(1+s^2)},$$
$$a_{13} = \frac{2\sqrt{2}k\sqrt{s(1-s^2)}}{1+2s-s^2},$$
$$a_{34} = -\frac{\sqrt{2}k(1-s^2)(1-2s-s^2)}{(1+2s-s^2)(1+s^2)},$$

$\varepsilon, \eta = \pm 1$. For $k = 1/\sqrt{2}$ and $s = 1/5$, we have $a_{12} = 35/221$, $a_{34} = -84/221$, $a_{13} = 2\sqrt{30}/17$.

Now, we explain the computation of the formula in Theorem 2.4. To parametrize the curve $\mathcal{V}_{\mathbb{C}}(F_S)$ using elliptic functions, we introduce new variables $U, V, W$ by

$$U = k(x^2 - y^2), \quad V = \frac{z(z - kx + ky)}{k}, \quad W = (z + kx - ky)(x + y).$$

The inverse of this birational transformation is given by

$$x = \frac{1}{2k}(2U^2 + UV - 3UW + W^2), \quad y = \frac{1}{2k}(2U^2 - UV - 3UW + W^2), \quad z = V(W - U).$$

The quartic curve $F_S(x,y,z) = 0$ is birationally transformed into the non-singular cubic curve $G(U,V,W) = 0$ where

$$\begin{aligned} G(U,V,W) = {} & (s^2+1)^4 W^3 - 4(s^2+1)^4 UW^2 + (5s^8 + 4s^6 + 62s^4 + 4s^2 + 5)U^2 W \\ & -2(s^4 - 6s^2 + 1)U^3 - (s^2+1)^4 V^2 W. \end{aligned}$$

We perform numerical computations for $k = 1/\sqrt{2}, s = 1/5$. The points $Q_j$ on the curve $F_S(x, y, z) = 0$ are transformed into the points

$$[Q_1] = \left(1, -\frac{50}{169} + \frac{5\sqrt{2}}{13}, 1 + \frac{5\sqrt{2}}{13}\right), \quad [Q_2] = \left(1, -\frac{50}{169} - \frac{5\sqrt{2}}{13}, 1 - \frac{5\sqrt{2}}{13}\right),$$
$$[Q_3] = \left(1, -\frac{288}{169} + \frac{12\sqrt{2}}{13}, 1 + \frac{12\sqrt{2}}{13}\right), \quad [Q_4] = \left(1, -\frac{288}{169} - \frac{12\sqrt{2}}{13}, 1 - \frac{12\sqrt{2}}{13}\right)$$

on the curve

$$\begin{aligned} G(U, V, W) = 16\,(&28{,}561W^3 - 114{,}244UW^2 + 128{,}405U^2W \\ &- 28{,}322U^3 - 28{,}561V^2W) = 0. \end{aligned}$$

By the transformation

$$U = -\widetilde{U} + \frac{128{,}405}{84{,}966}W, \quad V = \frac{119}{169\sqrt{2}}\widetilde{V},$$

the cubic curve $G(U, V, W) = 0$ turns into the Weierstrass canonical form

$$-\widetilde{V}^2W + 4\widetilde{U}^3 - g_2\widetilde{U}W^2 - g_3W^3 = 0$$

with

$$g_2 = \frac{6{,}780{,}988{,}321}{601{,}601{,}763}, \quad g_3 = -\frac{556{,}790{,}665{,}176{,}719}{76{,}673{,}543{,}092{,}587}.$$

Thus the affine algebraic curve $F_S(x, y, 1) = 0$ is parametrized as

$$\begin{aligned} x = R_1(u) = &\frac{1}{10{,}110{,}954(84{,}966\mathcal{P}(u) - 43{,}439)\mathcal{P}'(u)} \\ &\times \big(-5{,}055{,}477\sqrt{2}(84{,}966\mathcal{P}(u) - 128{,}405)\mathcal{P}'(u) \\ &\quad + 676(42{,}483\mathcal{P}(u) - 42{,}961)(84{,}966\mathcal{P}(u) - 43{,}439)\big), \\ y = R_2(u) = &\frac{1}{10{,}110{,}954(84{,}966\mathcal{P}(u) - 43{,}439)\mathcal{P}'(u)} \\ &\times \big(5{,}055{,}477\sqrt{2}(84{,}966\mathcal{P}(u) - 128{,}405)\mathcal{P}'(u) \\ &\quad + 676(4{,}2483\mathcal{P}(u) - 42{,}961)(84{,}966\mathcal{P}(u) - 43{,}439)\big). \end{aligned}$$

