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A NEW LOOK AT TOTALLY POSITIVE MATRICES

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Abstract. A close relationship between the class of totally positive matrices and anti-Monge matrices is used for suggesting a new direction for investigating totally positive matrices. Some questions are posed and a partial answer in the case of Vandermonde-like matrices is given.

Keywords: totally positive matrix; Monge matrix; semigroup; Vandermonde-like matrix

MSC 2010: 15B57, 15B48

1. Introduction

Totally positive matrices will mean matrices all of whose square submatrices have positive determinant. In this paper, we intend to investigate their relationship with another class of special matrices.

In [4] and [3], the present author studied the so called anti-Monge matrices, i.e. real, possibly rectangular matrices $[a_{ik}]$, for which

$$a_{ij} + a_{kl} \geqslant a_{il} + a_{kj}, \tag{1.1}$$

whenever i, j, k, l satisfy $i < k$ and $j < l$.

It was shown there that if such matrix is square, it can always be equilibrated, i.e., can be brought to the form that all row sums and all column sums are equal to

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Remark from the Editors: This paper was accepted in early summer of 2015, and its galleys were approved by Miroslav Fiedler on October 26 that year, less than a month before his death. Obviously this special issue of CMJ is the right place for this paper. It still fills us with great sadness that its author will not already see it.

zero, by adding to rows and to columns constant vectors (i.e., multiples of the row or column vector of all ones).

A basic result in [4] about equilibrated anti-Monge matrices of the same order is that they *form a multiplicative semigroup.*

We call anti-Monge matrices *strict* if there is always a strict inequality in (1.1).

2. Totally positive matrices and product-equilibration

It is easy to see that there is a close relationship between the class of strict anti-Monge matrices and totally positive matrices. We use the Hadamard (entrywise) logarithmic or exponential function. We shall use the abbreviation TP for totally positive matrices and TP_2 for 2-subtotally positive matrices (cf. [2] or [1]), i.e. positive matrices, the determinants of all 2×2 submatrices of which are also positive.

It is useful to introduce some notation. If A is a matrix, we denote by $\log^\circ A$, similarly $\exp^\circ A$, or $A^{\circ k}$, the Hadamard logarithm and Hadamard exponential, or the kth Hadamard power of A. The Hadamard product of two matrices A and B of the same size is denoted by $A \circ B$.

Clearly, the Hadamard logarithm of a TP_2-matrix X is a strict anti-Monge matrix since all 2×2 submatrices of $\log^\circ X$ are positive. Conversely, the Hadamard exponential of a strict anti-Monge matrix is a TP_2- matrix. In this relationship, equilibrated anti-Monge matrices correspond to *product-equilibrated* TP-matrices defined below. We first formulate a simple theorem the proof of which is analogous to that of Lemma 2.3 in [3].

Theorem 2.1. *If X is an $n \times n$ positive matrix, then there exist diagonal matrices D_1 and D_2 with positive diagonal entries, such that the matrix*

$$\widetilde{X} = D_1 X D_2 \tag{2.1}$$

has products in all rows and in all columns equal to one.

Definition 2.2. We call a square positive matrix *product-equilibrated* if all row- and all column-products are equal to one. We denote by P_n the class of $n \times n$ such positive matrices.

Remark 2.3. The result of forming the product-equilibrated matrix $\widetilde{X}$ from the given positive matrix X as in Theorem 2.1 is unique although the diagonal matrices D_1 and D_2 are not uniquely determined.

For completeness, we present a simple procedure for finding the diagonal matrices D_1 and D_2 in (2.1). Let $X = [x_{ij}] \in P_n$. Denote by μ the positive square root of the geometric mean of all entries in X and define, for $i, j = 1, 2, \ldots, n$, the numbers $y_i = \prod_{k=1}^{n} x_{ik}$, $z_j = \prod_{k=1}^{n} x_{kj}$. Then the diagonal matrices $D_1 = \operatorname{diag}(\mu y_1^{-1/n}, \mu y_2^{-1/n}, \ldots, \mu y_n^{-1/n})$, $D_2 = \operatorname{diag}(\mu z_1^{-1/n}, \mu z_2^{-1/n}, \ldots, \mu z_n^{-1/n})$ satisfy (2.1).

