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On Weil Bundles of the First Order

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Abstract

The descriptions of Weil bundles, lifts of functions and vector fields are given. Actions of the automorphisms group of the Whitney sum of algebras of dual numbers on a Weil bundle of the first order are defined.

Key words: Weil bundles, Weil algebra, smooth manifolds.

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1 Basic concepts

Any manifold M_n bellow is supposed to be connected, n-dimensional and C^∞. We denote the algebra of smooth real-valued functions of class C^∞ given on the manifold M_n by $C^\infty(M_n)$. The works of A. P. Shirokov and V. V. Shurygin, V. V. Shurygin (Jr.), A. Morimoto, I. Kolár and other authors [1, 2] are devoted to the study of Weil bundles. The definition of the Weil algebra is given.

Weil bundles are constructed by means of smooth manifolds and Weil algebras. We state the definition of a Weil algebra as follows:

Definition 1 The finite-dimensional linear algebra $\mathbb{A}$ over the field of real numbers $\mathbb{R}$ is called a Weil algebra if the following conditions are fulfilled:

(1) the algebra $\mathbb{A}$ is commutative, associative, has a unit;

(2) there is an ideal $\mathbb{I}$ such that $\mathbb{I}^p \neq \{0\}$, and $\mathbb{I}^{p+1} = \{0\}$;

(3) the factoralgebra $\mathbb{A}\backslash\mathbb{I}$ is isomorphic to the algebra of real numbers $\mathbb{R}$.

Following the recent definition we introduce the concepts of height and width of a Weil algebra putting height($\mathbb{A}$) $= p$ and width($\mathbb{A}$) $= \dim(\mathbb{I}/\mathbb{I}^2)$. It follows from the condition (3) of the previous definition that any $a \in A$ can be represented only as a sum $a = a_0\delta + b$, where δ is the unit of $\mathbb{A}$ and $b \in \mathbb{I}$. The set of the elements of the form $a_0\delta$ forms a subalgebra isomorphic to $\mathbb{A}$. Analogously,

$\mathbb{A}$ is isomorphic to the half-line sum of $\mathbb{R} + \mathbb{I}$. Therefore the unit of $\mathbb{A}$ can be identified with unit of $\mathbb{R}$.

We select a basis in $\mathbb{A}$ so that $\varepsilon^0 = 1$, $\varepsilon^\alpha \in \mathbb{I}(\alpha = 1, 2, \dots, m)$. We denote the structural constants of algebras $\mathbb{A}(\sigma, \tau, \nu = 0, 1, 2, \dots, m)$ by $\gamma_\nu^{\sigma\tau}$. According to condition (2) of definition 1 there is at least one product type $\varepsilon^{\alpha_1}\varepsilon^{\alpha_2}\dots\varepsilon^{\alpha_p}(\alpha_1, \alpha_2, \dots, \alpha_p = 1, 2, \dots, m)$, which is nonzero, and the product of any $p+1$ of the elements of the basis, taken from the ideal $\mathbb{I}$, is equal 0.

Following A. Weil we say that the homomorphism $j_q \colon \mathrm{C}^\infty(M_n) \to \mathbb{A}$ is an $\mathbb{A}$-close point-to-point $q \in M_n$, if $j_q(f) = f(q)(\mathrm{mod}\mathbb{I})$ for all $f \in \mathrm{C}^\infty(M_n)$. Many various $\mathbb{A}$-close points to the point $q \in M_n$ are denoted by $(M_n)_q^\mathbb{A}$ and a combination

$$\bigcup_{q \in M_n} (M_n)_q^\mathbb{A} = M_n^\mathbb{A}$$

is constructed. The mapping $\pi \colon M_n^\mathbb{A} \to M_n$ defined by the condition $\pi(j_q) = q$ is called a canonical projection.

A smooth C^∞-structure generated by that of the manifold M_n appears on $M_n^\mathbb{A}$ as well.

Definition 2 Let $\mathbb{A}$ be a Weil algebra satisfying width$(\mathbb{A}) = p$. Then the triple $(M_n^\mathbb{A}, \pi, M_n)$ is said to be a Weil bundle of order p.