The half-periods of the Weierstrass $P$-function are approximately

$$\omega_1 = 1.849{,}847{,}0, \quad \omega_2 = 0.921{,}393{,}5\mathrm{i}.$$

Their ratio $\tau = \omega_2/\omega_1$ is approximately 0.498,091,74i.

The cubic curve $-\widetilde{V}^2 + 4\widetilde{U}^3 - g_2\widetilde{U} - g_3 = 0$ is parametrized as

$$\widetilde{U} = \mathcal{P}(u, \{g_2, g_3\}), \quad \widetilde{V} = \mathcal{P}'(u, \{g_2, g_3\}).$$

The cubic and the line $\widetilde{V} = 0$ intersect at

$$(\widetilde{U}_1, 0) = \Big(\frac{42{,}961}{42{,}483}, 0\Big), \quad (\widetilde{U}_2, 0) = \Big(\frac{78{,}719}{84{,}966}, 0\Big), \quad (\widetilde{U}_3, 0) = \Big(-\frac{16{,}4641}{84{,}966}, 0\Big).$$

These three points correspond respectively to points $\omega_1, \omega_1 + \omega_2, \omega_2$ on the torus $\mathbb{C}/(2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z})$. The points $Q_j$ are transformed into the points

$$\begin{aligned}
\widetilde{Q}_1 &= \Big(\frac{7{,}739}{84{,}966} + \frac{65\sqrt{2}}{119}, \frac{28{,}470}{14{,}161} - \frac{16{,}900\sqrt{2}}{14{,}161}, 1\Big), \\
\widetilde{Q}_2 &= \Big(\frac{7{,}739}{84{,}966} - \frac{65\sqrt{2}}{119}, -\frac{28{,}470}{14{,}161} - \frac{16{,}900\sqrt{2}}{14{,}161}, 1\Big), \\
\widetilde{Q}_3 &= \Big(\frac{249{,}071}{84{,}966} - \frac{156\sqrt{2}}{119}, -\frac{142{,}584}{14{,}161} + \frac{97{,}344\sqrt{2}}{14{,}161}, 1\Big), \\
\widetilde{Q}_4 &= \Big(\frac{249{,}071}{84{,}966} + \frac{156\sqrt{2}}{119}, \frac{142{,}584}{14{,}161} + \frac{97{,}344\sqrt{2}}{14{,}161}, 1\Big)
\end{aligned}$$

on the cubic curve $-\widetilde{V}^2W + 4\widetilde{U}^3 - g_2\widetilde{U}W^2 - g_3 = 0$.

The two points $\widetilde{Q}_3, \widetilde{Q}_4$ lie on the pseudo line of the real part of the cubic curve, and the two points $\widetilde{Q}_1, \widetilde{Q}_2$ lie on the oval of the real part of the cubic curve. Under the elliptic curve group operation

$$(\mathcal{P}(u_1), \mathcal{P}'(u_1)) + (\mathcal{P}(u_2), \mathcal{P}'(u_2)) = (\mathcal{P}(u_1 + u_2), \mathcal{P}'(u_1 + u_2)),$$

the points $\widetilde{Q}_j$ satisfy

$$2\widetilde{Q}_1 = 2\widetilde{Q}_2 = 2\widetilde{Q}_3 = 2\widetilde{Q}_4 = \Big(\frac{128{,}405}{84{,}966}, \frac{169\sqrt{2}}{119}\Big).$$

We also have

$$2\Big(\frac{128{,}405}{84{,}966}, \frac{169\sqrt{2}}{119}\Big) = \Big(\frac{42{,}961}{42{,}483}, 0\Big).$$