Lemma 2.4. *The Hadamard (entrywise) product of matrices in P_n is also in P_n. The same holds for Hadamard division. The class P_n contains a distinguished matrix J_n of all ones.*

Let us add two observations about matrices in P_n:

Lemma 2.5. *Let $X \in P_n$. If Y is the Hadamard inverse matrix of X, then Y is also in P_n.*

Lemma 2.6. *If $A \in P_n$, then $AJ_n \geqslant nJ_n$ as well as $J_nA \geqslant nJ_n$.*

Proof. Follows from the A-G inequality in the rows and columns. □

Observe that the product-equilibrated matrices in TP_2 also belong to TP_2. The theorem in Introduction about the multiplicative semigroup of equilibrated anti-Monge matrices corresponds then to the following:

Theorem 2.7. *Let X_1 and X_2 be product-equilibrated* TP_2*-matrices of the same order. Then there exists a unique* TP_2*-matrix X_3 of the same order, such that for the Hadamard logarithms,*

$$\log^\circ X_3 = (\log^\circ X_1)(\log^\circ X_2).$$

The matrix X_3 is then also product-equilibrated.

The following question then arises:

Question 1. Is it true that if in Theorem 2.7 both the matrices X_1 and X_2 are product-equilibrated TP-matrices, then the matrix X_3 is also a TP-matrix (in that case also product-equilibrated)?

3. e-multiplication

Let X_1, X_2 be positive matrices of the same order. Form the matrix X_3 as the Hadamard exponential function of the matrix obtained by the usual multiplication of the Hadamard logarithms $(\log^\circ X_1)(\log^\circ X_2)$:

$$X_3 = \exp^\circ((\log^\circ X_1)(\log^\circ X_2)).$$

We shall call the operation in the class of square positive matrices of a fixed order which assigns to matrices X_1 and X_2, in this order, the matrix X_3 as described, *operation of* e*-multiplication*. The result will be called the e*-product*, denoted by $X_1 \square X_2$.

It is clear that e-multiplication is associative but apparently in general not commutative. For us, the following is important:

Lemma 3.1. *The operation of* e*-multiplication preserves positivity of matrices as well as the product-equilibration property.*

Proof. Let X_1, X_2 be positive matrices of the same order. The matrices $\log^\circ X_1$ and $\log^\circ X_2$ are then real matrices which can be multiplied and the matrix $X_1 \square X_2 = \exp^\circ((\log^\circ X_1)(\log^\circ X_2))$ is positive.

If both X_1 and X_2 are product-equilibrated, then both $\log^\circ X_1$ and $\log^\circ X_2$ are real equilibrated matrices. This means, if e is the column vector of all ones, that $(\log^\circ X_1)\mathrm{e} = 0$, $(\log^\circ X_2)\mathrm{e} = 0$, $\mathrm{e}^{\mathrm{T}} \log^\circ X_1 = 0$, $\mathrm{e}^{\mathrm{T}} \log^\circ X_1 = 0$, which implies that the same is true for their product $(\log^\circ X_1)(\log^\circ X_2)$. Thus $X_1 \square X_2$ which is the Hadamard exponential of $(\log^\circ X_1)(\log^\circ X_2)$ is product-equilibrated. □

Using the notation of P_n, Lemma 3.1 can be formulated as the following implication: $X_1 \in P_n$ and $X_2 \in P_n$ implies $X_1 \square X_2 \in P_n$.

The transpose matrix A^{T} to an equilibrated anti-Monge matrix A is also an equilibrated anti-Monge matrix and the symmetric mean $(A + A^{\mathrm{T}})/2$ as well. This property can be transformed to the class P_n as follows:

If $X \in P_n$, then X^{T} is also in P_n. Their e-product is the Hadamard exponential of a positive semidefinite matrix which is also positive semidefinite. Thus the positive semidefinite square root of $X \square X^{\mathrm{T}}$ is in P_n as well and can be viewed as the corresponding e*-symmetric mean*.

It is well known that square $n \times n$ TP-matrices form a class closed with respect to multiplication. By Lemma 3.1, the same is true for e-multiplication of square product-equilibrated TP-matrices.

It is not the purpose of this note to build the theory of e-multiplication of matrices in P_n. It would be good to have answers to several further open questions.

Question 2. Is the e-symmetric mean of a TP-matrix in P_n also in TP?

Question 3. Is it true that there exists a general e-power of a TP matrix with exponent greater than one? If so, is it TP?

Question 4. If we define, in addition, the e-sum of two positive matrices X_1 and X_2 as the matrix C for which $C = \exp^\circ(\log^\circ X_1 + \log^\circ X_2)$, i.e., $C = X_1 \circ X_2$, we obtain a maybe interesting non-commutative e-algebra over the set of positive matrices. What are its properties?