In a natural way, we define the A-continuations of functions from $\mathrm{C}^\infty(M_n)$ to functions from the algebra $\mathrm{C}^\infty(M_n^\mathbb{A})$, which are holomorphic over the Weil algebra $\mathbb{A}$ in the sense of Sheffers.

Definition 3 For any $f \in \mathrm{C}^\infty$, the function $f \circ \pi$ is said to be its the vertical lift and the function $f^\mathbb{A}$ satisfying the condition $f^\mathbb{A}(j_q) = j_q(f)$ for each point $q \in M_n$, is called its A-continuation.

The space of $\mathbb{R}$-valued linear forms is denoted by $\mathbb{A}^*$. Then the composition $a^* \circ f^\mathbb{A}$, where $a^* \in \mathbb{A}^*$ is a real-valued function defined on $M_n^\mathbb{A}$. We introduce the designation: $f_{(a^*)} = a^* \circ f^\mathbb{A}$. We will use the following notation: $f_{(\varepsilon_0)} = f_{(0)}$, $f_{(\varepsilon_\alpha)} = f_{(\alpha)}$ $(\alpha = 1, 2, \dots, m)$ for linear forms $\varepsilon_0, \varepsilon_\alpha(\alpha = 1, 2, \dots, m)$, the dual basis to the basis $\varepsilon^0, \varepsilon^\alpha(\alpha = 1, 2, \dots, m)$. Let us note that $f_{(0)} = f \circ \pi$ [4].

2 Lifts of functions and vector fields to the first-order Weil bundles

The Weil bundles of the first order $(M_n^\mathbb{A}, \pi, M_n)$ are constructed by using the Weil algebra $\mathbb{A}$ of height 1. Let the weight of the algebra $\mathbb{A}$ be equal m,

The construction of the first-order Weil bundles is based on Weil algebras of height equal to 1. Let height$(\mathbb{A}) = m$ and $\varepsilon^1, \varepsilon^2, \dots, \varepsilon^m$ be elements of the pseudobasis, $\varepsilon^0 = 1$ be a unit of this algebra. Then $\mathbb{I} = \{0\}$, therefore $\varepsilon^\alpha\varepsilon^\beta = 0$ for all $\alpha, \beta = 1, 2, \dots, m$. This implies that the algebra $\mathbb{A}$ is isomorphic to the Whitney sum of m copies of the algebra of dual numbers $D = \{a + b\varepsilon \mid a, b \in$

$\mathbb{R}, \varepsilon^2 = 0\}$. The total space $M_n^{\mathbb{A}}$ will be a Whitney sum of m copies of tangent bundles $T(M_n)$.

The system of elements $\varepsilon_0 = 1, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ forms a basis of the algebra $\mathbb{A}$. The elements of the dual basis in the space $\mathbb{A}^*$ will be denoted in terms of $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$.

Let $f \in \mathrm{C}^\infty(M_n)$. The function $f_{(0)} \in \mathrm{C}^\infty(M_n^{\mathbb{A}})$, defined by the condition $f_{(0)} = f \circ \pi$ is called a vertical lift of the function f. The correlations

$$(f+g)_{(0)} = f_{(0)} + g_{(0)}, (\lambda f)_{(0)} = \lambda f_{(0)}, (fg)_{(0)} = f_{(0)} g_{(0)}$$

are obvious for any $f, g \in \mathrm{C}^\infty(M_n)$ and $\lambda \in \mathbb{R}$. In other words, the mapping $(0)\colon f \mapsto f_{(0)}$ is a homomorphism $\mathrm{C}^\infty(M_n)$ to the algebra $\mathrm{C}^\infty(M_n^{\mathbb{A}})$. For the linear form $a^* \in \mathbb{A}^*$ the condition $(fg)_{(a^*)} = f_{(a^*)} g_{(a^*)}$ for any $f, g \in \mathrm{C}^\infty(M_n)$ is fulfilled if and only if $a^* = 0$ or $a^* = \varepsilon_0$. To prove this, we can use a local representation of the functions $f_{(a^*)}$, obtained in the following way.