Then we find that the point of the torus $\mathbb{C}/(2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z})$ corresponding to $2\widetilde{Q}_j$ is $3/2\omega_1$. Each difference $\widetilde{Q}_j - \widetilde{Q}_k$ $(j \neq k)$ satisfies

$$2(\widetilde{Q}_j - \widetilde{Q}_k) = 0$$

with respect to the elliptic curve group structure. By computing the tangent line passing through $\widetilde{Q}_j, \widetilde{Q}_k$, we find that

$$\begin{aligned}
\widetilde{Q}_2 - \widetilde{Q}_1 &= \widetilde{Q}_4 - \widetilde{Q}_3 = (\widetilde{U}_1, 0), \\
\widetilde{Q}_3 - \widetilde{Q}_2 &= \widetilde{Q}_4 - \widetilde{Q}_1 = (\widetilde{U}_2, 0), \\
\widetilde{Q}_3 - \widetilde{Q}_1 &= \widetilde{Q}_4 - \widetilde{Q}_2 = (\widetilde{U}_3, 0).
\end{aligned}$$

Using these relations, we find that the respective points on the torus $\mathbb{C}/(2\omega_1\mathbb{Z}+2\omega_2\mathbb{Z})$ and the normalized torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ are

$$Q'_1 = \frac{3}{4}\omega_1 + \omega_2, \quad Q'_2 = \frac{7}{4}\omega_1 + \omega_2, \quad Q'_3 = \frac{3}{4}\omega_1, \quad Q'_4 = \frac{7}{4}\omega_1,$$

and

$$Q''_1 = \frac{3}{8} + \frac{1}{2}\tau, \quad Q''_2 = \frac{7}{8} + \frac{1}{2}\tau, \quad Q''_3 = \frac{3}{8}, \quad Q''_4 = \frac{7}{8}.$$

Then the even Riemann theta functions $\theta_2$, $\theta_3$, $\theta_4$ on the normalized Jacobi variety $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ satisfy the equations

$$\begin{aligned}
&\theta_2(Q''_2 - Q''_1) = \theta_2(Q''_4 - Q''_3) = \theta_2\Big(\frac{1}{2}\Big) = 0, \\
&\theta_3(Q''_1 - Q''_4) = \theta_3\Big(-\frac{1}{2} + \frac{\tau}{2}\Big) = 0, \\
&\theta_3(Q''_2 - Q''_3) = \theta_3\Big(\frac{1}{2} + \frac{\tau}{2}\Big) = 0, \\
&\theta_4(Q''_1 - Q''_3) = \theta_4(Q''_2 - Q''_4) = \theta_4\Big(\frac{\tau}{2}\Big) = 0.
\end{aligned}$$

The elliptic functions $\theta_\delta/\theta_1$ over the normalized Jacobi variety take on the following approximate values at the points $Q''_j - Q''_k$:

$$\begin{aligned}
&\frac{\theta_2}{\theta_1}(Q''_3 - Q''_1) = \frac{\theta_2}{\theta_1}(Q''_4 - Q''_2) = \frac{\theta_2}{\theta_1}\Big(-\frac{\tau}{2}\Big) = 2.428{,}571{,}4\mathrm{i} \approx \frac{17}{7}\,\mathrm{i}, \\
&\frac{\theta_2}{\theta_1}(Q''_4 - Q''_1) = \frac{\theta_2}{\theta_1}(Q''_3 - Q''_2) = \frac{\theta_2}{\theta_1}\Big(\pm\frac{1}{2} - \frac{\tau}{2}\Big) = 0.411{,}764{,}71\mathrm{i} \approx \frac{7}{17}\,\mathrm{i}, \\
&\frac{\theta_3}{\theta_1}(Q''_2 - Q''_1) = \frac{\theta_3}{\theta_1}(Q''_4 - Q''_3) = \frac{\theta_3}{\theta_1}\Big(\frac{1}{2}\Big) = 0.414{,}778{,}33, \\
&\frac{\theta_3}{\theta_1}(Q''_3 - Q''_1) = \frac{\theta_3}{\theta_1}(Q''_4 - Q''_2) = \frac{\theta_3}{\theta_1}\Big(-\frac{\tau}{2}\Big) = 2.410{,}926{,}4\mathrm{i}, \\
&\frac{\theta_4}{\theta_1}(Q''_2 - Q''_1) = \frac{\theta_4}{\theta_1}(Q''_4 - Q''_3) = \frac{\theta_4}{\theta_1}\Big(\frac{1}{2}\Big) = 1.007{,}318{,}8, \\
&\frac{\theta_4}{\theta_1}(Q''_3 - Q''_2) = \frac{\theta_4}{\theta_1}\Big(-\frac{1}{2} - \frac{\tau}{2}\Big) = -0.992{,}734{,}38, \\
&\frac{\theta_4}{\theta_1}(Q''_4 - Q''_1) = \frac{\theta_4}{\theta_1}\Big(\frac{1}{2} - \frac{\tau}{2}\Big) = 0.992{,}734{,}38.
\end{aligned}$$

