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ON SOME NEW SHARP EMBEDDING THEOREMS IN MINIMAL AND PSEUDOCONVEX DOMAINS

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Abstract. We present new sharp embedding theorems for mixed-norm analytic spaces in pseudoconvex domains with smooth boundary. New related sharp results in minimal bounded homogeneous domains in higher dimension are also provided. Last domains we consider are domains which are direct generalizations of the well-studied so-called bounded symmetric domains in $\mathbb{C}^n$. Our results were known before only in the very particular case of domains of such type in the unit ball. As in the unit ball case, all our proofs are heavily based on nice properties of the r-lattice. Some results of this paper can be also obtained in some unbounded domains, namely tubular domains over symmetric cones.

Keywords: embedding theorem; minimal domain; pseudoconvex domain; Bergman-type space

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1. Introduction

The theory of analytic spaces on general admissible domains has been well-developed by various authors during last decades (see [3], [6], [8], [10], [12], [14], [15], [16], [20], [23], [26], [29], [30] and various references there). In this partially expository paper we will turn to the study of certain new embedding theorems for some new mixed norm analytic classes in strictly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary. We add such type sharp theorems also in other domains based on same ideas (bounded symmetric domains and their direct generalizations). In this paper also we extend some theorems from [17] and [18] where they can be seen in context of less general unit ball. Proving estimates and embedding theorems in pseudoconvex domains with smooth boundary we heavily use the technique which was

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developed recently in [2], [1]. For similar results in bounded symmetric domains and their direct generalizations we are based on a series of recent subtle results of Yamaji (see [32], [33] and various references there). Note that pseudoconvex domains with smooth boundary are not symmetric, tubular domains are not bounded. Minimal bounded homogeneous domains serve as direct extensions of bounded symmetric domains (see [32], [33] and references there).

The motivation of this paper was to provide some sharp embedding theorems from [30] in more general form and also add new results and discussions on embedding theorems in pseudoconvex domains making the picture more complete. New related results on sharp embeddings on other domains will be also presented in this paper. Untill now there are only several sharp embedding theorems in analytic function spaces in domains with complex structure in higher dimension.

Proofs of the last theorems in minimal domains repeat the proofs of Theorems 3.1–3.3 and they will be omitted.

In our embeddings theorems for analytic function spaces in pseudoconvex domains with smooth boundary and minimal bounded homogeneous domains the so-called Carleson type measures constantly appear. We turn to some history related to this problem. Carleson measures were introduced by Carleson [5] in his solution of the corona problem in the unit disk of the complex plane, and, since then, have become an important tool in analysis, and an interesting object of study per se. Let A be a Banach space of analytic functions on a domain $D \subset \mathbb{C}^n$. Given $p \geqslant 1$, a finite positive Borel measure μ on D is a *Carleson measure* of A (for p) if there is a continuous inclusion $A \hookrightarrow L^p(\mu)$, that is there exists a constant $C > 0$ such that

$$\int_D |f(z)|^p \,\mathrm{d}\mu(z) \leqslant C\|f\|_A^p, \quad f \in A.$$

A finite positive Borel measure μ is a Carleson measure of $H^p(\Delta)$ Hardy space if and only if there exists a constant $C > 0$ such that $\mu(S_{\theta_0,h}) \leqslant Ch$ for all sets

$$S_{\theta_0,h} = \{r\mathrm{e}^{\mathrm{i}\theta} \in \Delta\colon\ 1 - h \leqslant r < 1,\ |\theta - \theta_0| < h\}$$

(see, also, [5], [22]). The set of Carleson measures of $H^p(\Delta)$ does not depend on p.

In [11] (see also [21] and [22] for a result of such type) the author obtained a similar description for the Carleson measures of the Bergman spaces $A^p(\Delta)$. It was obtained in terms of the special sets $S_{\theta_0,h}$. In [7] the authors characterized Carleson measures for Bergman spaces in the unit ball $B^n \subset \mathbb{C}^n$, and Cima and Mercer [6] found description of Carleson measures of Bergman spaces in strongly pseudoconvex domains with smooth boundary, showing in particular that the set of Carleson measures of $A^p(D)$ is independent of $p \geqslant 1$. We turn to more details. In [7] a characterization of Carleson measures of Bergman spaces is formulated in terms of more general sets than $S_{\theta_0,h}$. We will use the one expressed via the intrinsic Kobayashi geometry of the domain. Let $z_0 \in D$ and $0 < r < 1$, let $B_D(z_0, r)$ denote the ball of center z_0 and radius $\frac{1}{2}\log((1+r)/(1-r))$ for the Kobayashi distance k_D of D (that is, of radius r with respect to the pseudohyperbolic distance $\varrho = \tanh(k_D)$; see Section 2 for the necessary definitions). It is known (see [19] for $D = \Delta$, [34] for $D = B^n$, and [2], [1] for D strongly pseudoconvex) that a finite positive measure μ is a Carleson measure of $A^p(D)$ for p if and only if for some (and hence all) $0 < r < 1$ there is a constant $C_r > 0$ such that

$$\mu(B_D(z_0, r)) \leqslant C_r \nu(B_D(z_0, r))$$

for all $z_0 \in D$. The proof of this we see in [2] relied on Cima and Mercer's characterization (see also [6]).

We say that a finite positive Borel measure μ is a (geometric) θ-*Carleson measure* if for some (and hence all) $0 < r < 1$ there is a constant $C_r > 0$ such that

$$\mu(B_D(z_0, r)) \leqslant c_r \nu(B_D(z_0, r))^\theta$$

for all $z_0 \in D$. Note a 1-Carleson measures are the usual Carleson measures of $A^p(D)$, and we know in pseudoconvex domains (see [2], [1]) that θ-Carleson measures are exactly the Carleson measures of weighted Bergman spaces. Note also that when $D = B^n$, a q-Carleson measure in the sense of [34] is a $(1 + q/(n+1))$-Carleson measure in our sense. We refer the reader to [9] and various references there for various (not only sharp) embedding theorems and related results in case of polydisk for analytic Bergman type and Besov type spaces in higher dimension and for various related mixed norm spaces of harmonic functions of several variables.

In this paper we are however more interested in Carleson type measures for some new analytic Bergman type mixed norm spaces in strongly pseudoconvex domains with smooth boundary. Note the literature concerning various one dimensional embeddings is very large. In recent papers of Yamaji (see [32], [33] and references there) new subtle estimates from below for the Bergman kernel and weighted Bergman kernel (see definitions below) on balls forming r-lattices (and some other nice properties of r-lattices) were provided in context of bounded minimal homogeneous domains. We will use them to get complete analogues of some of our theorems formulated below, in context of pseudoconvex domains with smooth boundary in minimal bounded homogeneous domains. Similarly to pseudoconvex domains with smooth boundary in minimal bounded homogeneous domains some sharp Carleson type embeddings for Bergman type spaces and mixed norm spaces will be also fully characterized in terms of Carleson type measures of minimal bounded homogeneous domains (see definitions of Carleson type measures for these domains below). Note these two scales of complicated domains in higher dimension are different. The minimal bounded homogeneous domains can be viewed as direct extensions of bounded symmetric domains, while bounded pseudoconvex domains with smooth boundary generally speaking are not even symmetric (see [32], [33] and various references there).