Let (U, x^i) be a map of the smooth atlas on M_n, $(\pi^{-1}(U), x^i_\alpha)$ be a map of the smooth atlas on $M_n^{\mathbb{A}}$, where $x^i_\alpha = \varepsilon_\alpha \circ (x^i)^{\mathbb{A}}$. Then for any function $f \in \mathrm{C}^\infty(M_n)$, its prolongation $f^{\mathbb{A}}$ in local coordinates is of the form

$$f^{\mathbb{A}} = f_{(0)} \varepsilon^0 + (\partial_j f)_{(0)} x^j_\lambda \varepsilon^\lambda$$

(the sum is considered with respect to λ). Therefore, for any element $a^* = a^0 \varepsilon_0 + a^\lambda \varepsilon_\lambda$ from $\mathbb{A}^*$, we get

$$f_{(a^*)} = a^0 f_{(0)} + a^\lambda (\partial_j f)_{(0)} x^j_\lambda.$$

Let X be a vector field defined on M_n, $a \in \mathbb{A}$, $X = X^i \frac{\partial}{\partial x^i}$ be its local representation. The only vector field $X^{(a)}$ on $M_n^{\mathbb{A}}$, satisfying

$$X^{(a)} f_{(b^*)} = (Xf)_{(b^* \cdot a)}$$

for all $f \in \mathrm{C}^\infty(M_n)$ is called an (a)-lift of the vector field X from M_n to $M_n^{\mathbb{A}}$.

If $a = \varepsilon^\alpha$, then we will write $X^{(\alpha)} (\alpha = 0, 1, \ldots, m)$ instead of $X^{(\varepsilon^\alpha)}$. The vector field $X^{(0)}$ is called a complete lift of the vector field X. In local coordinates we have

$$X^{(0)} = (X^i)_{(0)} \partial^0_i + (\partial_j X^i)_{(0)} x^j_\lambda \partial^\lambda_i (\lambda \neq 0),$$

where $\partial^\alpha_i = (\partial_i)^{(\alpha)} (\alpha = 0, 1, 2, \ldots, m)$. As for the vector field $X^{(\lambda)}$, we have set in local coordinates as follows $X^{(\lambda)} = (X^i)_{(0)} \partial^\lambda_i$. For any $a, b \in \mathbb{A}$ and vector fields X, Y on M_n we have the equality

$$[X^{(a)}, Y^{(b)}] = [X, Y]^{(ab)}.$$

3 Action of the automorphism group of a Weil algebra of height 1

Definition 4 The automorphism of the algebra $\mathbb{A}$ is a non-degenerate linear operator $\varphi\colon \mathbb{A} \to \mathbb{A}$, satisfying $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathbb{A}$.

The set of all automorphisms of the algebra $\mathbb{A}$ forms a group with respect to the composition of the operators. This group is denoted by $\mathrm{Aut}\,\mathbb{A}$. In our case, the algebra $\mathbb{A}$ is isomorphic to the Whitney sum of m copies of the algebra of dual numbers: $\mathbb{A} = \{a_0 + a_\lambda \varepsilon^\lambda | a_0, a_\lambda \in \mathbb{R}, \varepsilon^\lambda \varepsilon^\mu = 0\}, (\lambda, \mu = 1, 2, \ldots, m)$.

For any automorphism $\varphi \in \mathrm{Aut}\,\mathbb{A}$, we have

$$\varphi(1) = 1, \varphi(\varepsilon^\lambda) = \varphi^\lambda_\mu \varepsilon^\mu,$$

$(\lambda, \mu = 1, 2, \ldots, m$, the sum is considered with respect to $\mu)$. The matrix of the automorphisms φ is of the form

$$M(\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & B_\varphi \end{pmatrix}, \tag{3.1}$$

where $B_\varphi = \varphi^\lambda_\mu \in GL(m, \mathbb{R})$. Conversely, every linear operator $\varphi\colon \mathbb{A} \to \mathbb{A}$, having a matrix (3.1) in reference to the basis $(\varepsilon^0, \varepsilon^\lambda)$ is an automorphism of the algebra $\mathbb{A}$. Therefore, the dimension of the group $\mathrm{Aut}\,\mathbb{A}$ is equal m^2.