The numerical values of $\theta_1(0)$, $\theta_2(0)$, $\theta_3(0)$, $\theta_4(0)$ are given respectively by

$$3.693{,}259{,}2, \quad 1.411{,}753{,}8, \quad 1.422{,}086{,}2, \quad 0.585{,}564{,}89.$$

The values $\mathrm{d}(R_1/R_2)(Q'_j)$ are given numerically by

$$\mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q'_1) = \mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q'_2) = -\mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q'_3) = -\mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q'_4) = 1.414{,}213{,}6.$$

We then find that both the values

$$\frac{\theta'_1(0)}{2\omega_1\,\theta_2(0)} \frac{1}{\sqrt{\mathrm{d}\big(\frac{R_1}{R_2}\big)(Q'_1)}\sqrt{\mathrm{d}\big(\frac{R_1}{R_2}\big)(Q'_3)}}, \quad \frac{\theta'_1(0)}{2\omega_1\,\theta_2(0)} \frac{1}{\sqrt{\mathrm{d}\big(\frac{R_1}{R_2}\big)(Q'_1)}\sqrt{\mathrm{d}\big(\frac{R_1}{R_2}\big)(Q'_4)}},$$

are approximated by $-0.500{,}000{,}000\,\mathrm{i}$.

Now, for the main diagonals of $C$, it can be easily deduced from (2.2) that $c_{11} = c_{22} = c_{33} = c_{44} = 0$ for $\delta = 2, 3, 4$. We have the equations

$$\begin{aligned} \beta_3 - \beta_1 &= \frac{12}{13} - \frac{5}{13} = \frac{7}{13}, \quad \beta_4 - \beta_2 = -(\beta_3 - \beta_1) = -\frac{7}{13}, \\ \beta_4 - \beta_1 &= -\frac{12}{13} - \frac{5}{13} = -\frac{17}{13}, \quad \beta_3 - \beta_2 = -(\beta_4 - \beta_1) = \frac{17}{13}, \end{aligned}$$

and

$$\begin{aligned} &\Big(\frac{\theta_\delta}{\theta_1}\Big)(Q''_3 - Q''_1) = \Big(\frac{\theta_\delta}{\theta_1}\Big)(Q''_4 - Q''_2), \quad \delta = 2, 3, 4, \\ &\Big(\frac{\theta_\delta}{\theta_1}\Big)(Q''_4 - Q''_1) = \varepsilon_\delta \Big(\frac{\theta_\delta}{\theta_1}\Big)(Q''_3 - Q''_2), \end{aligned}$$

where $\varepsilon_2 = \varepsilon_3 = 1$, $\varepsilon_1 = -1$, and

$$\begin{aligned} &\sqrt{\mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q'_1)}\sqrt{\mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q'_3)} = \sqrt{\mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q'_2)}\sqrt{\mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q'_4)} \\ &= \sqrt{\mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q'_1)}\sqrt{\mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q'_4)} = \sqrt{\mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q'_2)}\sqrt{\mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q'_3)}. \end{aligned}$$

For $\delta = 2, 3$, we have that $c_{23} = -c_{14}$, $c_{24} = -c_{13}$.