Throughout this paper constants are denoted by C and C_i, $i \in \mathbb{N}$ or by C with other indexes. They are positive and need not be the same at each occurrence.

2. Preliminaries on geometry of strongly pseudoconvex domains with smooth boundary and minimal boounded homogeneous domains

In this section we provide a chain of facts, properties and estimates on the geometry of strongly convex domains which we will use heavily in all our proofs below. Practically all of them are taken from recent interesting papers of Abate and coauthors (see [2], [1]). In particular, following these papers we provide several results on the boundary behavior of Kobayashi balls, and formulate a vital submean property for nonnegative plurisubharmonic functions in Kobayashi balls. Then at the end of this section we will also add some basic notation taken from recent papers of S. Yamaji to formulate our sharp embedding theorems for analytic mixed norm spaces in minimal bounded homogeneous domains in higher dimension. These assertions are complete analogues of our lemmas below in context of bounded pseudoconvex domains with smooth boundary. Some related results, lemmas will be also given in this section to make the picture more complete. We now first recall the standard definition and the main properties of the Kobayashi distance which can be seen in various books and papers (we refer, for example, to [2], [1], [13], [14] for details). Let k_Δ denote the Poincaré distance on the unit disk $\Delta \subset \mathbb{C}^n$. If X is a complex manifold, the Lempert function $\delta_X\colon X \times X \to \mathbb{R}^+$ of X is defined by

$$\begin{aligned}\delta_X(z,w) = \inf\{k_\Delta(\zeta,\eta)\colon\ &\text{there exists a holomorphic } \varphi\colon \Delta \to X\\ &\text{with } \varphi(\zeta) = z \text{ and } \varphi(\eta) = w\}\end{aligned}$$

for all z, $w \in X$. The Kobayashi pseudodistance $k_X\colon X \times X \to \mathbb{R}^+$ of X is the smallest pseudodistance on X bounded below by δX. We say that X is (Kobayashi) hyperbolic if k_X is a true distance and in that case it is known that the metric topology induced by k_X coincides with the manifold topology of X (see, e.g., [2], [1]). For instance, all bounded domains are hyperbolic (see, e.g., [2], [1]). The following properties are well-known in literature. The Kobayashi (pseudo)distance is contracted by holomorphic maps: if $f\colon X \to Y$ is a holomorphic map then

$$k_Y(f(z), f(w)) \leqslant k_X(z,w), \quad z,w \in X.$$

Next, the Kobayashi distance is invariant under biholomorphisms and decreases under inclusions: if $D_1 \subset D_2 \subset\subset \mathbb{C}^n$ are two bounded domains we have $k_{D_2}(z,w) \leqslant k_{D_1}(z,w)$ for all $z, w \in D_1$. Further, the Kobayashi distance of the unit disk coincides with the Poincare distance. Also, the Kobayashi distance of the unit ball $B^n \subset \mathbb{C}^n$ coincides with the well-known in many applications the so-called Bergman distance (see [2], [1], [18], [34]).

If X is a hyperbolic manifold, $z_0 \in X$ and $r \in (0,1)$ we shall denote by $B_X(z_0, r)$ the Kobayashi ball of center z_0 and radius $\frac{1}{2}\log((1+r)/(1-r))$:

$$B_X(z_0, r) = \{z \in X\colon \ \tanh k_X(z_0, z) < r\}.$$

We can see that $\varrho_X = \tanh k_X$ is still a distance on X, because tanh is a strictly convex function on $\mathbb{R}^+$. In particular, ϱ_{B^n} is the pseudohyperbolic distance of B^n.

The Kobayashi distance of bounded strongly pseudoconvex domains with smooth boundary has several important properties. First of all, it is complete (see [2], [1]), and hence closed Kobayashi balls are compact. It is vital that we can describe the boundary behavior of the Kobayashi distance: if $D \subset\subset \mathbb{C}^n$ is a strongly pseudoconvex bounded domain and $z_0 \in D$, there exist $c_0, C_0 > 0$ such that

$$c_0 - \frac{1}{2}\log d(z, \partial D) \leqslant k_D(z_0, z) \leqslant C_0 - \frac{1}{2}\log d(z, \partial D), \quad z \in D$$

where $d(\cdot, \partial D)$ denotes the Euclidean distance from the boundary of D (see [2], [1]). We provide some facts on Kobayashi balls of B^n (for proofs see [2], [1] and [34]). The ball $B_{B^n}(z_0, r)$ is given by

$$B_{B^n}(z_0, r) = \Big\{z \in B^n\colon \ \frac{(1-\|z_0\|^2)(1-\|z\|^2)}{|1-\langle z, z_0\rangle|^2} > 1 - r^2\Big\}.$$

Geometrically, it is an ellipsoid of (Euclidean) center

$$c = \frac{1-r^2}{1-r^2\|z_0\|^2} z_0,$$

its intersection with the complex line C_{z_0} is an Euclidean disk of radius

$$r\frac{1-\|z_0\|^2}{1-r^2\|z_0\|^2},$$

and its intersection with the affine subspace through z_0 orthogonal to z_0 is a Euclidean ball of the larger radius

$$r\sqrt{\frac{1-\|z_0\|^2}{1-r^2\|z_0\|^2}}.$$

Let ν denote the Lebesgue volume measure of $\mathbb{R}^{2n}$, normalized so that $\nu(B^n) = 1$. We denote by the same letter the Lebesgue measure on the pseudoconvex domain D. Then the volume of a Kobayashi ball $B_{B^n}(z_0, r)$ is given by (see [34])

$$\nu(B_{B^n}(z_0, r)) = r^{2n}\Big(\frac{1 - \|z_0\|^2}{1 - r^2\|z_0\|^2}\Big)^{n+1}.$$

A similar estimate is valid for the volume of Kobayashi balls in strongly pseudoconvex bounded domains:

Lemma A ([2], [1]). *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex bounded domain with smooth boundary. Then there exist $c_1 > 0$ and $C_{1,r} > 0$ for each $r \in (0,1)$, depending on r such that*

$$c_1 r^{2n} d(z_0, \partial D)^{n+1} \leqslant \nu(B_D(z_0, r)) \leqslant C_{1,r} d(z_0, \partial D)^{n+1}$$

for every $z_0 \in D$ and $r \in (0,1)$.

Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$. We shall use the following notation:

- ▷ $\delta\colon D \to \mathbb{R}^+$ will denote the Euclidean distance from the boundary, that is $\delta(z) = d(z, \partial D)$;
- ▷ given two nonnegative functions $f, g\colon D \to \mathbb{R}^+$ we shall write $f \preceq g$ to say that there is $C > 0$ such that $f(z) \leqslant Cg(z)$ for all $z \in D$. The constant C is independent of $z \in D$, but it might depend on other parameters (r, θ, etc.);
- ▷ given two strictly positive functions $f, g\colon D \to \mathbb{R}^+$ we shall write $f \approx g$ if $f \preceq g$ and $g \preceq f$, that is if there is $C \geqslant 1$ such that $C^{-1}g(z) \leqslant f(z) \leqslant Cg(z)$ for all $z \in D$;
- ▷ $H(D)$ will denote the space of holomorphic functions on D, endowed with the topology of uniform convergence on compact subsets;
- ▷ given $1 \leqslant p \leqslant \infty$, the *Bergman space* $A^p(D)$ is the Banach space $L^p(D) \cap H(D)$, endowed with the L^p-norm;
- ▷ more generally, given $\beta \in \mathbb{R}$ we introduce the *weighted Bergman space*

$$A^p_\beta(D) = A^p(D, \beta) = L^P(\delta^\beta \nu) \cap H(D)$$

endowed with the norm

$$\|f\|_{p,\beta} = \Big[\int_D |f(\zeta)|^p \delta^\beta(\zeta)\, \mathrm{d}\nu(\zeta)\Big]^{1/p}$$

if $1 \leqslant p < \infty$, and with the norm

$$\|f\|_{\infty,\beta} = \|f\delta^\beta\|_\infty$$

if $p = \infty$;

- ▷ $K\colon D \times D \to \mathbb{C}$ will be the Bergman kernel of D. Further, K_t is a kernel of type t defined in a standard manner with help of the well-known Henkin-Ramirez function, see [4]. Note we have $K = K_{n+1}$ (see [1], [4]);
- ▷ given $r \in (0,1)$ and $z_0 \in D$, we shall denote by $B_D(z_0, r)$ the Kobayashi ball of center z_0 and radius $\frac{1}{2}\log((1+r)/(1-r))$.

See, e.g., [2], [1], [13], [14] for definitions, basic properties and applications to the geometric function theory of the Kobayashi distance and [13], [14], [24] for definitions and basic properties of the Bergman kernel. Let us now recall a number of results proved in [2]. The first one gives information about the shape of Kobayashi balls.

Let further $\mathrm{d}\nu_t(z) = (\delta(z))^t\,\mathrm{d}\nu(z)$, $t > -1$.

Lemma B ([2], Lemma 2.2)**.** *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then there is $C > 0$ such that*

$$\frac{C}{1-r}\delta(z_0) \geqslant \delta(z) \geqslant \frac{1-r}{C}\delta(z_0)$$

for all $r \in (0,1)$, $z_0 \in D$ and $z \in B_D(z_0, r)$.

Definition 2.1. Let $D \subset\subset \mathbb{C}^n$ be a bounded domain, and $r > 0$. An r-lattice in D is a sequence $\{a_k\} \subset D$ such that $D = \bigcup\limits_k B_D(a_k, r)$ and there exists $m > 0$ such that any point in D belongs to at most m balls of the form $B_D(a_k, R)$, where $R = (1+r)/2$.

The existence of r-lattices in bounded strongly pseudoconvex domains is ensured by the following.

Lemma C ([2], Lemma 2.5)**.** *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Then for every $r \in (0,1)$ there exists an r-lattice in D, that is there exist $m \in N$ and a sequence $\{a_k\} \subset D$ of points such that $D = \bigcup\limits_{k=0}^{\infty} B_D(a_k, r)$ and no point of D belongs to more than m of the balls $B_D(a_k, R)$, where $R = (1+r)/2$, $\nu_\alpha(B_D(a_k, R)) = (\delta^\alpha(a_k))\nu(B_D(a_k, R))$, $\alpha > -1$; this equality follows directly from the properties of r-lattices on Kobayashi balls we listed above and the definition of the weighted Lebegues measure.*

We will sometimes call r-lattice the family of balls $B_D(a_k, r)$. Let $K(z,\xi)$ be a measurable function on $\overline{D} \times \overline{D}$ and let t be a positive number. We say that $K = K_t$

(or $\widetilde{K}_t$) is a kernel of Bergman type t for all $z \in D$, if $|K(z,\xi)| \leqslant C(|\widetilde{\Phi}(z,\xi)|^{-t})$ where $\widetilde{\Phi}$ is the Henkin-Ramirez function. So, if K is a kernel of Bergman type t then K^s is a kernel of type st, $s > 0$.

Dealing with an unweighted Bergman kernel K, $K = K_{n+1}$, we always assume $|K(z,a_k)| \asymp |K(a_k,a_k)|$ for any $z \in B_D(a_k,r)$, $r \in (0,1)$ (see [2], [1]). Based on the definition of the Bergman kernel via the Henkin-Ramirez function (see [4]), it is easy to see this assertion is valid also for all K_t kernels, $t = m(n+1)$, $m \in \mathbb{N}$.

The key ingredient of proofs in embeddings in analytic Herz type spaces below is the assumption that a little bit stronger condition holds, namely, $|K(z,a_k)|$ is equivalent to $|K(w,a_k)|$ for any Bergman kernel of type t for any w and z in $B_D(a_m,r)$ and any a_k, $k \in \mathbb{N}$, where m is any natural number. This is valid in the unit ball (see [34]) and also plays a key role in the main theorems (see [34]).

This condition, the additonal condition on the Bergman kernel we need in the proofs on Herz type spaces (Theorem 3.2) below, can probably be dropped.

We shall use a submean estimate for nonnegative plurisubharmonic functions on Kobayashi balls:

Lemma D ([2], Corollary 2.8)**.** *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain. Given $r \in (0,1)$, set $R = (1+r)/2 \in (0,1)$. Then there exists a $C_r > 0$ depending on r such that*

$$\chi(z) \leqslant \frac{C_r}{\nu(B_D(z_0,r))} \int_{B_D(z_0,R)} \chi \,\mathrm{d}\nu, \quad z_0 \in D,\ z \in B_D(z_0,r)$$

for every nonnegative plurisubharmonic function $\chi\colon D \to \mathbb{R}^+$.

We will use this lemma for $\chi = |f(z)|^q$, $f \in H(D)$, $q \in (0,\infty)$.