We define the operation of multiplication on $\mathrm{Aut}\,\mathbb{A}$ putting $\varphi\psi = \psi \circ \varphi$ for any $\varphi, \psi \in \mathrm{Aut}\,\mathbb{A}$. Let the upper indices be formed by line numbers while the lower ones by the column numbers. For the product $\varphi\psi$ we will have $\varphi\psi(1) = 1$, $\varphi\psi(\varepsilon^\lambda) = \psi(\varphi(\varepsilon^\lambda)) = \psi(\varphi^\lambda_\mu \varepsilon^\mu) = (\varphi^\lambda_\mu \psi^\mu_\nu)\varepsilon^\nu (\lambda, \mu, \nu = 1, 2, \ldots, m)$. It follows that

$$B_{\varphi\psi} = B_\varphi B_\psi.$$

Therefore the mapping $B\colon \mathrm{Aut}\,\mathbb{A} \to GL(m, \mathbb{R})$, defined by the condition $B(\varphi) = B\varphi$ is an isomorphism.

We define the action of the group of automorphisms $\mathrm{Aut}\,\mathbb{A}$ on the bundle $M_n^{\mathbb{A}}$ according to the rule

$$(\varphi, j_q) \mapsto j_p\varphi = \varphi \circ j_p.$$

Then for any automorphisms $\varphi, \psi \in \mathrm{Aut}\,\mathbb{A}$ we obtain

$$j_p(\varphi\psi) = (\varphi\psi) \circ j_p = (\psi \circ \varphi) \circ j_p = \psi \circ (\phi \circ j_p) = \psi \circ (j_p\varphi) = (j_p\varphi)\psi.$$

It follows that the inducted action of the group $\mathrm{Aut}\,\mathbb{A}$ on the bundle $M_n^{\mathbb{A}}$ is right. This action is effective. Let us prove this statement. Let j_q be an arbitrary point of the manifold $M_n^{\mathbb{A}}$, (U, x^i) be a map on M_n such that $p \in U$. In the map $(\pi^{-1}(U), x^i_\alpha)$ the automorphism φ generates conversion $\Phi\colon \pi^{-1}(U) \to \pi^{-1}(U)$, defined by the condition $\Phi(j_p) = j_p\varphi$. Let $j_p\varphi(x^i) = y^i_0(j_p\varphi) + y^i_\lambda(j_p\varphi)\varepsilon^\lambda$. On the other hand,

$$j_p\varphi(x^i) = \varphi(\jmath_p(x^i)) = \varphi(x^i(p) + x^i_\lambda(j_p)\varepsilon^\lambda = x^i_0(j_p) + x^i_\lambda(j_p)\varphi^\lambda_\mu \varepsilon^\mu.$$

Therefore, the transformation Φ in local coordinates is defined by

$$y^i_0 = x^i_0, y^i_\lambda = \varphi^\mu_\lambda x^i_\mu (\lambda, \mu = 1, 2, \ldots, m), \tag{3.2}$$

and $det\|\varphi^\mu_\lambda\| \neq 0$.

Let $j_p\varphi = j_p$ for all $j_P \in M_n^{\mathbb{A}}$. Then $p_\lambda^i = \varphi_\lambda^\mu p_\mu^i$ for all p_λ^i, where $p_\lambda^i = x_\lambda^i(j_p)$ $(\lambda, \mu = 1, 2, \ldots, m)$.

Then, $\varphi_\lambda^\mu = \delta_\lambda^\mu$ and consequently, $\varphi = \mathrm{id}_{\mathbb{A}}$.

We find the infinitesimal transformations of the group of transformations generated by the action of the group $\operatorname{Aut}\mathbb{A}$ applying the formula (3.2):

$$E_\mu^\lambda = x_\mu^i \partial_i^\lambda \quad (\lambda, \mu = 1, 2, \ldots, m).$$

The following identities

$$[E_\mu^\lambda, E_\tau^\sigma] = \delta_\tau^\lambda E_\mu^\sigma - \delta_\mu^\sigma E_\tau^\lambda,$$

$$[E_\mu^\lambda, X^{(0)}] = 0,$$

$$[E_\mu^\lambda, X^{(\sigma)}] = -\delta_\mu^\sigma X^{(\lambda)} \quad (\lambda, \mu, \sigma = 1, 2, \ldots, m)$$

are related with the infinitesimal transformation E_μ^λ.

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