Suppose that $\delta = 2$. Then

$$\theta_2(Q''_2 - Q''_1) = \theta_2(Q''_4 - Q''_3) = 0$$

and $c_{12} = c_{34} = 0$. This implies that $c_{24} = -c_{13}$, $c_{23} = -c_{13}$, and

$$\begin{aligned} c_{13} &= \frac{\mathrm{i}}{2} \times (\beta_3 - \beta_1) \times \frac{\theta_2}{\theta_1}(Q''_3 - Q''_1) = \frac{\mathrm{i}}{2} \times \frac{7}{13} \times \frac{17\,\mathrm{i}}{7} = -\frac{17}{26}, \\ c_{14} &= \frac{\mathrm{i}}{2} \times (\beta_4 - \beta_1) \times \frac{\theta_2}{\theta_1}(Q''_4 - Q''_1) = \frac{\mathrm{i}}{2} \times \Big(-\frac{17}{13}\Big) \times \frac{7\,\mathrm{i}}{17} = \frac{7}{26}. \end{aligned}$$

The formula (2.5) then produces a real symmetric matrix

$$C = \begin{bmatrix} 0 & 0 & -\frac{17}{26} & \frac{7}{26} \\ 0 & 0 & -\frac{7}{26} & \frac{17}{26} \\ -\frac{17}{26} & -\frac{7}{26} & 0 & 0 \\ \frac{7}{26} & \frac{17}{26} & 0 & 0 \end{bmatrix}$$

admitting the representation $F_S(x, y, z) = \det(zI_4 + yB + xC)$.

Suppose that $\delta = 3$. Then $\theta_3(Q_4'' - Q_1'') = \theta_3(Q_3'' - Q_2'') = 0$ and $c_{14} = c_{23} = 0$. By the relations

$$\frac{\theta_3}{\theta_1}(Q_4'' - Q_3'') = \frac{\theta_3}{\theta_1}(Q_2'' - Q_1'')$$

and

$$\sqrt{\mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q_1')}\sqrt{\mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q_2')} = \sqrt{2}, \quad \sqrt{\mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q_3')}\sqrt{\mathrm{d}\Big(\frac{R_1}{R_2}\Big)(Q_4')} = -\sqrt{2},$$

we have

$$c_{12} = -\frac{\beta_2 - \beta_1}{\beta_4 - \beta_3} c_{34} = -\frac{5}{12} c_{34}.$$

Numerical computation, yields that

$$\frac{\theta_1'(0)}{2\omega_1\,\theta_3(0)} \times \frac{\theta_3(Q_4'' - Q_3'')}{\theta_1(Q_4'' - Q_3'')} \times \frac{1}{\sqrt{\mathrm{d}\big(\frac{R_1}{R_2}\big)(Q_3')}\sqrt{\mathrm{d}\big(\frac{R_1}{R_2}\big)(Q_4')}} \approx -0.205{,}882{,}35,$$

which is approximately $-7/34$, and thus

$$c_{34} = (\beta_4 - \beta_3) \times \frac{7}{34} = \frac{13}{24} \times \frac{7}{34} = \frac{84}{221}.$$

We also have

$$\frac{\theta_1'(0)}{2\omega_1\,\theta_3(0)} \times \frac{\theta_3(Q_3'' - Q_1'')}{\theta_1(Q_3'' - Q_1'')} \times \frac{1}{\sqrt{\mathrm{d}\big(\frac{R_1}{R_2}\big)(Q_1')}\sqrt{\mathrm{d}\big(\frac{R_1}{R_2}\big)(Q_3')}} \approx -1.196{,}704{,}7,$$

which is approximately $-26\sqrt{30}/119$, and the value leads to

$$c_{13} = (\beta_3 - \beta_1) \times \Big(-\frac{26\sqrt{30}}{119}\Big) = -\frac{2\sqrt{30}}{17}.$$

The formula (2.5) produces another real symmetric matrix

$$C = \begin{bmatrix} 0 & -\frac{35}{221} & -2\frac{\sqrt{30}}{17} & 0 \\ -\frac{35}{21} & 0 & 0 & \frac{2\sqrt{30}}{7} \\ -\frac{2\sqrt{30}}{17} & 0 & 0 & \frac{84}{221} \\ 0 & \frac{2\sqrt{30}}{17} & \frac{84}{221} & 0 \end{bmatrix}$$

satisfying the representation $F(x, y, z) = \det(zI_4 + yB + xC)$.

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