Using properties of Kobayashi balls $\{B_D(a_k,r)\}$ we will have the following estimates for the Bergman space $A^p_\alpha(D)$:

$$\begin{aligned}\|f\|^p_{A^p_\alpha} &= \int_D |f(w)|^p \delta^\alpha(w)\,\mathrm{d}\nu(w) \asymp \sum_{k=1}^{\infty} \Big[\max_{z\in B_D(a_k,r)} |f(z)|^p\Big] \nu_\alpha B_D(a_k,r) \\ &\asymp \sum_{k=1}^{\infty} \int_{B_D(a_k,R)} |f(z)|^p \delta^\alpha(z)\,\mathrm{d}\nu(z), \quad 0 < p < \infty,\ \alpha > -1.\end{aligned}$$

Let now

$$A(p,q,\alpha) = \Big\{ f \in H(D)\colon \sum_{k=1}^{\infty} \Big(\int_{B_D(a_k,r)} |f(z)|^p \delta^\alpha(z)\,\mathrm{d}\nu(z) \Big)^{q/p} < \infty \Big\},$$

where $0 < p, q < \infty$, $\alpha > -1$. These are Banach spaces if $\min(p,q) \geqslant 1$ and complete metric spaces for other values of parameters.

These $A(p,q,\alpha)$ spaces (or their multifunctional generalizations) can be viewed as natural extensions of classical Bergman spaces in strictly pseudoconvex domains with smooth boundary for which the family $\{B_D(a_k, r)\}$ related to the r-lattice $\{(a_k)\}$ exists (see [2], [1]). It is natural to consider the problem of extension of classical results on $A^p_\alpha(D)$ Bergman spaces to these $A(p,q,\alpha)$ spaces. Some of our results are motivated by this problem.

The next result is the main result of this section and contains the weighted L^p-estimates we shall need. Sometimes we denote the unweighted Bergman kernel K_{n+1} by K.

Theorem A ([1], [10])**.** *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain, and let $z_0 \in D$ and $1 \leqslant p < \infty$. Then*

$$\int_D |K(\zeta, z_0)|^p \delta^\beta(\zeta)\,\mathrm{d}\nu(\zeta) \preceq \begin{cases} \delta^{\beta-(n+1)(p-1)}(z_0), & \text{for } -1 < \beta < (n+1)(p-1); \\ |\log \delta(z_0)|, & \text{for } \beta = (n+1)(p-1); \\ 1, & \text{for } \beta > (n+1)(p-1). \end{cases}$$

In particular:

(i) *$\|K(\cdot, z_0)\|_{p,\beta} \preceq \delta^{\beta/p-(n+1)/q}(z_0)$ and $\|k_{z_0}\|_{p,\beta} \preceq \delta^{(n+1)/2+\beta/p-(n+1)/q}(z_0)$ when $-1 < \beta < (n+1)(p-1)$, where $q > 1$ is the conjugate exponent of p (and $(n+1)/q = 0$ when $p = 1$);*

(ii) *$\|K(\cdot, z_0)\|_{p,\beta} \preceq 1$ and $\|k_{z_0}\|_{p,\beta} \preceq \delta^{(n+1)/2}(z_0)$ when $\beta > (n+1)(p-1)$;*

(iii) *$\|K(\cdot, z_0)\|_{p,(n+1)(p-1)} \preceq \delta^{-\varepsilon}(z_0)$ and $\|k_{z_0}\|_{p,(n+1)(p-1)} \preceq \delta^{(n+1)/2-\varepsilon}(z_0)$ for all $\varepsilon > 0$.*

Furthermore,

(iv) *$\|K(\cdot, z_0)\|_{\infty,\beta} \approx \delta^{\beta-(n+1)}(z_0)$ and $\|k_{z_0}\|_{\infty,\beta} \approx \delta^{\beta-(n+1)/2}(z_0)$ for all $0 \leqslant \beta < n+1$; and $\|K(\cdot, z_0)\|_{\infty,\beta} \approx 1$ and $\|k_{z_0}\|_{\infty,\beta} \approx \delta^{(n+1)/2}(z_0)$ for all $\beta \geqslant n+1$.*

A complete analogue of this theorem is valid also for general K_t type kernels, $t > 0$ (see [4], [10]).

We add now shortly some basic facts on minimal bounded homogeneous domains which we will use partially in our paper (see [32], [33]).

We say the bounded $\mathcal{U}$ domain in $\mathbb{C}^n$ is a minimal domain with a center $t \in \mathcal{U}$ if the following condition is satisfied: for every biholomorphism $\psi\colon \mathcal{U} \to \mathcal{U}'$ with $\det J(\psi, t) = 1$ we have $\mathrm{Vol}(\mathcal{U}') \geqslant \mathrm{Vol}(\mathcal{U})$ where $J(\psi, t)$ denotes the complex Jacobi matrix of ψ at t (see [33]). We fix a minimal bounded homogeneous domain $\mathcal{U}$ with center t.

Let $\mathrm{d}V_\beta(z) = K_u(z,z)^{-\beta}\,\mathrm{d}V(z)$, $\beta \in \mathbb{R}$ and let $\mathrm{d}V$ denote the Lebesgue measure on $\mathcal{U}$, (see [33]). Let $L^p_{a,\beta}(\mathcal{U}, \mathrm{d}V_\beta) = L^p(\mathcal{U}, \mathrm{d}V_\beta) \cap H(\mathcal{U})$, $0 < p \leqslant \infty$, where $H(\mathcal{U})$ is the class of all analytic functions on $\mathcal{U}$. These spaces are nontrivial if and only if $\beta > \beta_{\min}$ for some fixed $\beta_{\min}$ (see [33]). Note we will always assume this below. These are Banach spaces for $p \geqslant 1$.

We below denote by $K^{(\beta)}_{\mathcal{U}}$ the reproducing kernel of $L^2_a(\mathcal{U}, \mathrm{d}V_\beta)$. Further L^2_a is the Bergman space on $\mathcal{U}$ (unweighted) and $L^2_a(\mathcal{U}, \mathrm{d}V) = L^2(\mathcal{U}, \mathrm{d}V) \cap H(\mathcal{U})$. It is known that $K_\beta = K^{(\beta)}_{\mathcal{U}}(z,w) = C_\beta K_u(z,w)^{1+\beta}$ for some positive constant C_β, (see [33]).

The Bergman kernel $K_u(z,w)$ of $\mathcal{U}$ is playing a very important role in our theorems below. Let $d_{\mathcal{U}}(\cdot,\cdot)$ be the Bergman distance on $\mathcal{U}$. For any $z \in \mathcal{U}$, $r > 0$, let $B(z,r) = \{w \in \mathcal{U}\colon\ d_{\mathcal{U}}(z,w) \leqslant r\}$ be the Bergman metric disk with center z and radius r.