4. Vandermonde-like matrices

To find an example for considering partial answers to Questions in Section 2 and 3, let us investigate the case of *Vandermonde-like matrices*, i.e., matrices of the form $X = [x_i^{n-k}]$, $i, k = 1, 2, \ldots, n$, where $x_1 > x_2 > \ldots > x_n > 0$, and their multiples by diagonal matrices with positive diagonal entries from both sides. Such matrix is well known [1] to be a TP-matrix. Suppose that in the matrix X the product $x_1 x_2 \ldots x_n$ equals one; then the column-products are all one, and if we multiply X by the diagonal matrix $D = \operatorname{diag}(x_1^{(-1/2)(n-1)}, x_2^{(-1/2)(n-1)}, \ldots, x_n^{(-1/2)(n-1)})$ from the left, the resulting matrix $\widetilde{X}$ will also be a TP-matrix, this time even product-equilibrated since the row-products are one as well.

Definition 4.1. Denote by TP_0 the class of $n \times n$ matrices formed as matrices $\widetilde{X}$. If $x_1 > x_2 > \ldots > x_n$ satisfying $x_1 x_2 \ldots x_n = 1$ are the numbers generating such $\widetilde{X}$, we denote $\widetilde{X} = V[x_1, x_2, \ldots, x_n]$. We call this class TP_0 the class of *Vandermonde-like product-equilibrated matrices.*

Thus matrices in TP_0 are exactly matrices of the form $V[x_1, x_2, \ldots, x_n]$.

It is easy to see:

Theorem 4.2. *The class* TP_0 *of Vandermonde-like product-equilibrated matrices is closed with respect to forming Hadamard products as well as Hadamard powers with real exponent greater than zero.*

We are able to show:

Theorem 4.3. *The operation of* e-*multiplication preserves the class* TP_0.

P r o o f. Suppose $X = V[x_1, x_2, \ldots, x_n]$ and $Y = V[y_1, y_2, \ldots, y_n]$ belong to TP_0. The matrix $\log^\circ X$ is easily computed as the product $D_x e v^{\mathrm{T}}$, where $D_x = \operatorname{diag}(\log x_1, \log x_2, \ldots, \log x_n)$, e is the column vector of all ones and $v^{\mathrm{T}} = [(n-1)/2, (n-3)/2, \ldots, -(n-3)/2, -(n-1)/2]$. Similarly, $\log Y = D_y e v^{\mathrm{T}}$ with $D_y = \operatorname{diag}(\log y_1, \log y_2, \ldots, \log y_n)$ and the same vectors e and v^{T}. The e-product Z of X and Y is thus

$$\begin{aligned} Z &= \exp^\circ((\log^\circ X)(\log^\circ Y)) = \exp^\circ((D_x e v^{\mathrm{T}})(D_y e v^{\mathrm{T}})) = \exp^\circ((v^{\mathrm{T}} D_y e) D_x e v^{\mathrm{T}}) \\ &= \exp^\circ((v^{\mathrm{T}} D_y e) \log^\circ X) = X^{\circ w}, \end{aligned}$$

where $w = (v^{\mathrm{T}} D_y e)$. Since $w > 0$, $Z \in \mathrm{TP}_0$ by Theorem 4.2. In addition, the Hadamard powers are also equilibrated. □

Remark 4.4. In this case, e-multiplication is even commutative.

Corollary 4.5. *In the class of* TP_0*-matrices,* e*-multiplication is the usual Hadamard multiplication.*

Theorems 4.2 and 4.3, together with Remark 4.4, raise many questions.

Question 5. If we denote by $\widetilde{\mathrm{TP}}_n$ the class of $n \times n$ product-equilibrated TP-matrices, is $\widetilde{\mathrm{TP}}_n$ also closed with respect to e-multiplication? Is this multiplication even commutative?

One can easily answer the last question negatively. Indeed, the class $\mathrm{TP}_0^{\mathrm{T}}$ of transpose matrices to product-equilibrated Vandermonde-like matrices has, of course, properties analogous to TP_0. For $n > 2$, there are examples for which the e-multiplication having one matrix in TP_0 and the other in $\mathrm{TP}_0^{\mathrm{T}}$ does not commute.

Observe that all diagonal entries of matrices in TP_0 are greater than or equal to one. This is no longer true for matrices in $\widetilde{\mathrm{TP}}_n$ for $n > 3$. This shows that the example of Vandermonde-like product-equilibrated matrices is very special. Apparently, the answer to Question 1 is negative and one has to enlarge the class of product-equilibrated TP-matrices by constructing some envelope for which this question has an affirmative answer.

There are other simple classes of TP-matrices which deserve an investigation similar to that we did for the Vandermonde-like matrices, e.g. TP Cauchy matrices. These are matrices of the form $[1/(x_i + y_k)]$, where the numbers $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$, in the case of order n, satisfy $0 < x_1 < x_2 < \ldots < x_n$, $0 < y_1 < y_2 < \ldots < y_n$.

One could ask many questions about TP-matrices, such as about commutativity, other canonical forms, etc.

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