The existence of the so-called Bergman sampling sequence can be seen in [33] (see also Lemma G). This sequence and estimates of the Bergman kernel on $\{B(z_k,\varrho)\}$ balls are very vital for this paper. We denote below the Lebesgue measure of the ball $B(z,\varrho)$ by Vol. We denote by $\mathrm{Vol}(E)$ the volume of the set E.

We supply three lemmas from [33] which are crucial for the proofs of theorems relating to the minimal domains (Theorems 3.4 and 3.5). Analogues in tube and pseudoconvex domains can bee seen in [3], [18], [29], [30].

Lemma E ([33]). *Take $\varrho > 0$. Then there exists $C_\varrho > 0$ such that*

$$C_\varrho^{-1} \leqslant \Bigl|\frac{K_{\mathcal{U}}(z,a)}{K_{\mathcal{U}}(a,a)}\Bigr| \leqslant C_\varrho, \quad z,a \in \mathcal{U},\ \beta_{\mathcal{U}}(z,a) \leqslant \varrho, \tag{2.1}$$

where $\beta_{\mathcal{U}}$ means the Bergman distance on $\mathcal{U}$.

Lemma F ([33]). *There exists a positive constant C such that*

$$|f(a)|^p \leqslant \frac{C}{\mathrm{Vol}(B(a,\varrho))} \int_{B(a,\varrho)} |f(z)|^p\,\mathrm{d}V(z), \tag{2.2}$$

$f \in H(\mathcal{U})$, $p \geqslant 1$, $a \in \mathcal{U}$.

Lemma G ([33]). *There exists a sequence $\{w_j\} \in \mathcal{U}$ satisfying the conditions*

$$\mathcal{U} = \bigcup_{j=1}^{\infty} B(w_j,\varrho), \quad B\Bigl(w_i,\frac{\varrho}{4}\Bigr) \cap B\Bigl(w_j,\frac{\varrho}{4}\Bigr) = \emptyset, \quad i \neq j.$$

There exists a positive integer n such that each point $z \in \mathcal{U}$ belongs to at most n sets $B(w_j, 2\varrho)$.

3. On some new sharp embedding theorems for mixed norm spaces in strictly pseudoconvex domain with smooth boundary and in minimal bounded homogeneous domain

This main part of our work contains formulations of all main results of this work and also the proofs of our main results in bounded strongly pseudoconvex domains with smooth boundary and in minimal bounded homogeneous domains. The theory of analytic spaces in bounded strictly pseudoconvex domains has developed rapidly during the last decades (see [6], [8], [14], [15], [20], [23]). Several Carleson type sharp embedding theorems for such spaces are known today (see [1], [6] and references there). The goal of this paper is to add to this list several new sharp assertions. We alert the reader that we extend our previous results in the unit ball of $\mathbb{C}^n$ from [18]. And the proofs are rather similar. Hovewer, we found our general results interesting enough to put them in a separate paper. We need for all our proofs as previously in the unit ball case various properties of r-lattices of D domain, which we listed in the previous section, and various properties of analytic functions on Kobayashi balls from recent papers [2] and [1] which we also listed above.

During the past decades the theory of Bergman spaces in strictly pseudoconvex domains with smooth boundary was developed in many papers by various authors. Here we consider direct generalizations of such spaces. For the Bergman space theory in the unit disk and in the unit ball we refer the reader to [9], [34]. One of the goals of this paper is to extend some results of standard weighted Bergman spaces in the strictly pseudoconvex domains in $\mathbb{C}^n$ to the case of more general $A(p,q,\alpha)$ classes of Bergman type classes.

Definition 3.1. Let D be a bounded domain with an r-lattice. Let μ be a positive Borel measure in D, $0 < p, q < \infty$, $s > -1$. Fix $r \in (0;\infty)$ and an r-lattice $\{a_k\}_{k=1}^{\infty}$. The analytic space $A(p,q,\mathrm{d}\mu)$ is the space of all holomorphic functions f such that

$$\|f\|^q_{A(p,q,\mathrm{d}\mu)} = \sum_{k=1}^{\infty}\left(\int_{B(a_k,r)} |f(z)|^p \,\mathrm{d}\mu(z)\right)^{q/p} < \infty.$$

If $\mathrm{d}\mu = \delta^s(z)\,\mathrm{d}\nu(z)$ then we will denote by $A(p,q,s)$ the space $A(p,q,\mathrm{d}\mu)$. This is a Banach space for $\min(p,q) \geqslant 1$. It is clear that $A(p,p,s) = A^p_s$.

Remark 3.1. It is clear now from the discussion above and the definition of $A(p,p,s)$ spaces that these classes are independent of $\{a_k\}$ and r. But in the general case of $A(p,q,s)$ spaces the answer is unknown. For simplicity we denote $\|f\|_{A(p,q,s,a_k,r)}$ by $\|f\|_{A(p,q,s)}$.

We also have the following estimates using the r-lattice:

$$\begin{aligned}\|f\|^q_{A(p,q,s)} &= \sum_{k=1}^{\infty}\Big(\int_D \chi_{B_D(a_k,r)}(z)|f(z)|^p\delta^s(z)\,\mathrm{d}\nu(z)\Big)^{q/p}\\ &\leqslant C\Big(\int_D |f(z)|^p\delta^s(z)\,\mathrm{d}\nu(z)\Big)^{q/p} = C\|f\|^q_{A^p_s}, \quad q\geqslant p,\ s>-1.\end{aligned}$$

So, finally, we have

$$\|f\|_{A(p,q,s)} \leqslant C\|f\|_{A^p_s}, \quad q\geqslant p,\ s>-1.$$

Motivated by this estimate we pose the following very natural and more general problem (as in the case of the unit ball).

Problem. Let μ be a positive Borel measure in D and let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence such that $B_D(a_k,r)$ is an r-lattice for a strictly pseudoconvex domain D with smooth boundary in $\mathbb{C}^n$. Let X be a quasinormed subspace of $H(D)$ and $p,q\in(0,\infty)$. Describe all positive Borel measures such that

$$\|f\|_{A(p,q,\mathrm{d}\mu)} \leqslant C\|f\|_X.$$

The following theorem gives a solution for Bergman spaces. It is known (see [30]), but we put it here with vital remarks after it to complete the picture.

Theorem 3.1. *Let $0<q,\ p<\infty$, $0<s\leqslant p<\infty$, $\beta>-1$. Let μ be a positive Borel measure on D. Then we have*

$$\|f\|_{A(q,p,\mathrm{d}\mu)} \leqslant c_1\|f\|_{A^s_\beta}$$

if and only if

$$\mu(B_D(a_k,r)) \leqslant c_2(\delta(a_k))^{q(n+1+\beta)/s} \tag{3.1}$$

for some constants $c_1,c_2>0$, $k\in\mathbb{N}$.

Remark 3.2. It is interesting that Theorem 3.1 can be extended even to more general mixed norm spaces (see [23]) if we replace the Bergman space norm on the right hand side of the estimate by the mixed norm space norm. This procedure was done for some other embedding theorems recently in a paper [25]. The ideas are similar to those used in the paper [25].

The following theorem gives a solution of the above mentioned problem for Herz type spaces. In the particular case of $q = q_1$ our Theorem 3.2 was shown in a recent paper [30].

Theorem 3.2. *Let $0 < q,\ s < \infty,\ q \geqslant s,\ q \geqslant q_1,\ \alpha > -1$. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence forming an r-lattice in D. Let μ be a positive Borel measure in D. Then*

$$\left(\int_D |f(z)|^q \,\mathrm{d}\mu(z)\right)^{1/q} \leqslant c_1\left(\int_D \left(\int_{B_D(z,r)} |f(w)|^s \,\mathrm{d}\nu_\alpha(w)\right)^{q_1/s} \mathrm{d}\nu(z)\right)^{1/q_1}$$

if and only if

$$\mu(B_D(a_k,r)) \leqslant c_2(\delta(a_k))^{q((n+1+\alpha)/s+(n+1)/q_1)} \tag{3.2}$$

for some constants $c_1, c_2 > 0$, $k \in \mathbb{N}$.

Remark 3.3. The unit ball case of Theorem 3.2 was considered before in [28], [31].

Remark 3.4. We denote the right hand side of the estimate in Theorem 3.2 by $D(f,s,\alpha,q)$ and by $D(f,s,\mu,q)$ replacing the Lebegues measure by any positive Borel measure. The problem of finding conditions on the positive Borel measure such that $D(f,s,\mu,q)$ is less than the A_β^P norm of f also appears naturally and some sufficient and necessary conditions can be found using the methods of this paper. In analytic Herz type spaces this type of problems was considered in the unit ball by the first author in [28], [31].

Finally we formulate a sharp result for multifunctional analytic function spaces in bounded pseudoconvex domains with smooth boundary.

Theorem 3.3. *Let μ be a positive Borel measure on D and $\{a_k\}$ a Kobayashi sampling sequence forming an r-lattice. Let $\alpha_i > \alpha_0$, for large enough α_0, $f_i \in H(D)$, $0 < p_i < q_i < \infty$, $i = 1, \ldots, m$, so that $\sum\limits_{i=1}^{m} 1/q_i = 1$. Let $(n+1+\alpha_i)/(p_i(n+1))$ be integer for all i. Then*

$$\int_D \prod_{i=1}^{m} |f_i(z)|^{p_i} \,\mathrm{d}\mu(z) \leqslant c_1 \prod_{i=1}^{m}\left[\sum_{k=1}^{\infty}\left(\int_{B(a_k,R)} |f_i(z)|^{p_i}\delta^{\alpha_i}(z)\,\mathrm{d}\nu(z)\right)^{q_i}\right]^{1/q_i},$$

$$R = \frac{1+r}{2}, \quad r > 0$$

if and only if

$$\mu(B_D(a_k,r)) \leqslant c_2\delta^{m(n+1)+\sum_{i=1}^{m}\alpha_i}(a_k) \tag{3.3}$$

for some constants $c_1, c_2 > 0$, $k \in \mathbb{N}$.

Remark 3.5. We wish to formulate also one more general functional case of Theorem 3.3. Let $q \geqslant s$ and $p \geqslant (q,s)$, $\alpha > \alpha_0$, for large enough α_0. Then

$$\left(\int_D |f(z)|^p \,\mathrm{d}\mu\right)^{1/p} \leqslant C \sum \left(\int_{B_D(a_k,R)} |f(z)|^s \delta^{\alpha}(z)\,\mathrm{d}\nu(z)\right)^{q/s}$$

for a positive Borel measure μ in D is valid if and only if $\mu(B_D(a_k,R)) \leqslant C\delta^{\tau_0}(a_k)$ for some constant C and τ_0 depending on p,q,s,α.

The proof of this assertion can be performed similarly in minimal bounded domains as well. Using the fact that $q \geqslant s$ we embed the right hand side of this estimate into the Bergman space norm and then continue as in [2] and [32].

Remark 3.6. The additional condition relating p_i and α_i in Theorem 3.3 as we can see from the proof below can be dropped, but in this case the proof is simpler.

Remark 3.7. The assertion of Theorem 3.3 can be found in the paper [18] for the case of the unit ball in $\mathbb{C}^n$. For $q_i = 1,\ p_i = p,\ m = 1$, it can be seen in [34] in the unit ball for $\alpha_j = \alpha,\ j = 1,\ldots,m$. Theorem 3.1 and Theorem 3.3 were given in [30] without detailed proofs in a sketchy form.

As was mentioned above we intend to give in this paper much more general versions of our earlier results proved before in the case of the unit ball in $\mathbb{C}^n$ in bounded strictly pseudoconvex domains with smooth boundary. We heavily use for this purpose the new vital technique which was developed in very recent vital papers [2], [1], where the so-called r-lattice was introduced and studied for bounded strictly pseudoconvex domains.

Note also that again here as before in the case of the unit ball all our proofs are heavily based on nice properties of the r-lattice, which we listed in the previous sections, mentioned above and which will not be mentioned again concretely below.

P r o o f of Theorem 3.1. Suppose (3.1) holds. Then we use the properties of the r-lattice, which we listed in the previous sections (Lemmas A–D):

$$\begin{aligned}
&\left(\sum_{k=1}^{\infty}\left[\int_{B_D(a_k,r)} |f(z)|^q \,\mathrm{d}\mu(z)\right]^{p/q}\right)^{s/p} \\
&\quad\leqslant C_1\left(\sum_{k=1}^{\infty} \max_{z\in B_D(a_k,r)} |f(z)|^p \delta^{p(n+1+\beta)/s}(a_k)\right)^{s/p} \\
&\quad\leqslant C_2 \sum_{k=1}^{\infty} \max_{z\in B_D(a_k,r)} |f(z)|^s \delta^{(n+1+\beta)}(a_k) \\
&\quad\leqslant C_3 \int_D |f(z)|^s \delta^{\beta}(z)\,\mathrm{d}\nu(z) \leqslant C_4\|f\|^s_{A^s_{\beta}(D)}, \quad \beta > -1,\ 0 < s < \infty.
\end{aligned}$$

Conversely, using an appropriate test function $f_k(z)$ and estimates from below of Bergman type kernel K_{n+1}, $|K(z,a_k)| \geqslant C_5(\delta(a_k))^{n+1}$, from [2], [1] and using also properties of an r-lattice which we listed in previous section, for the test function

$$f_k(z) = \delta^{(n+\tau+1)/s}(a_k) K^r_{n+1}(z,a_k), \quad z \in D,\ k \in \mathbb{N},\ r = \frac{t}{s(n+1)},$$
$$t = 2(n+1+\tau),$$

we can choose τ such that r is a natural number large enough (we follow the proof of the unit ball case) and noting that

$$\left(\int_{B_D(a_k,r)} |f(z)|^q \,\mathrm{d}\mu(z)\right)^{1/q} \leqslant C_6 \left[\sum_{k=1}^{\infty} \left(\int_{B_D(a_k,r)} |f(z)|^q \,\mathrm{d}\mu(z)\right)^{p/q}\right]^{1/p} \leqslant C_7 \|f\|_{A^s_\beta}, \tag{3.4}$$

we get what we need.

Indeed, putting f_k into (3.4) and using the fact that $\sup\limits_k \|f_k\|_{A^s_\beta} \leqslant C\delta^{\beta-\tau}(a_k)$ which follows from Theorem A (see also [4]) we get what we need. The proof is complete. $\square$

P r o o f of Theorem 3.2. Let (3.2) hold. We have for the same $\{a_k\}$ sequence and using the properties of the r-lattice, which we listed in the previous sections (Lemmas A–D)

$$\int_D |f(w)|^q \,\mathrm{d}\mu(w) \leqslant \sum_{k=1}^{\infty} \sup_{w \in B_D(a_k,r)} |f(w)|^q \mu(B_D(a_k,r))$$
$$\leqslant C_1 \sum_{k=1}^{\infty} \left(\sup_{w \in B_D(a_k,r)} |f(w)|^s\right)^{q/s} \delta^{q((n+1+\alpha)/s+(n+1)/q_1)}(a_k).$$

Then we have $\delta(w) \asymp \delta(z)$, $z \in B_D(w,r)$ (see [2], [1]) and hence

$$\int_{B_D(a_k,R)} |f(z)|^s \,\mathrm{d}\nu(z) \leqslant C_2 \int_{B_D(a_k,2R)} \left(\int_{B_D(z,r)} |f(\widetilde{w})|^s \,\mathrm{d}\nu_\alpha(\widetilde{w})\right) \frac{\mathrm{d}\nu(z)}{\delta^{n+1+\alpha}(z)}.$$

Hence we have now $t = q((n+1+\alpha)/s + (n+1)/q_1)$,

$$\int_D |f(z)|^q \,\mathrm{d}\mu(z)$$
$$\leqslant C_3 \sum_{k=1}^{\infty} \left(\int_{B_D(a_k,R)} |f(z)|^s \,\mathrm{d}\nu(z) \frac{1}{\delta^{n+1}(a_k)}\right)^{q/s} (\delta(a_k))^t$$
$$\leqslant C_4 \sum_{k=1}^{\infty} \left(\int_{B_D(a_k,R)} \int_{B_D(z,r)} |f(\widetilde{w})|^s \,\mathrm{d}\nu_\alpha(\widetilde{w}) \,\mathrm{d}\nu(z)\right)^{q/s} (\delta^{n+1}(a_k))^{q/q_1}.$$

By Hölder's inequality we have, using the properties of the r-lattice (Lemmas A–D)

$$\left(\int_{B_D(a_k,R)}\int_{B_D(z,r)}|f(\widetilde{w})|^s\,\mathrm{d}\nu_\alpha(\widetilde{w})\frac{\mathrm{d}\nu(z)}{\delta^{n+1}(z)}\right)^{q/s}$$
$$\leqslant\int_{B_D(a_k,R)}\left(\int_{B_D(z,r)}|f(\widetilde{w})|^s\,\mathrm{d}\nu_\alpha(\widetilde{w})\right)^{q/s}(\delta^{-(n+1)s/q}(a_k))\,\mathrm{d}\nu(z),\quad R=\frac{1+r}{2}.$$

Combining all the above estimates we get the desired results. We show the reverse. We have for $\{a_k\}$, $z\in D$, $k=1,2,\dots$ and β which is big enough.

$$f_k(z)=\delta^{\beta-(n+1+\alpha)/s-(n+1)/q_1}(a_k)[K_{n+1}(z,a_k)]^{\widetilde{\beta}},\quad \widetilde{\beta}=\frac{\beta}{n+1},$$
$$\tau=\beta q_1-q_1\frac{n+1+\alpha}{s}-(n+1)$$

($\widetilde{\beta}$ can be chosen to be a large positive integer). Then by Theorem A and Lemmas A–D we have

$$\int_D\left(\int_{B_D(w,r)}|f_k(z)|^s\,\mathrm{d}\nu_\alpha(z)\right)^{q_1/s}\mathrm{d}\nu(w)\leqslant\left[C(\delta^\tau(a_k))\left(\frac{1}{\delta^\tau(a_k)}\right)\right]\leqslant\text{const}.$$

Then we have, using the estimate from below of the Bergman kernel as we did above

$$\int_D|f_k(z)|^q\,\mathrm{d}\mu(z)\geqslant\mu(B_D(a_k,r))[\delta^{-q((n+1+\alpha)/s+(n+1)/q_1)}(a_k)].$$

The rest is clear (see also [18]). $\square$

Note that in all the proofs we repeat the arguments from the case of the unit ball (see, for example, [18]).

P r o o f of Theorem 3.3. We assume $(n+1+\alpha_j)/(p_i(n+1))$ is integer for all i. First suppose that (3.3) holds. Then using properties of r-lattices which we listed in the previous sections and the Kobayashi balls we have (we put $\alpha_j=\alpha$ for all j and the general case is the same here)

$$\int_D\prod_{i=1}^m|f_i(z)|^{p_i}\,\mathrm{d}\mu(z)$$
$$\leqslant C_1\sum_{k=1}^\infty\mu(B_D(a_k,r))\prod_{i=1}^m\sup_{z\in B_D(a_k,r)}|f_i(z)|^{p_i}\int_D\prod_{i=1}^m|f_i(z)|^{p_i}\,\mathrm{d}\mu(z)$$
$$\leqslant C_2\sum_{k=1}^\infty\frac{\mu(B_D(a_k,r))}{\delta^{m(n+1+\alpha)}(a_k)}\prod_{i=1}^m\int_{B_D(a_k,R)}|f_i(w)|^{p_i}\delta^\alpha(w)\,\mathrm{d}\nu(w)$$
$$\leqslant C_3\sum_{k=1}^\infty\prod_{i=1}^m\int_{B_D(a_k,R)}|f_i(w)|^{p_i}\delta^\alpha(w)\,\mathrm{d}\nu(w).$$

Using the condition $\sum\limits_{i=1}^{m} 1/q_i = 1$ and Hölder's inequality for m functions we get what we need. The reverse follows from the chain of equalities and estimates based again on the properties of the r-lattice, which we listed in the previous section. Indeed we have as above for the test function f_i

$$f_i(z) = \delta^{(n+1+\alpha_i)/p_i}(a_k) K^{\tau_i}_{n+1}(a_k, z), \quad \tau_i = \frac{2(n+1+\alpha_i)}{(n+1)p_i},\ i = 1, \dots, m.$$

We choose α_i such that τ_i is a large enough positive integer.

By the properties of the r-lattice, which we listed in the previous sections (Lemmas A–D) we have

$$\begin{aligned}\int_D \prod_{i=1}^{m} |f_i(z)|^{p_i} \, \mathrm{d}\mu(z) &\geqslant \int_{B_D(a_k,r)} \delta^{m(n+1)+\sum_{j=1}^{m}\alpha_j}(a_k) K_\tau(a_k, z) \, \mathrm{d}\mu(z) \\ &\geqslant \frac{\mu(B_D(a_k,r))}{\delta^{m(n+1)+\sum_{j=1}^{m}\alpha_j}(a_k)}.\end{aligned}$$

Hence we get what we need. Indeed we have the estimates

$$\begin{aligned}&\prod_{i=1}^{m}\Bigl(\sum_{k=1}^{\infty}\Bigl(\int_{B_D(a_k,R)} |f_i(z)|^{p_i}\delta^{\alpha_i}(z)\,\mathrm{d}\nu(z)\Bigr)^{q_i}\Bigr)^{1/q_i} \\ &\qquad \leqslant \prod_{i=1}^{m}\sum_{k=1}^{\infty}\int_{B_D(a_k,R)} |f_i(z)|^{p_i}(\delta^{\alpha_i}(z))\,\mathrm{d}\nu(z) \\ &\qquad \leqslant C_4 \prod_{i=1}^{m}\int_D |f_i(z)|^{p_i}(\delta^{\alpha_i}(z))\,\mathrm{d}\nu(z) \leqslant C_5, \quad R = \frac{1+r}{2}.\end{aligned}$$

□

The careful analysis of proofs we provided above shows various similarities with our previously mentioned work in the unit ball. Nevertheless, bounded strictly pseudoconvex domains are much more general as domains than the unit balls.

The goal of this subsection is to obtain also new sharp results on Bergman type analytic spaces in minimal bounded homogeneous domains. Our results were known before only in the very particular case of domains of such type in the unit ball. Our results are heavily based on a series of subtle new estimates obtained recently in [33]. We note domains we consider here are direct generalizations of the well-studied so-called bounded symmetric domains in $\mathbb{C}^n$ (see [33]). Note, also, that all the above mentioned domains and even the polydisk are examples of minimal domains.

Proofs of our last theorems are simply copies of our previous parallel theorems in bounded pseudoconvex domains with smooth boundary (see above) and we omit the details of these proofs for that reason.

The following theorem is a new sharp result on embeddings in $L^2_a(\mathcal{U}, \mathrm{d}V_\beta)$ analytic function spaces.

Theorem 3.4. *Let $1 < p,\ q < \infty$, $1 < s \leqslant p < \infty$, $\beta > \beta_{\min}$, $\varrho > 0$. Let $\{z_k\}$ be a sampling sequence in $\mathcal{U}$, let μ be a positive Borel measure on $\mathcal{U}$. Then*

$$\sum_{k=1}^{\infty}\Big(\int_{B(z_k,\varrho)} |f(z)|^q \,\mathrm{d}\mu(z)\Big)^{p/q} \leqslant c_1 \|f\|_{L^s_a(\mathcal{U},\mathrm{d}V_\beta)} \tag{3.5}$$

if and only if

$$\mu(B(z_k,\varrho) \leqslant c_2(\mathrm{Vol})(B(z_k,\varrho))^{\beta_0} \tag{3.6}$$

for all $\{z_k\} \in \mathcal{U}$, $\varrho > 0$, for some fixed β_0, $\beta_0 = \beta_0(\beta, q, s, n)$ and for some constants $c_1, c_2 > 0$, $k \in \mathbb{N}$.

Note it was shown in [33] that a condition similar to (3.6) holds if and only if

$$\int_{\mathcal{U}} |f(z)|^p \,\mathrm{d}\mu(z) \leqslant \widetilde{C} \int_{\mathcal{U}} |f(z)|^p \,\mathrm{d}V_\beta(z) \tag{3.7}$$

for all $p > 0$ and for all $f \in L^p_a(\mathcal{U}, \mathrm{d}V_\beta)$.

The proof of this result and those of Theorems 3.4, 3.5 are similar. Note the proofs of theorems of this paper can be obtained after careful study of estimates of the proof of the unit ball case and parallel estimates obtained recently in the case of bounded minimal homogeneous domains in $\mathbb{C}^n$ (see [18], [27], [26], [33]).

The following theorem is another new sharp result on embeddings in $L^2_a(\mathcal{U}, \mathrm{d}V_\beta)$ analytic function spaces in minimal bounded homogeneous domain in $\mathbb{C}^n$. The base of proof are the Forelly-Rudin estimates and a lower estimate for the Bergman kernel.

Theorem 3.5. *Let μ be a positive Borel measure on $\mathcal{U}$, and $\{z_k\}$ a Bergman sampling sequence. Let $\alpha > \alpha_{\min}$, $f_i \in H(\mathcal{U})$, $1 < p_i, q_i < \infty$, $i = 1, \ldots, m$ so that $\sum\limits_{i=1}^{m} 1/q_i = 1$. Then*

$$\int_{\mathcal{U}} \prod_{i=1}^{m} |f_i(z)|^{p_i} \,\mathrm{d}\mu(z) \leqslant c_1 \prod_{i=1}^{m} \Big[\sum_{k=1}^{\infty}\Big(\int_{B(z_k,2r)} |f_i(z)|^{p_i} \,\mathrm{d}V_\alpha(z)\Big)^{q_i}\Big]^{1/q_i} \tag{3.8}$$

if and only if $\mu(B(z_k, r)) \leqslant c_2(\mathrm{Vol}(B(z_k, r)))^{\alpha_0}$ for every $k \in \mathbb{N}$, $r > 0$, for some fixed α_0, $\alpha_0(m, n, \alpha)$ and for some constants c_1, $c_2 > 0$.

Similar results with similar proofs were obtained by the first author in tubular domains over symmetric cones (unbounded domains) and bounded strictly pseudoconvex (nonsymmetric) domains (see [3], [29], [30] and references there).

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